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Matrix-exponential distributions: Closure properties

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Abstract

Analysing the properties of a probability distribution is a question of general interest. In this paper we describe the properties of the matrix-exponential class of distributions, developing some properties for the discrete case and proving the closure properties, which for the case of phase-type distributions are extended to the matrix-exponential case, this not being an immediate consequence. Given the structure of this class of distributions, we were able to achieve the results in both matrix and algorithmic form. These results can be used in stochastic modelling, and in the latter case, the model can be analysed in matrix and algorithmic form.

Keywords: *Matrix-exponential distribution, Phase-type distribution, Closure properties*

1 Introduction

Analysing the properties of a probability distribution is a question of immediate interest. These properties enable us to determine the internal structure of the distribution and can be used to study various associated measures. In this paper we analyse certain properties of matrix-exponential distributions (ME), which were preceded by the phase-type (PH) class. This class of distributions, with an underlying Markov structure, enables us to express in algorithmic and algebraic form certain results and measures that are intractable when complex probability models are developed in a classical form. This class of distributions has been used in several fields, such as telecommunications, queuing theory, survival analysis and reliability theory. It was introduced by Neuts (1975)[1] and a comprehensive treatment of its properties was provided in Neuts (1981) [2]. Among these properties is the fact that the class of phase-type distributions is closed under a variety of operations; this has been considered in the modelling of systems in fields such as reliability and queuing theory. Some examples are given in Shaked and Shantikumar (1985)[3]; more recently, Pérez-Ocón and Ruiz-Castro (2004)[4] investigated general repairable systems involving phase-type distributions. One important property of phase-type distributions is that they are dense in the class of all distributions defined on non-negative real numbers. A characterization of this class of distributions is given in O'Cinneide (1990)[5], following the structure of the Laplace-Stieltjes transform.

The class of ME distributions is wider than that of phase-type distributions. Both classes have distributions with the same form but representations of ME distributions do not necessarily have a simple probabilistic interpretation. Given that the representations have a similar form, when a ME distribution is considered in complex analysis, the results can also be achieved in an algorithmic form.

On the other hand, the ME class can be considered a class with a rational Laplace-Stieltjes transform. This fact characterizes this class of distributions, and was proven by Asmussen and Bladt (1997)[6]. A study of this class of distributions is given in the latter paper and in Asmussen and O'Cinneide (1998)[7]. ME distributions have been applied in several fields, particularly in queuing theory, and can be interpreted in various ways. Thus, Bladt and Neuts (2003)[8] consider an interpretation via flows, using random stopping times of deterministic flows. In addition, they introduced a generalization of the Markovian arrival process (MAP) to the setting of ME distributions, thus obtaining the Rational arrival process (RAP). Other studies have also been performed with ME distributions; Fackrell (2005)[9] developed a characterization for ME distributions and used it for fitting data by maximum likelihood estimation. In this paper we analyse the overparameterizing of ME distributions.

We study ME distribution and develop its properties. The ME distribution has useful properties that allow it to be used in the construction of general analytical models. For the discrete case, a characterization is made using the probability generating function of a discrete ME distribution. The closure properties of phase-type distributions are extended to the ME class of distributions. This extension is not an immediate consequence from the phase-type case.

The Kronecker product and its properties are commonly used in matrix-analytical methods when PH distributions are presented, and also with ME distributions. We define this and present some of its properties.

Definition 1.1. *If $\mathbf{M}$ and $\mathbf{N}$ are rectangular matrices with dimensions $m_1 \times m_2$ and $n_1 \times n_2$ respectively, then their Kronecker product $\mathbf{M} \otimes \mathbf{N}$ is a matrix with dimension $m_1 n_1 \times m_2 n_2$ written in a block partitioned form as $(M_{ij}\mathbf{N})$ for $i = 1, \ldots, m_1$ and $j = 1, \ldots, m_2$.*

Kronecker product property. If $\mathbf{M}$, $\mathbf{N}$, $\mathbf{U}$ and $\mathbf{V}$ are rectangular matrices with appropriate dimensions for defining the products $\mathbf{MU}$ and $\mathbf{NV}$ then

$$(\mathbf{M} \otimes \mathbf{N})(\mathbf{U} \otimes \mathbf{V}) = \mathbf{MU} \otimes \mathbf{NV}.$$

Definition 1.2. *If $\mathbf{M}$ and $\mathbf{N}$ are rectangular matrices of order m and n respectively and $\mathbf{I}_m$ and $\mathbf{I}_n$ identity matrices of order m and n respectively then their Kronecker sum $\mathbf{M} \oplus \mathbf{N}$ is*

$$\mathbf{M} \oplus \mathbf{N} = \mathbf{M} \oplus \mathbf{I}_n + \mathbf{I}_m \oplus \mathbf{N}.$$

Kronecker sum property. If $\mathbf{M}$ and $\mathbf{N}$ are rectangular matrices of order m and n respectively, then

$$\exp(\mathbf{M} \oplus \mathbf{N}) = \exp(\mathbf{M})) \otimes \exp(\mathbf{N}).$$

The paper is organized as follows: the matrix-exponential distribution is defined in Section 2 for the discrete and continuous case, and its properties are shown. Section 3 focuses on analysing the closure properties of ME distributions. Finally, concluding remarks are shown in Section 4.

