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Research paper

Comparison of estimates using censored samples from Gompertz model: Bayesian, E-Bayesian, hierarchical Bayesian and empirical Bayesian schemes

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Abstract

This paper aims to introduce a comparative study for the E-Bayesian criteria with three various Bayesian approaches; Bayesian, hierarchical Bayesian and empirical Bayesian. This study is concerned to estimate the shape parameter and the hazard function of the Gompertz distribution based on type-II censoring. All estimators are obtained under symmetric loss function [squared error loss (SELF))] and three different asymmetric loss functions [quadratic loss function (QLF), entropy loss function (ELF) and LINEX loss function (LLF)]. Comparisons among all estimators are achieved in terms of mean square error (MSE) via Monte Carlo simulation.

Keywords: *Bayes estimates; E-Bayes estimates; Empirical Bayes estimates; Gompertz distribution; Hierarchical Bayes estimates.*

1. Introduction

The Gompertz distribution has great importance in modeling human mortality and actuarial tables. It has many applications, particularly in medical and actuarial studies. Also, it used as a survival model in reliability. Historically, the Gompertz distribution was first proposed by Gompertz [1]. The probability density function (pdf), cumulative distribution function (cdf), and hazard function $h(t)$ of the two-parameter Gompertz distribution are given, respectively, by

$$f(x;\lambda,\theta)=\lambda\theta\exp\left[\lambda x-\theta(e^{\lambda x}-1)\right],\qquad x>0,\ \lambda,\theta>0, \qquad (1\text{-}1)$$

$$F(x;\lambda,\theta)=1-\exp\left[-\theta(e^{\lambda x}-1)\right],\qquad x>0,\ \lambda,\theta>0 \qquad (1\text{-}2)$$

And

$$h(t;\lambda,\theta)=\lambda\theta\exp(\lambda t),\qquad t>0,\ \lambda,\theta>0 \qquad (1\text{-}3)$$

Where λ and θ are the scale and shape parameters respectively. Recently, many authors have studied the Gompertz distribution; for example, Grag [2] discussed the properties of the Gompertz distribution and estimate its parameters by using the maximum likelihood method. Chen [3] reproduced an exact confidence interval and exact joint confidence region for the parameters associated to the Gompertz distribution based on type-II censoring. Jaheen [4] constructed the Bayesian technique for the Gompertz distribution under record values. Wu et al [5] obtained the point and interval estimators for the unknown parameters corresponding to the Gompertz distribution based on progressive type-II censored samples. Gohary [6] introduced the bivariate Gompertz distribution and completed the analysis for the mixture of components of the proposed distribution. Saracoglu et al [7] compared the non-Bayes and Bayes estimates for the unknown parameters of the Gompertz distribution. Ismail [8] derived point and interval estimates for the Gompertz distribution based on partially accelerated life tests with type-II censoring. Feroze and Aslam [9] obtained point and interval estimates for the parameters of the two-component mixture of the Gompertz model based on Bayes Method along with posterior predictions for the future value from model. Sarabia et al [10] exploded several properties of the Gompertz distribution when lifetime or other kinds of data available fully observed.
The E-Bayesian estimation is a new method of estimation first introduced by Han [11]. Han [12] derived the E-Bayes and hierarchical Bayes estimates of the reliability parameter for testing data from products with exponential distribution under type-I censoring and by considering the quadratic loss function. He proved that via simulation, the E-Bayesian estimator is efficient and easy to operate. Han [13] obtained the E-Bayesian estimation of the failure probability based on type-I censored data and by using the quadratic loss function. Yin and Liu [14] applied the E-Bayesian estimation and hierarchical Bayesian estimation methods for estimating the unknown reliability parameter of the geometric distribution under scaled squared loss function in complete samples. They deducted that the E-Bayes criteria is more stability and convenient in terms of calculation complexity than the hierarchical Bayes method. Han [15] obtained the E-Bayes and hierarchical Bayes estimates of reliability for testing data from products with binomial distribution under type-I censoring and by considering the quadratic loss function. He showed that by using simulation the E-Bayes technique is much simpler than the hierarchical Bayes method to operate. Wei et al [16] constructed the minimum risk equivariant estimation and E-Bayes estimation methods for estimating the unknown parameter of the Burr-XII distribution based on entropy loss function in complete samples. They deducted that

E-Bayes estimates have most accuracy. Jaheen and Okasha [17] compared the Bayesian and E-Bayesian estimators for the parameters and reliability function of the Burr Burr-XII distribution based on type-II censoring and by considering the squared error loss and LINEX loss functions. They deducted that the overall performance of the E-Bayes estimates are better than the similar obtained by using the Bayes technique. Cai et al [18] applied the E-Bayesian estimation method for forecasting of security investment. Okasha [19] constructed the maximum likelihood, Bayesian and E-Bayesian methods for estimating the unknown scale parameter and reliability and hazard functions of the Weibull distribution under type-2 censored samples and by considering the squared error loss function. He concluded that the E-Bayes estimates were more efficient than the maximum likelihood estimates or the Bayes estimates. Wu [20] introduced the Bayesian estimation and E-Bayesian estimation techniques in a new integral interval for estimating the failure probability under zero-failure data and by considering the quadratic loss function. Azimi et al [21] estimated the parameter and reliability function of the generalized half Logistic distribution by using the Bayes and E-Bayes methods based on progressively type-II censoring and by considering the squared error loss and LINEX loss functions. They deducted that the E-Bayes criteria generally is more efficient than the Bayes criteria. Javadkani et al [22] applied the Bayes, empirical Bayes and E-Bayes techniques for estimating the unknown shape parameter and the reliability function of the two parameter bathtub-shaped lifetime distribution based on progressively first-failure-censored samples and by considering the minimum expected loss and LINEX loss functions. Liu et al [23] used the E-Bayes and hierarchical methods for estimating the unknown parameter of the Rayleigh distribution under q-symmetric entropy loss function in complete samples. They deducted that the two techniques were close to each other when the sample size is large enough and the E-Bayes estimation was more convenient in terms of calculation complexity. Okasha [24] constructed the Bayesian and the E-Bayesian methods for estimating the scale parameter, reliability and hazard functions of the Lomax distribution based on type-2 censored and by considering the balanced squared error loss function. He pointed out that the performance of the E-Bayes estimates is generally better than the Bayes estimates. Reyad and Othman [25] obtained the Bayesian and E-Bayesian estimates for the shape parameter of the Gumbell type-II distribution based on type-II censoring and by considering squared error, LINEX, Degroot, Quadratic and minimum expected loss functions. They deducted that the E-Bayes estimates were generally much better than the other estimates.

The goal of this paper is to introduce a statistical comparison between the E-Bayesian criteria versus other three techniques of Bayesian approaches; Bayesian, hierarchical and empirical Bayesian to illustrate the potential usefulness of the E-Bayesian estimates which are simple in calculations and efficient. The resulting estimates are obtained based on symmetric and different asymmetric loss functions and the all outcomes obtained in this article can be generalized to use in complete sample.

The layout of the paper is as follow. In Section 2, the Bayes estimates of the parameter θ and the hazard function $h(t)$ based on type-II censored sample are derived under SELF, QLF, ELF and LLF. The E- Bayes estimates are obtained of the parameter θ and the hazard function $h(t)$ based on type-II censored sample under SELF, QLF, ELF and LLF in Section 3. In Sections 4, 5, the hierarchical Bayes estimates and empirical Bayes of the parameter θ and the hazard function $h(t)$ are derived based on type-II censored sample under SELF, QLF, ELF and LLF respectively. In Section 6, a Monte Carlo simulation is done to compare the behavior of the resulting estimators. Some concluding remarks have been given in the last Section.