2 Probability of matrix-exponential distributions

In this section matrix-exponential distributions (ME), in the discrete and continuous cases, are defined. Throughout the paper, $\mathbf{I}$ denotes the identity matrix and $\mathbf{e}$ a column vector of ones, in both cases with the appropriate order. Also, the transpose of a matrix A is denoted as A'.

2.1 Continuous ME distributions

Definition 2.1. *A continuous non-negative random variable is distributed according to a ME distribution with representation $(\beta, \mathbf{S}, \nu)$ if its density function has the form*

$$g(t) = \beta exp(\mathbf{S}t)\nu \quad ; \quad t \geq 0$$

where β is a row vector, ν is a column vector and $\mathbf{S}$ is a matrix in which complex entries are allowed. The order of the matrix $\mathbf{S}$ is the dimension of the representation.

Matrix $\mathbf{S}$ must be non singular and the following conditions are necessary: $-\beta\mathbf{S}^{-1}\nu = 1$ and $\beta\nu \geq 0$.

The distribution function is given by

$$F(t) = 1 + \beta \exp(\mathbf{S}t)\mathbf{S}^{-1}\nu = 1 + \beta\mathbf{S}^{-1}\exp(\mathbf{S}t)\nu \quad ; \quad t \geq 0,$$

and the moment generating function (mgf) is

$$\hat{B}(s) = \beta(-s\mathbf{I} - \mathbf{S})^{-1}\nu.$$

Given the mgf the Laplace-Stieltjes transform is

$$\phi(s) = \beta(s\mathbf{I} - \mathbf{S})^{-1}\nu \quad ; \quad Re(s) > 0. \tag{1}$$

The $n-th$ moment has the expression

$$(-1)^{n-1} n! \beta \mathbf{S}^{-n-1}\nu.$$

From the above expression, the expected value of a ME distribution with representation $(\beta, \mathbf{S}, \nu)$ is $\beta\mathbf{S}^{-2}\nu$.

2.2 Discrete ME distributions

Definition 2.2. *A discrete non-negative random variable is distributed according to a ME distribution with representation* $(\alpha, \mathbf{T}, \mathbf{s})$ *if its probability mass function (pmf) is*

$$p_k = \alpha \mathbf{T}^{k-1} \mathbf{s} \quad ; \quad k \geq 1.$$

The matrix $\mathbf{I} - \mathbf{T}$ must be non-singular and it is verified that $\alpha(\mathbf{I} - \mathbf{T})^{-1}\mathbf{s} = 1$; $\alpha\mathbf{s} \geq 0$ and $\alpha\mathbf{T}^{k-1}\mathbf{s} \geq 0$, for $k \geq 1$. The order of the matrix $\mathbf{T}$ is the dimension of the representation. The distribution function of this distribution is given by

$$F(n) = \alpha(\mathbf{I} - \mathbf{T}^n)(\mathbf{I} - \mathbf{T})^{-1}\mathbf{s} = 1 - \alpha\mathbf{T}^n(\mathbf{I} - \mathbf{T})^{-1}\mathbf{s} \quad ; \quad n \geq 1,$$

and the survival one is

$$S(n) = \alpha\mathbf{T}^n(\mathbf{I} - \mathbf{T})^{-1}\mathbf{s} \quad ; \quad n \geq 1.$$

From the above definition, the probability generating function (pgf) is calculated and has the form

$$P(z) = z\alpha\left(\mathbf{I} - z\mathbf{T}\right)^{-1}\mathbf{s} = \alpha\left(z^{-1}\mathbf{I} - \mathbf{T}\right)^{-1}\mathbf{s}, \tag{2}$$

with factorial moments

$$P^{(k)}(1) = k!\alpha\mathbf{T}^{k-1}\left(\mathbf{I} - \mathbf{T}\right)^{-k-1}\mathbf{s} = k!\alpha\left(\mathbf{I} - \mathbf{T}\right)^{-k-1}\mathbf{T}^{k-1}\mathbf{s},$$

where $P^{(k)}(1) = \frac{d^k}{d^k z}P(z)\Big|_{z=1}$.

Therefore, the expected value of a ME distribution with representation $(\alpha, \mathbf{T}, \mathbf{s})$ is

$$\mu = P^{(1)}(1) = \alpha\left(\mathbf{I} - \mathbf{T}\right)^{-2}\mathbf{s}.$$

In this section, two useful properties are proven. A characterization is made using the probability generating function (Proposition 2.3) and any discrete ME distribution can have a representation $(\alpha, \mathbf{T}, \mathbf{u})$ with $\mathbf{u} = \mathbf{e} - \mathbf{T}\mathbf{e}$, proven in Proposition 2.4.