2. Bayesian estimation

In this section, we will obtain the Bayes estimates of the shape parameter θ and the hazard function $h(t)$ of the Gompertz distribution by considering symmetric loss function (SELF)) and three asymmetric loss functions (QLF, ELF and LLF). Based on type-II censored samples of size r obtained from a life test of n items from the Gompertz in (1-1) and (1-2) distribution, the likelihood function can be written as

$$L(\theta|\underline{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\lambda\theta\exp\left[\lambda x_{(i)} - \theta(e^{\lambda x_{(i)}}-1)\right]\left[\exp\left(-\theta(e^{\lambda x_{(r)}}-1)\right)\right]^{n-r}$$

$$\propto \theta^{r}\exp\left[-\theta Q\right] \qquad (2\text{-}1)$$

Where

$$Q = \left\{\exp\left[\lambda\sum_{i=1}^{r} x_{(i)}\right] + (n-r)\left[\exp\left(\lambda x_{(r)}\right)\right] - r\right\} \qquad (2\text{-}2)$$

Assuming λ is known, we can use the gamma distribution as an conjugate prior distribution of θ with shape and scale parameter a and b respectively and its pdf given by

$$g(\theta|a,b) = \frac{b^{a}}{\Gamma(a)}\theta^{a-1}\exp\left[-b\,\theta\right], \qquad \theta>0, \qquad a,b>0 \qquad (2\text{-}3)$$

Combining (2-1) and (2-3), from Bayesian theorem the posterior density function of θ can be obtained as

$$\pi(\theta|\underline{x}) = \frac{L(\theta|\underline{x})\,g(\theta|a,b)}{\int_0^{\infty} L(\theta|\underline{x})\,g(\theta|a,b)\,d\theta}$$

$$= \frac{(Q+b)^{r+a}}{\Gamma(r+a)}\theta^{r+a-1}e^{-(Q+b)}, \qquad \theta>0 \qquad (2\text{-}4)$$

That mean, the posterior distribution of θ obeys $\Gamma(r+a, Q+b)$.

2.1. Bayesian estimation under squared error loss function (SELF)

A commonly used loss function is the square error loss function (SELF) introduced by Mood et al (26) as follows:

$$L_1(\hat{\theta},\theta) = k(\hat{\theta}-\theta)^2, \qquad k>0 \qquad (2\text{-}5)$$

Where $\hat{\theta}$ is an estimator of θ and k is the scale of the loss function. The scale k is often taken equal to one which has no effect on the Bayes estimates. This loss function is symmetric in nature. i.e. it gives equal importance to both over and under estimation. The Bayes estimator of θ denoted by $\hat{\theta}_{BS}$ can be obtained as

$$\hat{\theta}_{BS} = E_{\pi}(\theta|\underline{x}) \qquad (2\text{-}6)$$

Where E_{π} indicated to the expectation of the posterior distribution. We can derived $\hat{\theta}_{BS}$ by using (2-4) in (2-6) to be

$$\hat{\theta}_{BS} = \frac{r+a}{Q+b} \qquad (2\text{-}7)$$

We can also obtain the Bayes estimator of $h(t)$ based on SELF denoted as $\hat{h}_{BS}$ by replacing $\hat{\theta}_{BS}$ given in (2-7) instead of θ given in (1-3) to be

$$\hat{h}_{BS} = \lambda\left(\frac{r+a}{Q+b}\right)e^{\lambda t} \tag{2-8}$$

2.2. Bayesian estimation under quadratic loss function (QLF)

Bhuiyan et al [27] defined the quadratic loss function (QLF) as follows:

$$L_2(\hat{\theta},\theta) = \left(\frac{\theta-\hat{\theta}}{\theta}\right) \tag{2-9}$$

The Bayes estimator of θ based on QLF denoted by $\hat{\theta}_{BQ}$ can be obtained as

$$\hat{\theta}_{BQ} = \frac{E_\pi(\theta^{-1}|\underline{x})}{E_\pi(\theta^{-2}|\underline{x})} \tag{2-10}$$

We can derived $\hat{\theta}_{BQ}$ by using (2-4) in (2-10) to be

$$\hat{\theta}_{BQ} = \frac{r+a-2}{Q+b} \tag{2-11}$$

We can also obtain the Bayes estimator of $h(t)$ based on SELF denoted as $\hat{h}_{BQ}$ by replacing $\hat{\theta}_{BQ}$ given in (2-11) instead of θ given in (1-3) to be

$$\hat{h}_{BQ} = \lambda\left(\frac{r+a-2}{Q+b}\right)e^{\lambda t} \tag{2-12}$$

2.3. Bayesian estimation under entropy loss function (ELF)

Day et al [28] have discussed the entropy loss function (ELF) of the form

$$L_3(\hat{\theta},\theta) \propto \left(\frac{\hat{\theta}}{\theta}\right) - \ln\left(\frac{\hat{\theta}}{\theta}\right) - 1 \tag{2-13}$$

The Bayes estimator of θ relative to ELF denoted by $\hat{\theta}_{BE}$ can be obtained as

$$\hat{\theta}_{BE} = \left[E_\pi(\theta^{-1}|\underline{x})\right]^{-1} \tag{2-14}$$

We can obtain $\hat{\theta}_{BE}$ by using (2-4) in (2-14) to be

$$\hat{\theta}_{BE} = \frac{r+a-1}{Q+b} \tag{2-15}$$

The Bayes estimator of $h(t)$ relative to ELF denoted as $\hat{h}_{BE}$ by replacing $\hat{\theta}_{BE}$ given in (2-15) instead of θ given in (1-3) to be

$$\hat{h}_{BE} = \lambda\left(\frac{r+a-1}{Q+b}\right)e^{\lambda t} \tag{2-16}$$

2.4. Bayesian estimation under LINEX loss function (LLF)

Zellner [29] represent the LINEX (linear-exponential) loss function (LLF) to be

$$L_4(\hat{\theta},\theta) = m\left\{\exp\left[s(\hat{\theta}-\theta)\right] - s(\hat{\theta}-\theta) - 1\right\} \tag{2-17}$$

With two parameters $m>0, s\neq 0,$ where m is the scale of the loss function and s determines its shape. Without loss of generality, we assume $m=1$. The Bayes estimator relative to LLF denoted by $\hat{\theta}_{BL}$ can be obtained as

$$\hat{\theta}_{BL} = \left(\frac{-1}{s}\right)\ln\left[E_\theta\left(e^{-s\theta}|\underline{x}\right)\right] \tag{2-18}$$

We can obtain $\hat{\theta}_{BL}$ by using (2-4) in (2-18) to be

$$\hat{\theta}_{BL} = \left(\frac{r+a}{s}\right)\ln\left[1+\frac{s}{Q+b}\right] \tag{2-19}$$

The Bayes estimator of $h(t)$ relative to LLF denoted as $\hat{h}_{BL}$ by replacing $\hat{\theta}_{BL}$ given in (2-19) instead of θ given in (1-3) to be

$$\hat{h}_{BL} = \lambda\left(\frac{r+a}{s}\right)\ln\left[1+\frac{s}{Q+b}\right]e^{\lambda t} \tag{2-20}$$

3. E-Bayesian estimation

In this section, we will derive the E-Bayes estimates of the shape parameter θ and the hazard function $h(t)$ of the Gompertz distribution based on symmetric loss function (SELF)) and three asymmetric loss functions (QLF, ELF and LLF). Based on Han [30], the prior parameters a and b must be choose to guarantee that $g(\theta|a,b)$ given in (2-3) is a decreasing function of θ. The derivative of $g(\theta|a,b)$ with respect to θ is

$$\frac{dg(\theta|a,b)}{d\theta} = \frac{b^a}{\Gamma(a)}\theta^{a-2}\left[\exp\left[-b\theta\right]\right]\left[(a-1)-b\theta\right] \tag{3-1}$$

Note that $a>0, b>0$ and $\theta>0$ leads to $0<a<1, b>0$ due to $\frac{dg(\theta|a,b)}{d\theta}<0$, and therefore $g(\theta|a,b)$ is a decreasing function of θ. Suppose that a and b are independent with bivariate density function

$$\pi(a,b) = \pi_1(a)\pi_2(b) \tag{3-2}$$

Then, the E-Bayesian estimate of θ (expectation of the Bayesian estimate of θ) can be written as

$$\hat{\theta}_{EB} = E(\theta|\underline{x}) = \iint_\Omega \hat{\theta}_B(a,b)\pi(a,b)\,da\,db \tag{3-3}$$

Where $\hat{\theta}_B(a,b)$ is the Bayes estimate θ of given by (2-7), (2-11), (2-15) and (2-19). For more details see Han [11, 31].