Proposition 2.3. *The probability generating function of a discrete Matrix-Exponential distribution can be written as*

$$P(z) = \frac{b_1 z^n + b_2 z^{n-1} + \cdots + b_{n-1}z^2 + b_n z}{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + 1} \tag{3}$$

for some $n \geq 1$ *and constants* $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$. *Moreover, the distribution has the following representation* $(\alpha, \mathbf{T}, \mathbf{u})$

$$\alpha = (b_1, b_2, \ldots, b_n),$$

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & -a_{n-4} & \cdots & -a_2 & -a_1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{4}$$

Proof. Let us consider a discrete ME distribution with representation $(\beta, \mathbf{S}, \nu)$. The probability generating function for a discrete ME distribution is given in (2) and the following expression is obtained from Proposition 2.3. given in Asmussen and Bladt (1997)[6]. The Laplace-Stieltjes transform given in (1) can be written as

$$\hat{f}(s) = \beta\left(s\mathbf{I} - \mathbf{S}\right)^{-1}\nu = \frac{b_1 + b_2 s + b_3 s^2 + \cdots + b_n s^{n-1}}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n},$$

for some $n \geq 1$ and constants $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$.

If it is considered then

$$\begin{aligned} P(z) &= z\beta\left(\mathbf{I} - z\mathbf{S}\right)^{-1}\nu = \beta\left(z^{-1}\mathbf{I} - \mathbf{S}\right)^{-1}\nu \\ &= \hat{f}(z^{-1}) = \frac{b_1 z^n + b_2 z^{n-1} + \cdots + b_{n-1}z^2 + b_n z}{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + 1}. \end{aligned}$$

It remains to be proved that the probability generating function (3) coincides with $z\alpha\left(\mathbf{I}-z\mathbf{T}\right)^{-1}\mathbf{u}$, where α , $\mathbf{T}$ and $\mathbf{u}$ are given in (4). The following expression, $\hat{f}(s)=\alpha\left(s\mathbf{I}-\mathbf{T}\right)^{-1}\mathbf{u}$, is proved in Proposition 2.3 in Asmussen and Bladt (1997)[6]. The proof is completed given that $P(z)=\hat{f}(z^{-1})$. □

Proposition 2.4. *For any discrete ME distribution, their exists a representation* $(\alpha,\mathbf{T},\mathbf{u})$ *such that* $\mathbf{Te}+\mathbf{u}=\mathbf{e}$.

Proof. A minimal representation $(\beta,\mathbf{S},\nu)$ of the ME distribution is considered and the probability mass function is equal to $f(k)=\beta\mathbf{S}^{k-1}\nu$.

Let $\mathbf{M}$ be a non-singular matrix; then

$$f(k)=\beta\mathbf{M}\mathbf{M}^{-1}\mathbf{S}^{k-1}\mathbf{M}\mathbf{M}^{-1}\nu=\beta\mathbf{M}(\mathbf{M}^{-1}\mathbf{S}\mathbf{M})^{k-1}\mathbf{M}^{-1}\nu.$$

This matrix must verify $\mathbf{M}^{-1}\nu+\mathbf{M}^{-1}\mathbf{SMe}=\mathbf{e}$ which is equivalent to $\mathbf{Me}=(\mathbf{I}-\mathbf{S})^{-1}\nu$.

Using the structure given in (4) from Proposition 2.3 we can consider the matrix $\mathbf{M}$ with the following form

$$\mathbf{M}=\frac{1}{1+\sum_i a_i}\mathbf{I}.$$

Then the result is obtained with $\alpha=\beta\mathbf{M}$, $\mathbf{T}=\mathbf{S}$ and $\mathbf{u}=\mathbf{M}^{-1}\nu$. □

Remark 2.5. *A discrete ME distribution with representation* $(\alpha,\mathbf{T},\mathbf{u})$*, such that* $\mathbf{u}=\mathbf{e}-\mathbf{Te}$*, will be denoted as* $(\alpha,\mathbf{T})$*. Analogously for the continuous case with the condition* $\mathbf{u}=-\mathbf{Te}$.

In general, for the discrete and continuous cases, the representation of a ME distribution is not unique as can be seen in Asmussen and Bladt (1997) [6]. The minimal one is defined as follows.

Definition 2.6. *A representation* $(\alpha,\mathbf{T},\mathbf{s})$ *of a matrix-exponential distribution is said to be minimal if it has the lowest possible dimension.*

3 Closure properties of ME distributions

The result of several operations involving ME distributions again has a ME distribution. This aspect is important when ME distributions are considered in applications and for studying properties for different probabilistic models. Complex operations that in general require numerical expressions are replaced by matrix ones with these properties. This is very useful because the results obtained are expressed in algorithmic and matrix forms, which makes it possible to implement the results computationally. For each case, the representation of the ME distribution is given.

3.1 Convolution and mixture operators

Proposition 3.1. *Let* $F(\cdot)$ *and* $G(\cdot)$ *continuous (or discrete) ME distributions with representations* $(\alpha_1,\mathbf{T}_1,\mathbf{s}_1)$ *and* $(\alpha_2,\mathbf{T}_2,\mathbf{s}_2)$ *respectively. Then the convolution* $F*G(\cdot)$ *is a ME distribution with representation* $(\gamma,\mathbf{L},\nu)$ *where* $\gamma=(\alpha_1,\mathbf{0})$*,* $\nu=(\mathbf{0},\mathbf{s}_2)'$ *and*

$$\mathbf{L}=\begin{pmatrix}\mathbf{T}_1 & \mathbf{s}_1\alpha_1\\ \mathbf{0} & \mathbf{T}_2\end{pmatrix}.$$

Proof. The Laplace-Stieltjes of $(\gamma,\mathbf{L},\nu)$ is given by $\phi(s)=\gamma(s\mathbf{I}-\mathbf{L})^{-1}\nu$. Given the expressions of the above matrices

$$\begin{aligned}\gamma\left(s\mathbf{I}-\mathbf{L}\right)^{-1}\mathbf{u} &= (\alpha_1,\mathbf{0})\begin{pmatrix}s\mathbf{I}-\mathbf{T}_1 & -s_1\alpha\\ \mathbf{0} & s\mathbf{I}-\mathbf{T}_2\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{0}\\ \mathbf{s}_2\end{pmatrix}\\ &= (\alpha_1,\mathbf{0})\begin{pmatrix}\left(s\mathbf{I}-\mathbf{T}_1\right)^{-1} & \left(s\mathbf{I}-\mathbf{T}_1\right)^{-1}\mathbf{s}_1\alpha_2\left(s\mathbf{I}-\mathbf{T}_2\right)^{-1}\\ \mathbf{0} & \left(s\mathbf{I}-\mathbf{T}_2\right)^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{0}\\ \mathbf{s}_2\end{pmatrix},\end{aligned}$$

which is equal to

$$\alpha_1\left(s\mathbf{I}-\mathbf{T}_1\right)^{-1}\mathbf{s}_1\alpha_2\left(s\mathbf{I}-\mathbf{T}_2\right)^{-1}\mathbf{s}_2.$$

□

Proposition 3.2. *Let $(p_1, p_2, \ldots, p_n)$ be a discrete probability distribution and $F_j(\cdot)$, $1 \geq j \geq n$ continuous ME distributions with representations $(\alpha_j, \mathbf{T}_j, \mathbf{s}_j)$. The mixture is ME distributed with representation*

$$\left[(p_1\alpha_1, p_2\alpha_2, \ldots, p_n\alpha_n), \begin{pmatrix} \mathbf{T}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{T}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T}_n \end{pmatrix}, \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \vdots \\ \mathbf{s}_n \end{pmatrix}\right].$$

Proof. The distribution of the mixture is given by

$$\begin{aligned} F(t) &= \sum_{i=1}^{n} p_i F_i(t) = \sum_{i=1}^{n} p_i \left(1 + \alpha_i \exp(\mathbf{T}_i t)\mathbf{T}_i^{-1}\mathbf{s}_i\right) = 1 + \sum_{i=1}^{n} p_i\alpha_i \exp(\mathbf{T}_i t)\mathbf{T}_i^{-1}\mathbf{s}_i \\ &= 1 + (p_1\alpha_1, p_2\alpha_2, \ldots, p_n\alpha_n) \\ &\cdot \exp\left[\begin{pmatrix} \mathbf{T}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{T}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T}_n \end{pmatrix} t\right] \begin{pmatrix} \mathbf{T}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{T}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T}_n \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \vdots \\ \mathbf{s}_n \end{pmatrix}, \end{aligned}$$

which completes the proof. □

A result with infinite mixtures is given in the following proposition.

Proposition 3.3. *Let $\{s_l\}$ be a discrete ME density with representation $(\beta, \mathbf{S}, \nu)$ and $F(\cdot)$ a continuous ME distribution with representation $(\alpha, \mathbf{T}, \mathbf{u})$. Then the mixture of the successive convolutions of $F(\cdot)$ distribution, $\sum_{l=0}^{\infty} s_l F^{(l)}(\cdot)$, is ME distributed with representation $(\gamma, \mathbf{L}, \mathbf{r})$ and*

$$\gamma = \alpha \otimes \beta \quad ; \quad \mathbf{L} = \mathbf{T} \otimes \mathbf{I} + \mathbf{u}\alpha \otimes \mathbf{S} \text{ and } \mathbf{r} = \mathbf{u} \otimes \nu.$$