3.1. E-Bayesian estimation under squared error loss function (SELF)

E-Bayesian estimates of θ are derived depending on three different distributions of the hyper-parameters a and b. These distributions are used to study the impact of the different prior distributions on the E-Bayesian estimation of θ. The following distributions of a and b may be used:

$$\pi_1(a,b) = \frac{2(c-b)}{c^2}, \qquad 0<a<1,\ 0<b<c \tag{3-4}$$

$$\pi_2(a,b) = \frac{1}{c}, \qquad 0<a<1,\ 0<b<c \tag{3-5}$$

$$\pi_3(a,b) = \frac{2b}{c^2}, \qquad 0<a<1,\ 0<b<c \tag{3-6}$$

We can obtained the E-Bayesian estimate of θ relative to SELF based on $\pi_1(a,b)$ which is denoted as $\hat{\theta}_{EBS1}$ by using (2-7) and (3-4) in (3-3) to be

$$\hat{\theta}_{EBS1} = \int_0^1\int_0^c \left(\frac{r+a}{Q+b}\right)\left[\frac{2(c-b)}{c^2}\right] db\, da$$

$$= \left(\frac{2r+1}{c}\right)\left[\left(1+\frac{Q}{c}\right)\ln\left(1+\frac{c}{Q}\right)-1\right] \tag{3-7}$$

Similarly, we can derive the E-Bayesian estimates of θ relative to SELF based on $\pi_2(a,b)$ and $\pi_3(a,b)$ which are denoted as $\hat{\theta}_{EBS2}, \hat{\theta}_{EBS3}$ by using (2-7), (3-5) in (3-3) and (2-7), (3-6) in (3-3) respectively to be

$$\hat{\theta}_{EBS2} = \int_0^1\int_0^c \left(\frac{r+a}{Q+b}\right)\left[\frac{1}{c}\right] db\, da = \left(\frac{2r+1}{2c}\right)\left[\ln\left(1+\frac{c}{Q}\right)\right] \tag{3-8}$$

And

$$\hat{\theta}_{EBS3} = \int_0^1\int_0^c \left(\frac{r+a}{Q+b}\right)\left[\frac{2b}{c^2}\right] db\, da = \left(\frac{2r+1}{c}\right)\left[1-\frac{Q}{c}\ln\left(1+\frac{c}{Q}\right)\right] \tag{3-9}$$

The E-Bayes estimates of $h(t)$ relative to SELF denoted as $\hat{h}_{EBSi}\ (i=1,2,3)$ can be obtained by replacing $\hat{\theta}_{EBSi}\ (i=1,2,3)$ given in (3-7), (3-8) and (3-9) instead of θ given in (1-3) to be

$$\hat{h}_{EBS1} = \lambda e^{\lambda t}\left(\frac{2r+1}{c}\right)\left[\left(1+\frac{Q}{c}\right)\ln\left(1+\frac{c}{Q}\right)-1\right], \tag{3-10}$$

$$\hat{h}_{EBS2} = \lambda e^{\lambda t}\left(\frac{2r+1}{2c}\right)\left[\ln\left(1+\frac{c}{Q}\right)\right] \tag{3-11}$$

And

$$\hat{h}_{EBS3} = \lambda e^{\lambda t}\left(\frac{2r+1}{c}\right)\left[1-\frac{Q}{c}\ln\left(1+\frac{c}{Q}\right)\right] \tag{3-12}$$

3.2. E-Bayesian estimation under quadratic loss function (QLF)

We can obtain the E-Bayesian estimate of θ relative to QLF based on $\pi_1(a,b)$ which is denoted as $\hat{\theta}_{EBQ1}$ by using (2-11) and (3-4) in (3-3) to be

$$\hat{\theta}_{EBQ1} = \int_0^1\int_0^c \left(\frac{r+a-2}{Q+b}\right)\left[\frac{2(c-b)}{c^2}\right] db\, da$$

$$= \left(\frac{2r-3}{c}\right)\left[\left(1+\frac{Q}{c}\right)\ln\left(1+\frac{c}{Q}\right)-1\right] \tag{3-13}$$

Also, we can derive the E-Bayesian estimates of θ relative to QLF based on $\pi_2(a,b)$ and $\pi_3(a,b)$ which are denoted as $\hat{\theta}_{EBQ2}, \hat{\theta}_{EBQ3}$ by using (2-11), (3-5) in (3-3) and (2-11), (3-6) in (3-3) respectively to be

$$\hat{\theta}_{EBQ2} = \int_0^1\int_0^c \left(\frac{r+a-2}{Q+b}\right)\left[\frac{1}{c}\right] db\, da = \left(\frac{2r-3}{2c}\right)\left[\ln\left(1+\frac{c}{Q}\right)\right] \tag{3-14}$$

And

$$\hat{\theta}_{EBQ3} = \int_0^1\int_0^c \left(\frac{r+a-2}{Q+b}\right)\left[\frac{2b}{c^2}\right] db\, da$$

$$= \left(\frac{2r-3}{c}\right)\left[1-\frac{Q}{c}\ln\left(1+\frac{c}{Q}\right)\right] \tag{3-15}$$

Similarly, the E-Bayes estimates of $h(t)$ based on QLF denoted as $\hat{h}_{EBQi}\ (i=1,2,3)$ can be obtained by replacing $\hat{\theta}_{EBQi}\ (i=1,2,3)$ given in (3-13), (3-14) and (3-15) instead of θ given in (1-3) to be

$$\hat{h}_{EBQ1} = \lambda e^{\lambda t}\left(\frac{2r-3}{c}\right)\left[\left(1+\frac{Q}{c}\right)\ln\left(1+\frac{c}{Q}\right)-1\right], \tag{3-16}$$

$$\hat{h}_{EBQ2} = \lambda e^{\lambda t}\left(\frac{2r-3}{2c}\right)\left[\ln\left(1+\frac{c}{Q}\right)\right] \tag{3-17}$$

And

$$\hat{h}_{EBQ3} = \lambda e^{\lambda t}\left(\frac{2r-3}{c}\right)\left[1-\frac{Q}{c}\ln\left(1+\frac{c}{Q}\right)\right] \tag{3-18}$$

3.3. E-Bayesian estimation under entropy loss function (ELF)

We can get the E-Bayesian estimate of θ relative to ELF based on $\pi_1(a,b)$ which is denoted as $\hat{\theta}_{EBE1}$ by using (2-15) and (3-4) in (3-3) to be

$$\hat{\theta}_{EBE1} = \int_0^1\int_0^c \left(\frac{r+a-1}{Q+b}\right)\left[\frac{2(c-b)}{c^2}\right] db\, da$$

$$= \left(\frac{2r-1}{c}\right)\left[\left(1+\frac{Q}{c}\right)\ln\left(1+\frac{c}{Q}\right)-1\right] \tag{3-19}$$

Also, we can derive the E-Bayesian estimates of θ relative to ELF based on $\pi_2(a,b)$ and $\pi_3(a,b)$ which are denoted as $\hat{\theta}_{EBE2}, \hat{\theta}_{EBE3}$ by using (2-15), (3-5) in (3-3) and (2-15), (3-6) in (3-3) respectively to be

$$\hat{\theta}_{EBE2} = \int_0^1\int_0^c \left(\frac{r+a-1}{Q+b}\right)\left[\frac{1}{c}\right] db\, da = \left(\frac{2r-1}{2c}\right)\left[\ln\left(1+\frac{c}{Q}\right)\right] \tag{3-20}$$

And

$$\hat{\theta}_{EBE3}=\int_0^1\int_0^c\left(\frac{r+a-1}{Q+b}\right)\left[\frac{2b}{c^2}\right]db\,da$$

$$=\left(\frac{2r-1}{c}\right)\left[1-\frac{Q}{c}\ln\left(1+\frac{c}{Q}\right)\right] \quad (3\text{-}21)$$

Also, the E-Bayes estimates of $h(t)$ relative to ELF denoted as $\hat{h}_{EBEi}\,(i=1,2,3)$ can be obtained by replacing $\hat{\theta}_{EBEi}\,(i=1,2,3)$ given in (3-19), (3-20) and (3-21) instead of θ given in (1-3) to be

$$\hat{h}_{EBE1}=\lambda e^{\lambda t}\left(\frac{2r-1}{c}\right)\left[\left(1+\frac{Q}{c}\right)\ln\left(1+\frac{c}{Q}\right)-1\right], \quad (3\text{-}22)$$

$$\hat{h}_{EBE2}=\lambda e^{\lambda t}\left(\frac{2r-1}{2c}\right)\left[\ln\left(1+\frac{c}{Q}\right)\right] \quad (3\text{-}23)$$