Proof. Let $P(z)$ and $f(z)$ be the probabilitites generating function and the density function of a discrete and continuous ME distribution respectively; then

$$\begin{aligned} P(z) &= z\beta\left(\mathbf{I} - z\mathbf{S}\right)^{-1}\nu \\ f(s) &= \alpha\left(s\mathbf{I} - \mathbf{T}\right)^{-1}\mathbf{u}. \end{aligned}$$

The Laplace-Stieltjes transform of the mixture is given by

$$P(f(s)) = f(s)\beta\left(\mathbf{I} - f(s)\mathbf{S}\right)^{-1}\nu.$$

This result is proved if,

$$\gamma\left(s\mathbf{I} - \mathbf{L}\right)^{-1}\mathbf{r} = f(s)\beta\left(\mathbf{I} - f(s)\mathbf{S}\right)^{-1}\nu.$$

Firstly, let us prove the inverse matrix. We denote it as $\mathbf{U} = \left(s\mathbf{I} - \mathbf{L}\right)^{-1}$, then

$$\mathbf{I} = \left(s\mathbf{I} - \mathbf{L}\right)^{-1}\mathbf{U} = s\mathbf{U} - \left(\mathbf{T} \otimes \mathbf{I}\right)\mathbf{U} - \left[\mathbf{u}\alpha \otimes \mathbf{S}\right]\mathbf{U}.$$

Given that $\mathbf{I} - \mathbf{T}$ is a non-singular matrix, $s\mathbf{I} - \mathbf{T} \otimes \mathbf{I} = (s\mathbf{I} - \mathbf{T}) \otimes \mathbf{I}$ is also non-singular and so the above expression can be written as

$$\left(s\mathbf{I} - \mathbf{T} \otimes \mathbf{I}\right)\mathbf{U} = \mathbf{I} + \left[\mathbf{u}\alpha \otimes \mathbf{S}\right]\mathbf{U},$$

and then

$$\left(s\mathbf{I} - \mathbf{T}\right)^{-1} \otimes \mathbf{I} = \left[\mathbf{I} - \left[\left(s\mathbf{I} - \mathbf{T}\right)^{-1} \otimes \mathbf{I}\right]\left[\mathbf{u}\alpha \otimes \mathbf{S}\right]\right]\mathbf{U}.$$

Therefore

$$\begin{aligned} \mathbf{U} &= \left[\mathbf{I} - \left[\left(s\mathbf{I} - \mathbf{T}\right)^{-1} \otimes \mathbf{I}\right]\left[\mathbf{u}\alpha \otimes \mathbf{S}\right]\right]^{-1}\left[\left(s\mathbf{I} - \mathbf{T}\right)^{-1} \otimes \mathbf{I}\right] \\ &= \sum_{i=0}^{\infty}\left[\left(s\mathbf{I} - \mathbf{T}\right)^{-1}\mathbf{u}\alpha \otimes \mathbf{S}\right]^{i}\left[\left(s\mathbf{I} - \mathbf{T}\right)^{-1} \otimes \mathbf{I}\right] \\ &= \sum_{i=0}^{\infty}\left[\left(s\mathbf{I} - \mathbf{T}\right)^{-1}\mathbf{u}\alpha\right]^{i}\left(s\mathbf{I} - \mathbf{T}\right)^{-1} \otimes \mathbf{S}^{i}. \end{aligned}$$

And finally

$$\begin{aligned}\gamma(s\mathbf{I}-\mathbf{L})^{-1}\mathbf{r} &= (\alpha\otimes\beta)\left[\sum_{i=0}^{\infty}\left[(s\mathbf{I}-\mathbf{T})^{-1}\mathbf{u}\alpha\right]^{i}(s\mathbf{I}-\mathbf{T})^{-1}\otimes\mathbf{S}^{i}\right](\mathbf{u}\otimes\nu)\\ &= \sum_{i=0}^{\infty}\alpha\left[(s\mathbf{I}-\mathbf{T})^{-1}\mathbf{u}\alpha\right]^{i}(s\mathbf{I}-\mathbf{T})^{-1}\mathbf{u}\otimes\beta\mathbf{S}^{i}\nu\end{aligned}$$

Given that $f(s)=\alpha(s\mathbf{I}-\mathbf{T})^{-1}\mathbf{u}$,

$$\alpha\left[(s\mathbf{I}-\mathbf{T})^{-1}\mathbf{u}\alpha\right]^{i}(s\mathbf{I}-\mathbf{T})^{-1}\mathbf{u}=\alpha\left[(s\mathbf{I}-\mathbf{T})^{-1}\mathbf{u}\alpha\right]^{i+1}=f(s)^{i+1},$$

and so

$$\gamma(s\mathbf{I}-\mathbf{L})^{-1}\mathbf{r}=\sum_{i=0}^{\infty}f(s)^{i+1}\otimes\beta\mathbf{S}^{i}\nu=\sum_{i=0}^{\infty}f(s)\beta\left(f(s)\mathbf{S}\right)^{i}\nu=f(s)\beta\left(\mathbf{I}-f(s)\mathbf{S}\right)^{-1}\nu.$$

□

An analogous result can be proved when $F(\cdot)$ is a discrete ME. We show it without proof.

Proposition 3.4. *Let $\{s_l\}$ and $\{p_j\}$ be discrete ME densities with representations $(\beta,\mathbf{S},\nu)$ and $(\alpha,\mathbf{T},\mathbf{u})$ of orders n and m respectively. Then the distribution $\sum_{l=0}^{\infty}s_l\{p_j\}^{(l)}$, is ME distributed with representation $(\gamma,\mathbf{L},\mathbf{r})$ and*

$$\gamma=\alpha\otimes\beta\quad;\quad\mathbf{L}=\mathbf{T}\otimes\mathbf{I}+\mathbf{u}\alpha\otimes\mathbf{S}\ \text{ and }\ \mathbf{r}=\mathbf{u}\otimes\nu.$$

3.2 Maximun and minimum operators

In several fields, such as reliability and queuing theory, maximun and minimun operators are frequently used. The closure properties for these operators have been studied for different classes of distributions.

Proposition 3.5. *Let $F_1(\cdot)$ and $F_2(\cdot)$ be ME distributions with representation $(\alpha_1,\mathbf{T}_1,\mathbf{s}_1)$ and $(\alpha_2,\mathbf{T}_2,\mathbf{s}_2)$ respectively. Then the operator $\max(F_1,F_2)$ is ME distributed with representation $(\gamma,\mathbf{L},\mathbf{u})$ being*

$$\gamma=(\alpha_1\otimes\alpha_2,\alpha_1,\alpha_2)\quad;\quad\mathbf{u}=\left(\left(\mathbf{T}_1^{-1}\oplus\mathbf{T}_2^{-1}\right)(\mathbf{s}_1\otimes\mathbf{s}_2),\mathbf{s}_1,\mathbf{s}_2\right)',$$

$$\mathbf{L}=\begin{pmatrix}\mathbf{T}_1\oplus\mathbf{T}_2 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{T}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{T}_2\end{pmatrix}.$$

Proof. The distribution function of a ME distribution with representation $(\gamma,\mathbf{L},\mathbf{u})$ is equal to

$$\begin{aligned}F(t) &= 1+(\alpha_1\otimes\alpha_2,\alpha_1,\alpha_2)\left[\exp\left(\begin{pmatrix}\mathbf{T}_1\oplus\mathbf{T}_2 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{T}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{T}_2\end{pmatrix}t\right)\right]\begin{pmatrix}\mathbf{T}_1\oplus\mathbf{T}_2 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{T}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{T}_2\end{pmatrix}^{-1}\\ &\quad\cdot\begin{pmatrix}\left(\mathbf{T}_1^{-1}\oplus\mathbf{T}_2^{-1}\right)(\mathbf{s}_1\otimes\mathbf{s}_2)\\ \mathbf{s}_1\\ \mathbf{s}_2\end{pmatrix}\\ &= 1+(\alpha_1\otimes\alpha_2,\alpha_1,\alpha_2)\begin{pmatrix}\exp\left((\mathbf{T}_1\oplus\mathbf{T}_2)t\right) & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \exp(\mathbf{T}_1t) & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \exp(\mathbf{T}_2t)\end{pmatrix}\\ &\quad\cdot\begin{pmatrix}(\mathbf{T}_1\oplus\mathbf{T}_2)^{-1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{T}_1^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{T}_2^{-1}\end{pmatrix}\begin{pmatrix}\left(\mathbf{T}_1^{-1}\oplus\mathbf{T}_2^{-1}\right)(\mathbf{s}_1\otimes\mathbf{s}_2)\\ \mathbf{s}_1\\ \mathbf{s}_2\end{pmatrix}\\ &= 1+(\alpha_1\otimes\alpha_2)\exp\left((\mathbf{T}_1\oplus\mathbf{T}_2)t\right)(\mathbf{T}_1\oplus\mathbf{T}_2)^{-1}(\mathbf{s}_1\otimes\mathbf{s}_2)+\alpha_1\exp(\mathbf{T}_1t)\mathbf{T}_1^{-1}\mathbf{s}_1\\ &\quad+\alpha_2\exp(\mathbf{T}_2t)\mathbf{T}_2^{-1}\mathbf{s}_2.\end{aligned}$$