And

$$\hat{h}_{EBE3}=\lambda e^{\lambda t}\left(\frac{2r-1}{c}\right)\left[1-\frac{Q}{c}\ln\left(1+\frac{c}{Q}\right)\right] \quad (3\text{-}24)$$

3.4. E-Bayesian estimation under LINEX Loss function (LLF)

We can get the E-Bayesian estimate of θ relative to LLF based on $\pi_1(a,b)$ which is denoted as $\hat{\theta}_{EBL1}$ by using (2-19) and (3-4) in (3-3) to be

$$\hat{\theta}_{EBL1}=\int_0^1\int_0^c\left(\frac{r+a}{s}\right)\ln\left[1+\frac{s}{Q+b}\right]\left[\frac{2(c-b)}{c^2}\right]db\,da$$

$$=\left(\frac{2r+1}{2}\right)\left\{\begin{array}{l}\left[\left(\frac{-(Q+c)^2}{c^2s}\right)\ln\left(1+\frac{c}{Q}\right)\right]+\\ \left[\left(\frac{(Q+s+c)^2}{c^2s}\right)\ln\left(1+\frac{c}{Q+s}\right)\right]\\ +\left[\frac{1}{s}\ln\left(1+\frac{s}{Q}\right)\right]-\left[\frac{1}{c}\right]\end{array}\right\} \quad (3\text{-}25)$$

Also, we can derive the E-Bayesian estimates of θ relative to LLF based on $\pi_2(a,b)$ and $\pi_3(a,b)$ which are denoted as $\hat{\theta}_{EBL2},\hat{\theta}_{EBL3}$ by using (2-19), (3-5) in (3-3) and (2-19), (3-6) in (3-3) respectively to be

$$\hat{\theta}_{EBL2}=\int_0^1\int_0^c\left(\frac{r+a}{s}\right)\ln\left[1+\frac{s}{Q+b}\right]\left[\frac{1}{c}\right]db\,da$$

$$=\left(\frac{2r+1}{2s}\right)\left\{\begin{array}{l}\left[\ln\left(1+\frac{s}{Q+c}\right)\right]+\\ \left[\left(\frac{Q+s}{c}\right)\ln\left(1+\frac{c}{Q+s}\right)\right]-\\ \left[\left(\frac{Q}{c}\right)\ln\left(1+\frac{c}{Q}\right)\right]\end{array}\right\} \quad (3\text{-}26)$$

And

$$\hat{\theta}_{EBL3}=\int_0^1\int_0^c\left(\frac{r+a}{s}\right)\ln\left[1+\frac{s}{Q+b}\right]\left[\frac{2b}{c^2}\right]db\,da$$

$$\therefore\hat{\theta}_{EBL3}=\left(\frac{2r+1}{2}\right)\left\{\begin{array}{l}\left[\left(\frac{-(Q+c)^2}{c^2s}\right)\ln\left(1+\frac{c}{Q+s}\right)\right]+\\ \left[\left(\frac{Q^2}{c^2s}\right)\ln\left(1+\frac{c}{Q}\right)\right]+\\ \left[\frac{1}{s}\ln\left(1+\frac{s}{Q+c}\right)\right]+\left[\frac{1}{c}\right]\end{array}\right\} \quad (3\text{-}27)$$

Also, the E-Bayes estimates of $h(t)$ relative to LLF denoted as $\hat{h}_{EBLi}\,(i=1,2,3)$ can be obtained by replacing $\hat{\theta}_{EBLi}\,(i=1,2,3)$ given in (3-25), (3-26) and (3-27) instead of θ given in (1-3) to be

$$\hat{h}_{EBL1}=\lambda e^{\lambda}\left(\frac{2r+1}{2}\right)\left\{\begin{array}{l}\left[\left(\frac{-(Q+c)^2}{c^2s}\right)\ln\left(1+\frac{c}{Q}\right)\right]+\\ \left[\left(\frac{(Q+s+c)^2}{c^2s}\right)\ln\left(1+\frac{c}{Q+s}\right)\right]\\ +\left[\frac{1}{s}\ln\left(1+\frac{s}{Q}\right)\right]-\left[\frac{1}{c}\right]\end{array}\right\}, \quad (3\text{-}28)$$

$$\hat{h}_{EBL2}=\lambda e^{\lambda t}\left(\frac{2r+1}{2s}\right)\left\{\begin{array}{l}\left[\ln\left(1+\frac{s}{Q+c}\right)\right]+\\ \left[\left(\frac{Q+s}{c}\right)\ln\left(1+\frac{c}{Q+s}\right)\right]-\\ \left[\left(\frac{Q}{c}\right)\ln\left(1+\frac{c}{Q}\right)\right]\end{array}\right\} \quad (3\text{-}29)$$

And

$$\hat{h}_{EBL3}=\lambda e^{\lambda t}\left(\frac{2r+1}{2}\right)\left\{\begin{array}{l}\left[\left(\frac{-(Q+c)^2}{c^2s}\right)\ln\left(1+\frac{c}{Q+s}\right)\right]+\\ \left[\left(\frac{Q^2}{c^2s}\right)\ln\left(1+\frac{c}{Q}\right)\right]+\\ \left[\frac{1}{s}\ln\left(1+\frac{s}{Q+c}\right)\right]+\left[\frac{1}{c}\right]\end{array}\right\} \quad (3\text{-}30)$$

4. Hierarchical Bayesian estimation

In this section, we will derive the hierarchical Bayes estimates of the shape parameter θ and the hazard function $h(t)$ of the Gompertz distribution based on symmetric loss function (SELF)) and three asymmetric loss functions (QLF, ELF and LLF). According to Lindley and Smith [32], if a and b are hyper-parameters in θ, the prior density function of θ is $g(\theta|a,b)$ given in (2-3) and the prior density functions of hyper-parameters a,b are given in (3-4), (3-5) and (3-6), then the corresponding hierarchical prior density functions of θ are given as the following:

$$\pi_4(\theta)=\int_0^1\int_0^c g(\theta|a,b)\,\pi_1(a,b)\,db\,da$$
$$=\frac{2}{c^2}\int_0^1\int_0^c\frac{b^a}{\Gamma(a)}(c-b)\,\theta^{a-1}e^{-b\theta}db\,da, \quad (4\text{-}1)$$

$$\pi_5(\theta)=\int_0^1\int_0^c g(\theta|a,b)\,\pi_2(a,b)\,db\,da$$
$$=\frac{1}{c}\int_0^1\int_0^c\frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}db\,da \quad (4\text{-}2)$$

And

$$\pi_6(\theta)=\int_0^1\int_0^c g(\theta|a,b)\,\pi_3(a,b)\,dbda$$
$$=\frac{2}{c^2}\int_0^1\int_0^c \frac{b^{a+1}}{\Gamma(a)}\theta^{a-1}e^{-b\theta}dbda \tag{4-3}$$

From Bayesian theorem, the hierarchical posterior density functions of θ can be derived by combining (2-1), (4-1), (4-2) and (4-3) to be

$$h_1(\theta|\underline{x})=\frac{L(\theta|\underline{x})\,\pi_4(\theta)}{\int_0^\infty L(\theta|\underline{x})\,\pi_4(\theta)d\theta}=\frac{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\,\theta^{r+a-1}e^{-\theta(Q+b)}dbda}{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\Gamma(r+a)\left[Q+b\right]^{-(r+a)}dbda}, \tag{4-4}$$

$$h_2(\theta|\underline{x})=\frac{L(\theta|\underline{x})\,\pi_5(\theta)}{\int_0^\infty L(\theta|\underline{x})\,\pi_5(\theta)d\theta}=\frac{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}\theta^{r+a-1}e^{-\theta(Q+b)}dbda}{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}\Gamma(r+a)\left[Q+b\right]^{-(r+a)}dbda}, \tag{4-5}$$

And

$$h_3(\theta|\underline{x})=\frac{L(\theta|\underline{x})\,\pi_6(\theta)}{\int_0^\infty L(\theta|\underline{x})\,\pi_6(\theta)d\theta}=\frac{\int_0^1\int_0^c \frac{b^{a+1}}{\Gamma(a)}\theta^{r+a-1}e^{-\theta(Q+b)}dbda}{\int_0^1\int_0^c \frac{b^{a+1}}{\Gamma(a)}\Gamma(r+a)\left[Q+b\right]^{-(r+a)}dbda} \tag{4-6}$$