The expression $\left(\mathbf{T}_1^{-1}\oplus\mathbf{T}_2^{-1}\right)(\mathbf{s}_1\otimes\mathbf{s}_2)$ can be expressed in the following way,

$$\begin{aligned}\left(\mathbf{T}_1^{-1}\oplus\mathbf{T}_2^{-1}\right)(\mathbf{s}_1\otimes\mathbf{s}_2) &= \mathbf{T}_1^{-1}\mathbf{s}_1\otimes\mathbf{s}_2+\mathbf{s}_1\otimes\mathbf{T}_2^{-1}\mathbf{s}_2\\ &= (\mathbf{T}_1\oplus\mathbf{T}_2)\left(\mathbf{T}_1^{-1}\mathbf{s}_1\otimes\mathbf{T}_2^{-1}\mathbf{s}_2\right).\end{aligned}\tag{5}$$

Therefore, the distribution function can be expressed as

$$\begin{aligned}
F(t) &= 1+(\alpha_1\otimes\alpha_2)\exp\left((\mathbf{T}_1\oplus\mathbf{T}_2)t\right)\left(\mathbf{T}_1^{-1}\mathbf{s}_1\otimes\mathbf{T}_2^{-1}\mathbf{s}_2\right)+\alpha_1\exp(\mathbf{T}_1t)\mathbf{T}_1^{-1}\mathbf{s}_1\\
&\quad+\alpha_2\exp(\mathbf{T}_2t)\mathbf{T}_2^{-1}\mathbf{s}_2\\
&= 1+\alpha_1\exp(\mathbf{T}_1t)\mathbf{T}_1^{-1}\mathbf{s}_1\alpha_2\exp(\mathbf{T}_2t)\mathbf{T}_2^{-1}\mathbf{s}_2+\alpha_1\exp(\mathbf{T}_1t)\mathbf{T}_1^{-1}\mathbf{s}_1\\
&\quad+\alpha_2\exp(\mathbf{T}_2t)\mathbf{T}_2^{-1}\mathbf{s}_2\\
&= \left[1+\alpha_1\exp(\mathbf{T}_1t)\mathbf{T}_1^{-1}\mathbf{s}_1\right]\left[1+\alpha_2\exp(\mathbf{T}_2t)\mathbf{T}_2^{-1}\mathbf{s}_2\right]\\
&= F_1(t)F_2(t).
\end{aligned}$$

□

Remark 3.6. *Another representation is given by* $(\gamma,\mathbf{L},\mathbf{u})$, *where*

$$\gamma=(\alpha_1\otimes\alpha_2,\mathbf{0},\mathbf{0})\qquad \mathbf{L}=\begin{pmatrix}\mathbf{T}_1\oplus\mathbf{T}_2 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{T}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{T}_2\end{pmatrix}\quad;\quad \mathbf{u}=(\mathbf{0},\mathbf{s}_1,\mathbf{s}_2)'.$$

Proposition 3.7. *Let* $F_1(\cdot)$ *and* $F_2(\cdot)$ *be ME distributions with representation* $(\alpha_1,\mathbf{T}_1,\mathbf{s}_1)$ *and* $(\alpha_2,\mathbf{T}_2,\mathbf{s}_2)$ *respectively. Then the operator* $\min(F_1,F_2)$ *is ME distributed with representation* $(\gamma,\mathbf{L},\mathbf{u})$ *being*

$$\gamma=(\alpha_1\otimes\alpha_2)\quad;\quad \mathbf{L}=\mathbf{T}_1\oplus\mathbf{T}_2\quad;\quad \mathbf{u}=-\left(\mathbf{T}_1^{-1}\oplus\mathbf{T}_2^{-1}\right)(\mathbf{s}_1\otimes\mathbf{s}_2).$$

Proof. The argument is similar to that described in Proposition 3.5. The distribution function of a ME distribution with representation $(\gamma,\mathbf{L},\mathbf{u})$ is

$$F(t)=1-(\alpha_1\otimes\alpha_2)\exp\left((\mathbf{T}_1\oplus\mathbf{T}_2)t\right)(\mathbf{T}_1\oplus\mathbf{T}_2)^{-1}\left(\mathbf{T}_1^{-1}\oplus\mathbf{T}_2^{-1}\right)(\mathbf{s}_1\otimes\mathbf{s}_2).$$

and after operating on it and considering (5) we have

$$\begin{aligned}
F(t) &= 1-(\alpha_1\otimes\alpha_2)\exp\left((\mathbf{T}_1\oplus\mathbf{T}_2)t\right)\left(\mathbf{T}_1^{-1}\mathbf{s}_1\oplus\mathbf{T}_2^{-1}\mathbf{s}_2\right)\\
&= 1-\alpha_1\exp(\mathbf{T}_1t)\mathbf{T}_1^{-1}\mathbf{s}_1\alpha_2\exp(\mathbf{T}_2t)\mathbf{T}_2^{-1}\mathbf{s}_2\\
&= 1-(1-F_1(t))(1-F_2(t)).
\end{aligned}$$

□

3.3 Integral operator

In fields such as reliability, given a probability distribution it is of interest to analyse the mean time by time t. To study this, the integral calculus must be considered. With the following results, the integrals can be replaced by algebraic operations, thus providing algebraic results.