4.1. Hierarchical Bayesian estimation under squared error loss function (SELF)

The hierarchical Bayes estimates of θ based on SELF denoted by $\hat{\theta}_{HBSi}\ (i=1,2,3)$ can be obtained as

$$\hat{\theta}_{HBSi}=E_{h_i}(\theta|\underline{x}),\qquad i=1,2,3 \tag{4-7}$$

Where E_{h_i} indicated to the expectation of the hierarchical posterior distribution. We can derived $\hat{\theta}_{HBSi}\ (i=1,2,3)$ by using (4-4), (4-5) and (4-6) in (4-7) to be

$$\hat{\theta}_{HBS1}=\frac{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\Gamma(r+a+1)\left[Q+b\right]^{-(r+a+1)}dbda}{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\Gamma(r+a)\left[Q+b\right]^{-(r+a)}dbda}, \tag{4-8}$$

$$\hat{\theta}_{HBS2}=\frac{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}\Gamma(r+a+1)\left[Q+b\right]^{-(r+a+1)}dbda}{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}\Gamma(r+a)\left[Q+b\right]^{-(r+a)}dbda} \tag{4-9}$$

And

$$\hat{\theta}_{HBS3}=\frac{\int_0^1\int_0^c \frac{b^{a+1}}{\Gamma(a)}\Gamma(r+a+1)\left[Q+b\right]^{-(r+a+1)}dbda}{\int_0^1\int_0^c \frac{b^{a+1}}{\Gamma(a)}\Gamma(r+a)\left[Q+b\right]^{-(r+a)}dbda} \tag{4-10}$$

Similarly, the hierarchical Bayes estimates of $h(t)$ based on SELF denoted as $\hat{h}_{HBSi}\ (i=1,2,3)$ can be obtained by replacing $\hat{\theta}_{HBSi}\ (i=1,2,3)$ given in (4-8), (4-9) and (4-10) instead of θ given in (1-3) to be

$$\hat{h}_{HBS1}=\frac{\lambda e^{\lambda t}\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\Gamma(r+a+1)\left[Q+b\right]^{-(r+a+1)}dbda}{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\Gamma(r+a)\left[Q+b\right]^{-(r+a)}dbda}, \tag{4-11}$$

$$\hat{h}_{HBS2}=\frac{\lambda e^{\lambda t}\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}\Gamma(r+a+1)\left[Q+b\right]^{-(r+a+1)}dbda}{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}\Gamma(r+a)\left[Q+b\right]^{-(r+a)}dbda} \tag{4-12}$$

And

$$\hat{h}_{HBS3}=\frac{\lambda e^{\lambda t}\int_0^1\int_0^c \frac{b^{a+1}}{\Gamma(a)}\Gamma(r+a+1)\left[Q+b\right]^{-(r+a+1)}dbda}{\int_0^1\int_0^c \frac{b^{a+1}}{\Gamma(a)}\Gamma(r+a)\left[Q+b\right]^{-(r+a)}dbda} \tag{4-13}$$

4.2. Hierarchical Bayesian estimation under quadratic loss function (QLF)

The hierarchical Bayes estimates of θ based on QLF denoted by $\hat{\theta}_{HBQi}\ (i=1,2,3)$ can be obtained as

$$\hat{\theta}_{HBQi}=\frac{E_{h_i}(\theta^{-1}|\underline{x})}{E_{h_i}(\theta^{-2}|\underline{x})}\qquad i=1,2,3 \tag{4-14}$$

We can derived $\hat{\theta}_{HBQi}\ (i=1,2,3)$ by using (4-4), (4-5) and (4-6) in (4-14) to be

$$\hat{\theta}_{HBQ1}=\frac{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)}dbda}{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\Gamma(r+a-2)\left[Q+b\right]^{-(r+a-2)}dbda}, \tag{4-15}$$

$$\hat{\theta}_{HBQ2}=\frac{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}\Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)}dbda}{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}\Gamma(r+a-2)\left[Q+b\right]^{-(r+a-2)}dbda} \tag{4-16}$$

And

$$\hat{\theta}_{HBQ3}=\frac{\int_0^1\int_0^c \frac{b^{a+1}}{\Gamma(a)}\Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)}dbda}{\int_0^1\int_0^c \frac{b^{a+1}}{\Gamma(a)}\Gamma(r+a-2)\left[Q+b\right]^{-(r+a-2)}dbda} \tag{4-17}$$

Similarly, the hierarchical Bayes estimates of $h(t)$ based on QLF denoted as $\hat{h}_{HBQi}\ (i=1,2,3)$ can be obtained by replacing $\hat{\theta}_{HBQi}\ (i=1,2,3)$ given in (4-15), (4-16) and (4-17) instead of θ given in (1-3) to be

$$\hat{h}_{HBQ1}=\frac{\lambda e^{\lambda t}\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)}dbda}{\int_0^1\int_0^c \frac{b^a}{\Gamma(a)}(c-b)\Gamma(r+a-2)\left[Q+b\right]^{-(r+a-2)}dbda}, \tag{4-18}$$

$$\hat{h}_{HBQ2} = \frac{\lambda e^{\lambda t} \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a-2)\left[Q+b\right]^{-(r+a-2)} dbda} \quad (4\text{-}19)$$

And

$$\hat{h}_{HBQ3} = \frac{\lambda e^{\lambda t} \int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)} dbda}{\int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a-2)\left[Q+b\right]^{-(r+a-2)} dbda} \quad (4\text{-}20)$$

4.3. Hierarchical Bayesian estimation under entropy loss function (ELF)

The hierarchical Bayes estimates of θ based on ELF denoted by $\hat{\theta}_{HBEi}\,(i=1,2,3)$ can be obtained as

$$\hat{\theta}_{HBEi} = \left[E_{h_i}(\theta^{-1}|\underline{x}) \right]^{-1} \qquad i=1,2,3 \quad (4\text{-}21)$$

We can derived $\hat{\theta}_{HBEi}\,(i=1,2,3)$ by using (4-4), (4-5) and (4-6) in (4-21) to be

$$\hat{\theta}_{HBE1} = \frac{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} (c-b)\Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} (c-b)\Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)} dbda}, \quad (4\text{-}22)$$

$$\hat{\theta}_{HBE2} = \frac{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)} dbda} \quad (4\text{-}23)$$

And

$$\hat{\theta}_{HBE3} = \frac{\int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)} dbda} \quad (4\text{-}24)$$

Similarly, the hierarchical Bayes estimates of $h(t)$ based on ELF denoted as $\hat{h}_{HBEi}\,(i=1,2,3)$ can be obtained by replacing $\hat{\theta}_{HBEi}\,(i=1,2,3)$ given in (4-22), (4-23) and (4-24) instead of θ given in (1-3) to be

$$\hat{h}_{HBE1} = \frac{\lambda e^{\lambda t} \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} (c-b)\Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} (c-b)\Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)} dbda}, \quad (4\text{-}25)$$

$$\hat{h}_{HBE2} = \frac{\lambda e^{\lambda t} \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)} dbda} \quad (4\text{-}26)$$

And

$$\hat{h}_{HBE3} = \frac{\lambda e^{\lambda t} \int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a-1)\left[Q+b\right]^{-(r+a-1)} dbda} \quad (4\text{-}27)$$

4.4. Hierarchical Bayesian estimation under LINEX loss function (LLF)

The hierarchical Bayes estimates of θ based on LLF denoted by $\hat{\theta}_{HBLi}\,(i=1,2,3)$ can be obtained as

$$\hat{\theta}_{HBEi} = \left(\frac{-1}{s}\right) \ln\left[E_{h_i}\left(e^{-s\theta}\middle|\underline{x}\right)\right] \qquad i=1,2,3 \quad (4\text{-}28)$$

We can derived $\hat{\theta}_{HBLi}\,(i=1,2,3)$ by using (4-4), (4-5) and (4-6) in (4-28) to be

$$\hat{\theta}_{HBL1} = \left(\frac{-1}{s}\right) \ln\left[\frac{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} (c-b)\Gamma(r+a)\left[Q+b+s\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} (c-b)\Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}\right], \quad (4\text{-}29)$$

$$\hat{\theta}_{HBL2} = \left(\frac{-1}{s}\right) \ln\left[\frac{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a)\left[Q+b+s\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}\right] \quad (4\text{-}30)$$

And

$$\hat{\theta}_{HBL3} = \left(\frac{-1}{s}\right) \ln\left[\frac{\int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a)\left[Q+b+s\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}\right] \quad (4\text{-}31)$$

Similarly, the hierarchical Bayes estimates of $h(t)$ based on LLF denoted as $\hat{h}_{HBLi}\,(i=1,2,3)$ can be obtained by replacing $\hat{\theta}_{HBLi}\,(i=1,2,3)$ given in (4-29), (4-30) and (4-31) instead of θ given in (1-3) to be

$$\hat{h}_{HBL1} = \left(\frac{-\lambda e^{\lambda t}}{s}\right) \ln\left[\frac{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} (c-b)\Gamma(r+a)\left[Q+b+s\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} (c-b)\Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}\right], \quad (4\text{-}32)$$

$$\hat{h}_{HBL2} = \left(\frac{-\lambda e^{\lambda t}}{s}\right) \ln\left[\frac{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a)\left[Q+b+s\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}\right] \quad (4\text{-}33)$$

And

$$\hat{h}_{HBL3} = \left(\frac{-\lambda e^{\lambda t}}{s}\right) \ln\left[\frac{\int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a)\left[Q+b+s\right]^{-(r+a)} dbda}{\int_0^1 \int_0^c \frac{b^{a+1}}{\Gamma(a)} \Gamma(r+a)\left[Q+b\right]^{-(r+a)} dbda}\right] \quad (4\text{-}34)$$

5. Empirical Bayesian estimation

The Bayes approach assumed that the hyper-parameters a and b are known. When a and b are unknown, we may use the empirical Bayes criteria to get its estimates from likelihood function and probability density function of the prior distribution [33].Now, from (2-1) and (2-3), the marginal distribution of x given a and b is obtained as:

$$f\left(x\,|a,b\right) \propto b^a \left[\Gamma(a)\right]^{-1} \Gamma(r+a)\left(Q+b\right)^{-(r+a)} \quad (5\text{-}1)$$

By taking the natural log for (5-1), we get

$$\log f\left(x\,|a,b\right) \propto a\log b - \log\Gamma(a) + \log\Gamma(r+a) - (r+a)\log\left(Q+b\right) \quad (5\text{-}2)$$

By taking the derivative for (5-3) and setting it equal to zero, we obtain

$$\frac{\partial \log f\left(x\,|a,b\right)}{\partial a}=\log b-\frac{\partial}{\partial a}\log\Gamma(a)+\frac{\partial}{\partial a}\log\Gamma(r+a)-\log\left(Q+b\right)=0 \quad (5\text{-}3)$$

$$\frac{\partial \log f\left(x\,|a,b\right)}{\partial b}=\frac{a}{b}-\frac{(r+a)}{Q+b}=0 \quad (5\text{-}4)$$

By solving (5-3) and (5-4) simultaneously, we can get the maximum likelihood estimators of a and b denoted by $\tilde{a}$ and $\tilde{b}$ to be

$$\tilde{a}=\log\left[\frac{aQ}{r}\right]-\frac{\partial}{\partial a}\log\Gamma(a)+\frac{\partial}{\partial a}\log\Gamma(r+a)-\log\left[Q\left(1+\frac{a}{r}\right)\right] \quad (5\text{-}5)$$

$$\tilde{b}=\frac{\tilde{a}Q}{r} \quad (5\text{-}6)$$

5.1. Empirical Bayesian estimation under squared error loss function (SELF)

The empirical Bayes estimates of θ and $h(t)$ based on SELF denoted as $\hat{\theta}_{eBS}$ and $\hat{h}_{eBS}$ respectively can be obtained by replacing $\tilde{a}$ and $\tilde{b}$ in (5-5), (5-6) instead of a and b in (2-7), (2-8) respectively to be

$$\hat{\theta}_{eBS}=\frac{r+\tilde{a}}{Q+\tilde{b}} \quad (5\text{-}7)$$

And

$$\hat{h}_{eBS}=\lambda\left(\frac{r+\tilde{a}}{Q+\tilde{b}}\right)e^{\lambda t} \quad (5\text{-}8)$$

5.2. Empirical Bayesian estimation under quadratic loss function (QLF)

The empirical Bayes estimates of θ and $h(t)$ relative to on QLF denoted as $\hat{\theta}_{eBQ}$ and $\hat{h}_{eBQ}$ respectively can be obtained by replacing $\tilde{a}$ and $\tilde{b}$ in (5-5), (5-6) instead of a and b in (2-11), (2-12) respectively to be

$$\hat{\theta}_{eBQ}=\frac{r+\tilde{a}-2}{Q+\tilde{b}} \quad (5\text{-}9)$$

And

$$\hat{h}_{eBQ}=\lambda\left(\frac{r+\tilde{a}-2}{Q+\tilde{b}}\right)e^{\lambda t} \quad (5\text{-}10)$$

5.3. Empirical Bayesian estimation under entropy Loss function (ELF)

The empirical Bayes estimates of θ and $h(t)$ relative to on ELF denoted as $\hat{\theta}_{eBE}$ and $\hat{h}_{eBE}$ respectively can be obtained by replacing $\tilde{a}$ and $\tilde{b}$ in (5-5), (5-6) instead of a and b in (2-15), (2-16) respectively to be

$$\hat{\theta}_{eBE}=\frac{r+\tilde{a}-1}{Q+\tilde{b}} \quad (5\text{-}11)$$

And

$$\hat{h}_{eBE}=\lambda\left(\frac{r+\tilde{a}-1}{Q+\tilde{b}}\right)e^{\lambda t} \quad (5\text{-}12)$$

5.4. Empirical Bayesian estimation under LINEX loss function (LLF)

The empirical Bayes estimates of θ and $h(t)$ relative to on LLF denoted as $\hat{\theta}_{eBL}$ and $\hat{h}_{eBL}$ respectively can be obtained by replacing $\tilde{a}$ and $\tilde{b}$ in (5-5), (5-6) instead of a and b in (2-19), (2-20) respectively to be

$$\hat{\theta}_{eBL}=\left(\frac{r+\tilde{a}}{s}\right)\ln\left[1+\frac{s}{Q+\tilde{b}}\right] \quad (5\text{-}13)$$

And

$$\hat{h}_{eBL}=\lambda\left(\frac{r+\tilde{a}}{s}\right)\ln\left[1+\frac{s}{Q+\tilde{b}}\right]e^{\lambda t} \quad (5\text{-}14)$$

6. Monte Carlo simulation

This section conducted a Monte Carlo simulation study to evaluate the performance of different estimators for the shape parameter and hazard function corresponding to the Gompertz distribution discussed in the preceding sections. The simulation structure consists of five basic steps which are:

Step (1): Set the default values (true values) of λ, s and c which are 0.4, 2 and 3 respectively. We considered different censoring schemes (different values of n, r) to observe their effect on the estimates in small, moderate and large dataset which are

	small samples	moderate samples	large samples
n	5, 10	15, 20, 25	50, 70
r	2, 3, 4, 5	7, 12, 13, 16, 18, 22	25, 30, 32, 35

Step (2): For these cases, we generate a, b from the uniform priors distributions (0, 1) and (0, c) respectively given in (3-4), (3-5) and (3-6). For given values of a and b, we generate θ from the gamma prior distribution given in (2-3).

Step (3): For known values of λ, type-II censored samples are generated from the Gompertz distribution with pdf and cdf given in (1-1) and (1-2) respectively through the adoption of inverse transformation method, by using the formula

$$t_i=F^{-1}(U_i)=\left(\frac{1}{\lambda}\right)\ln\left[1-\left(\frac{1}{\theta}\right)\ln\left(1-U_i\right)\right];\quad i=1,2,\ldots n$$

Where U is a random variable distributed according to uniform distribution on the period (0, 1).

Step (4): Calculate the Bayes, E-Bayes, hierarchical Bayes and empirical Bayes estimates of the unknown shape parameter and the hazard function associated to the Gompertz distribution according to the formulas that have been obtained.

Step (5): We repeated this process 10000 times and compute the Mean Square Error (MSE) for the estimates for different censoring schemes and given values of c, s, λ where

$$MSE(\hat{\theta})=\frac{1}{10000}\sum_{i=1}^{10000}(\hat{\theta}_i-\theta)^2$$

And $\hat{\theta}$ stands for an estimator of θ . The simulation results are displayed in Tables (1-8).