Proposition 3.8. *Let* $(\alpha,\mathbf{T},\mathbf{u})$ *be a minimal representation of a ME distribution* $F(\cdot)$. *Then* $F^*(t)=\frac{1}{\mu}\int_0^t(1-F(r))\,dr$ *is the distribution function of a ME distribution, where* μ *is the expected value of* $F(\cdot)$. *The representation of this new distribution is given by* $(\pi,\mathbf{T},\mathbf{u})$, *being* $\pi=-\mu^{-1}\alpha\mathbf{T}^{-1}$.

Proof.

$$\begin{aligned}
F^*(t) &= \frac{1}{\mu}\int_0^t(1-F(r))\,dr=\frac{1}{\alpha\mathbf{T}^{-2}\mathbf{u}}\alpha\mathbf{T}^{-1}\left(\mathbf{I}-\exp(\mathbf{T}t)\right)\mathbf{T}^{-1}\mathbf{u}\\
&= 1-\frac{1}{\alpha\mathbf{T}^{-2}\mathbf{u}}\alpha\mathbf{T}^{-1}\exp(\mathbf{T}t)\mathbf{T}^{-1}\mathbf{u}\\
&= 1+\pi\exp(\mathbf{T}t)\mathbf{T}^{-1}\mathbf{u}.
\end{aligned}$$

□

Remark 3.9. *The vector* π *is the unique solution of the system*

$$\pi(\mathbf{T}+\mathbf{u}\alpha)=\mathbf{0}\quad;\quad -\pi\mathbf{T}^{-1}\mathbf{u}=1$$

and verifies $\pi\mathbf{u}=\mu^{-1}$. *Moreover,* F *also has the representation* $(\pi',\mathbf{T},\nu)$, *where* $\pi'=-\mu^{-1}\alpha$ *and* $\nu=\mathbf{T}^{-1}\mathbf{u}$.

Corollary 3.10. *Let $(\alpha, \mathbf{T}, \mathbf{u})$ be a minimal representation of a ME distribution with expected value equal to μ and with distribution function $F(\cdot)$. Then, the function $F^*(t) = \frac{1}{\mu}\int_0^t (1 - F(r))\, dr$ is the distribution function of a ME distribution with representation $(\omega, \mathbf{S})$, being $\omega = -\mu^{-1}\alpha\mathbf{T}^{-1}\mathbf{M}$ and $\mathbf{S} = \mathbf{M}^{-1}\mathbf{T}\mathbf{M}$, where M is the matrix*

$$\mathbf{M} = \begin{pmatrix} \frac{1}{a_n} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

being a_n the value given in Corollary 2.1 in Asmussen and Bladt (1997)[6].

Proof. Let us consider the Corollary 2.1., given in Asmussen and Bladt (1997)[6], to express the ME distribution $(\alpha, \mathbf{T}, \mathbf{u})$ through the representation $(\beta, \mathbf{S})$, where $\beta = \alpha\mathbf{M}$ and $\mathbf{S} = \mathbf{M}^{-1}\mathbf{T}\mathbf{M}$. From this new representation and after operating, the function F^* has the following expression:

$$F^*(t) = 1 + \mu^{-1}\beta\mathbf{S}^{-1}\exp(\mathbf{S}t)\,\mathbf{e} = 1 + \mu^{-1}\alpha\mathbf{T}^{-1}\mathbf{M}\exp(\mathbf{S}t)\,\mathbf{e} = 1 - \omega\exp(\mathbf{S}t)\,\mathbf{e}.$$

$\square$

Remark 3.11. *The vector ω is the unique solution for the system $\omega(\mathbf{S} + \nu\beta) = \mathbf{0}$ verifying $\omega\mathbf{e} = 1$, where $\nu = -\mathbf{S}\mathbf{e}$. It is verified that $\omega\nu = \mu^{-1}$. The vectors ω and π are related through the expression $\omega = \pi\mathbf{M}$.*

4 Conclusions

Analogously to PH ones, ME distributions have a structured form, which allows us to work in an algorithmic form in stochastic models in which these distributions are involved. We show that the closure properties of PH distributions can be extended to the wider class of ME distributions, also in a well structured form. This extension is not an immediate consequence from the properties of PH distributions. In addition, we show a characterization for discrete ME distributions and some properties are proved. These properties enable us to facilitate stochastic modelling in several fields. Given that PH distributions are dense in the set of non-negative distributions, the ME class is also dense. For this reason, in stochastic modelling, when general distributions are involved, the latter can be approximated by ME distributions. This methodology allows us to develop the models and to deduce the measures in a tractable, algebraic and algorithmic form. We show that operations which require numerical integrations can be replaced by matrix operations. This class of distributions is not subject to probabilistic interpretation as are PH distributions, and they can be physically interpreted.

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