Table 1: Averaged Values of MSEs for Estimates of the Parameter Based on SELF

n	r	$\hat{\theta}_{BS}$	$\hat{\theta}_{EBS}$	$\hat{\theta}_{HBS}$	$\hat{\theta}_{eBS}$	Best estimator
			0.0923816	0.0900734		
	2	0.1154007	0.1098492	0.1034039	0.1081736	Hierarchical Bayes
5			0.1319542	0.1187886		
			0.28303229	0.7028934		
	3	0.1246743	0.1740581	0.6009574	0.2037672	E-Bayes
			0.1115517	0.2547409		
			0.0963787	0.0949064		
	4	0.1073193	0.1057411	0.1020377	0.1012551	Hierarchical Bayes
10			0.1169024	0.1096066		
			0.1170692	0.1247767		
	5	0.1102817	0.1122438	0.1290224	0.1113921	Bayes
			0.111367	0.1141872		
			0.1320453	0.1327131		
	7	0.1343433	0.1343069	0.134495	0.1340809	E-Bayes
15			0.1375919	0.135698		
			0.4895673	0.4935076		
	12	0.4877208	0.4882623	0.4933152	0.5068884	E-Bayes
			0.4872165	0.4916517		
			0.4390487	0.4398919		
	13	0.4387684	0.4388826	0.4399477	0.4604651	Bayes
20			0.4388002	0.4395233		
			0.5270064	0.5270607		
	16	0.5270217	0.5270204	0.5270596	0.5338379	E-Bayes
			0.5270347	0.5270582		
			0.5261164	0.5261638		
	18	0.5261348	0.5261334	0.5261661	0.5331317	E-Bayes
25			0.5261509	0.5261682		
			0.5385311	0.5385329		
	22	0.5385323	0.5385319	0.5385327	0.5388161	E-Bayes
			0.5385301	0.5385325		
			0.5323611	0.5324108		
	25	0.5323851	0.5323835	0.5324138	0.539386	E-Bayes
50			0.5324059	0.5324162		
			0.5530951	0.553102		
	30	0.5530953	0.5530952	0.5531014	0.5539138	E-Bayes
			0.5530954	0.5531009		
			0.5536152	0.5536406		
	32	0.5536197	0.5536186	0.5536383	0.5566048	E-Bayes
70			0.5536224	0.5536366		
			0.5564315	0.5564399		
	35	0.5564317	0.5564318	0.5564391	0.5574312	E-Bayes
			0.556432	0.5564385		

Table 2: Averaged Values of MSEs for Estimates of the Parameter θ Based on QLF

n	r	$\hat{\theta}_{BQ}$	$\hat{\theta}_{EBQ}$	$\hat{\theta}_{HBQ}$	$\hat{\theta}_{eBQ}$	Best estimator
			0.4238526	0.4633921		
	2	0.42377492	0.4236345	0.4658621	0.4417558	E-Bayes
5			0.4456019	0.4678147		
			0.2202763	0.2292603		
	3	0.2398602	0.2336599	0.2350935	0.2200502	Empirical
			0.2555785	0.2435989		
			0.2378877	0.2367211		
	4	0.2523542	0.2501698	0.2464958	0.2448517	Hierarchical
10			0.2630071	0.2548032		
			0.1798498	0.1791594		
	5	0.1899613	0.1884004	0.1856097	0.1842346	Hierarchical
			0.1985524	0.1918728		
			0.1895637	0.1887348		
	7	0.1960455	0.1953579	0.1932251	0.1956803	Hierarchical
15			0.2017024	0.1973158		
			0.4951878	0.497656		
	12	0.4940261	0.4943824	0.4974899	0.5118598	E-Bayes
			0.4937599	0.4962776		
			0.4503912	0.4509263		
	13	0.4503843	0.4504469	0.4510272	0.4712319	Bayes
20			0.4505636	0.4508205		
			0.5283811	0.5284392		
	16	0.5283964	0.5283947	0.5284373	0.5344438	E-Bayes
			0.5284085	0.5284358		
			0.5274266	0.5274779		
	18	0.5274446	0.5274432	0.5274792	0.5337475	E-Bayes
25			0.5274598	0.5274806		
			0.5385601	0.538563		Bayes
	22	0.5385601	0.5385601	0.5385628	0.5388208	=
			0.5385601	0.5385627		E-Bayes

50			0.5340203	0.5340734		
	25	0.5340429	0.5340413	0.5340749	0.5405435	E-Bayes
			0.5340623	0.5340762		
			0.5531793	0.5531863		
	30	0.5531795	0.5531794	0.5531857	0.5539447	E-Bayes
			0.5531796	0.5531852		
70			0.5541335	0.5541593		
	32	0.5541373	0.5541362	0.5541568	0.5569455	E-Bayes
			0.5541394	0.554155		
			0.5565402	0.5565487		
	35	0.5565404	0.5565404	0.5565479	0.5574841	E-Bayes
			0.5565407	0.5565473		

Table 3: Averaged Values of MSEs for Estimates of the Parameter θ Based on ELF

n	r	$\hat{\theta}_{BE}$	$\hat{\theta}_{EBE}$	$\hat{\theta}_{HBE}$	$\hat{\theta}_{eBE}$	Best estimator
5			0.2168677	0.2165015		
	2	0.2447842	0.2382808	0.2338617	0.2393492	Hierarchical Bayes
			0.2613634	0.2491744		
			0.1783073	0.286921		
	3	0.1381134	0.1505408	0.2454365	0.1489339	Bayes
			0.1464825	0.1578187		
10			0.1543769	0.1524457		
	4	0.1687689	0.1666153	0.1622049	0.1611572	Hierarchical Bayes
			0.1799419	0.1714139		
			0.1355164	0.1373493		
	5	0.1392281	0.1390392	0.1396337	0.1361098	E-Bayes
			0.1452053	0.1414759		
15			0.1561711	0.1558313		
	7	0.1609936	0.1605749	0.1592189	0.1607174	Hierarchical Bayes
			0.1657474	0.1622656		
			0.4922695	0.4954227		
	12	0.4907839	0.4912282	0.4952433	0.5093679	E-Bayes
			0.4904062	0.4938076		
20			0.4445716	0.4452436		
	13	0.4444365	0.4445235	0.4453234	0.4658121	Bayes
			0.4445488	0.4450136		
			0.5276915	0.5277477		
	16	0.5277067	0.5277054	0.5277462	0.5341407	E-Bayes
			0.5277195	0.5277448		
25			0.5267692	0.5268187		
	18	0.5267877	0.5267862	0.5268205	0.5334395	E-Bayes
			0.5268034	0.5268223		
			0.5385450	0.5385482		
	22	0.5385451	0.5385451	0.5385477	0.5388185	E-Bayes
			0.5385451	0.5385476		
50			0.5331892	0.5332406		
	25	0.5332126	0.5332109	0.5332428	0.5399644	E-Bayes
			0.5332327	0.5332447		
			0.5531372	0.5531442		
	30	0.5531374	0.5531373	0.5531435	0.5339292	Empirical Bayes
			0.5531375	0.5531431		
70			0.5538742	0.5538998		
	32	0.5538784	0.5538773	0.5538974	0.5567751	E-Bayes
			0.5538805	0.5538956		
			0.5564859	0.5564943		
	35	0.556486	0.5564861	0.5564935	0.5574577	E-Bayes
			0.5564863	0.5564929		

Table 4: Averaged Values of MSEs for Estimates of the Parameter θ Based on LLF

n	r	$\hat{\theta}_{BL}$	$\hat{\theta}_{EBL}$	$\hat{\theta}_{HBL}$	$\hat{\theta}_{eBL}$	Best estimator
5			0.1300703	0.1274319		
	2	0.1534217	0.1486116	0.1423646	0.1468433	Hierarchical Bayes
			0.1693468	0.1570616		
			0.1164396	0.1409541		
	3	0.0999737	0.1047927	0.1277961	0.1077884	Bayes
			0.1047618	0.1085985		
10			0.1162738	0.1143817		
	4	0.1283687	0.1267342	0.1226245	0.1218346	Hierarchical Bayes
			0.1383037	0.1307677		
			0.1115827	0.1128788		
	5	0.1145334	0.1144052	0.114787	0.1121342	E-Bayes
			0.1193545	0.1163856		
15			0.2035517	0.2019998		
	7	0.2109233	0.2101355	0.2071128	0.2108287	Hierarchical Bayes
			0.4591252	0.2122008		
	12	1.3658455	1.3655501	1.3660657	1.4051697	E-Bayes

			1.3658569 1.4517256	1.3662818 1.3663222		
20	13	0.7815429	0.7808361 0.7815032 0.8878425	0.7808361 0.7815032 0.7817255	0.8112455	Hierarchical Bayes = E-Bayes
	16	1.4826256	1.4825934 1.4826228 1.5017362	1.482692 1.4826932 1.4826956	1.4939107	E-Bayes
25	18	1.1256065	1.1255801 1.1256044 1.1415574	1.1256476 1.1256518 1.125656	1.1345492	E-Bayes
	22	2.1960561	3.4861319 2.1960560 3.7932213	2.1960631 2.1960626 2.1960622	2.196718	E-Bayes
50	25	0.5324297	0.5324057 0.5324283 0.5324504	0.5324556 0.5324584 0.5324607	0.539399	E-Bayes
	30	0.5530956	0.5584083 0.5530956 0.5565524	0.5531023 0.5531017 0.5531012	0.5539138	Bayes = E-Bayes
70	32	0.5536264	0.5536209 0.5536253 0.5536319	0.5536474 0.5536451 0.5536434	0.5566063	E-Bayes
	35	0.5564322	0.5661383 0.5564323 0.5653351	0.5564404 0.5564396 0.556439	0.5574313	Bayes

Table 5: Averaged Values of MSEs for Estimates of the Parameter $h(t)$ Based on SELF

n	r	$\hat{h}_{BS}$	$\hat{h}_{EBS}$	$\hat{h}_{HBS}$	$\hat{h}_{eBS}$	Best estimator
5	2	0.1476429	0.1258195 0.1426235 0.1606808	0.1227171 0.1363939 0.1502469	0.1412723	Hierarchical Bayes
	3	0.1413497	0.120067 0.1343846 0.1613064	0.1692736 0.172298 0.1511548	0.1178139	Empirical Bayes
10	4	0.1434964	0.1313219 0.141665 0.1525482	0.1290326 0.1372499 0.1455241	0.1371602	Hierarchical Bayes
	5	0.1701982	0.1565579 0.1679724 0.1806547	0.1537777 0.1621621 0.1717051	0.1633304	Hierarchical Bayes
15	7	0.1974726	0.1896829 0.1965934 0.2038466	0.1879971 0.1932869 0.198737	0.1974171	Hierarchical Bayes
	12	1.3652581	1.3653674 1.3653868 1.3655327	1.3668662 1.3670484 1.3667106	1.4049949	Bayes
20	13	0.7802063	0.7795411 0.7801771 0.7808441	0.7796862 0.7801322 0.7804643	0.8106249	E-Bayes
	16	1.4825668	1.4825332 1.4825633 1.4825935	1.4826317 1.4826333 1.4826361	1.4939071	E-Bayes
25	18	1.1255559	1.1255281 1.1255537 1.255794	1.1255963 1.1256008 1.1256054	1.1345459	E-Bayes
	22	2.1960559	2.1960558 2.1960559 2.1960559	2.196063 2.1960624 2.1960621	2.196718	E-Bayes
50	25	0.572003	0.5719854 0.5720018 0.5720182	0.5720236 0.5720256 0.5720272	0.5770019	E-Bayes
	30	0.7808449	0.7808447 0.7808449 0.780845	0.7808513 0.7808507 0.7808503	0.7815812	E-Bayes
70	32	0.5173336	0.5173308 0.5173329 0.5173351	0.5173478 0.5173462 0.517345	0.5192111	E-Bayes
	35	0.5868674	0.5868669 0.586867 0.5868672	0.586873 0.5868724 0.5868719	0.5875585	E-Bayes

Table 6: Averaged Values of MSEs for Estimates of the Parameter $h(t)$ Based on QLF

n	r	$\hat{h}_{BQ}$	$\hat{h}_{EBQ}$	$\hat{h}_{HBQ}$	$\hat{h}_{eBQ}$	Best estimator
			0.3389923	0.360396		
5	2	0.3459976	0.344487	0.3614346	0.3482932	E-Bayes
			0.3500319	0.3622528		
			0.3497848	0.3483162		
5	3	0.3857052	0.3750094	0.3682359	0.3586745	Hierarchical
			0.4025491	0.3885239		
			0.2417891	0.2410258		
10	4	0.2512266	0.2498273	0.2474453	0.2463663	Hierarchical
			0.2580322	0.2527922		
			0.2684134	0.2663228		
10	5	0.2822113	0.2798524	0.2753904	0.2752486	Hierarchical
			0.2918048	0.2840613		
			0.2591428	0.2578511		
15	8	0.2666107	0.2657306	0.2630432	0.2663315	Hierarchical
			0.2725026	0.2678566		
			1.3792834	1.3802569		
15	12	1.3793881	1.3794473	1.3804155	1.14138456	Empirical
			1.3796988	1.380229		
			0.7954954	0.7956202		
20	13	0.7961594	0.7961197	0.7960272	0.8232002	E-Bayes
			0.7967664	0.7963558		
			1.4849752	1.4850816		
20	16	1.4850051	1.4850023	1.4850805	1.4950112	E-Bayes
			1.4850295	1.485081		
			1.1273277	1.1274009		
25	18	1.1273531	1.1273514	1.1274034	1.1354064	E-Bayes
			1.1273744	1.1274061		
			2.1961274	2.1961347		
25	22	2.1961275	2.1961274	2.1961342	2.1967308	E-Bayes
			2.1961275	2.1961338		
			0.5732148	0.5732553		
50	25	0.5732312	0.5732345	0.5732562	0.5778575	E-Bayes
			0.5732453	0.573257		
			0.7809229	0.7809296		
50	30	0.7809230	0.7809230	0.7809293	0.7816112	E-Bayes
			0.7809231	0.7809285		
			0.5176671	0.5176846		
70	32	0.5176698	0.5176691	0.5176829	0.5194343	E-Bayes
			0.5176711	0.5176817		
			0.5869444	0.5869507		
70	35	0.5869445	0.5869446	0.5869501	0.5875973	E-Bayes
			0.5869447	0.5869496		

Table 7: Averaged Values of MSEs for Estimates of the Parameter $h(t)$ Based on ELF

n	r	$\hat{h}_{BE}$	$\hat{h}_{EBE}$	$\hat{h}_{HBE}$	$\hat{h}_{eBE}$	Best estimator
			0.2191796	0.2189966		
5	2	0.2363194	0.2324628	0.229798	0.2331493	Hierarchical Bayes
			0.2461973	0.2389718		
			0.2070463	0.2117466		
5	3	0.2446143	0.2331799	0.2281944	0.2129794	E-Bayes
			0.2657445	0.2453087		
			0.1820682	0.1804084		
10	4	0.1934148	0.1917172	0.1882323	0.1875475	Hierarchical Bayes
			0.201693	0.195379		
			0.2072122	0.2044472		
10	5	0.2216547	0.2192354	0.2137031	0.2144449	Hierarchical Bayes
			0.2321073	0.2234342		
			0.222526	0.2209862		
15	7	0.2303126	0.2294133	0.2263544	0.2301584	Hierarchical Bayes
			0.236558	0.231613		
			1.3722569	1.3734691		
15	12	1.3722642	1.372356	1.3736401	1.4094141	E-Bayes
			1.3725596	1.3733802		
			0.7874409	0.7875702		
20	13	0.7881094	0.7880746	0.7879983	0.8168902	E-Bayes
			0.7887349	0.7883314		
			1.4837526	1.4838549		
20	16	1.4837841	1.4837812	1.4838552	1.494459	E-Bayes
			1.4838109	1.4838568		
			1.1264267	1.1264973		
25	18	1.1264533	1.1264512	1.1265008	1.1349761	E-Bayes
			1.1264757	1.1265045		
			2.1960916	2.1960988		
25	22	2.1960917	2.1960917	2.1960983	2.1967244	E-Bayes
			2.1960917	2.1960979		

50	25	0.5726165	0.5725995 0.5726153 0.5726312	0.5726388 0.5726403 0.5726415	0.5774295	E-Bayes
	30	0.7808846	0.7808838 0.7808839 0.7808841	0.7808904 0.7808899 0.7808894	0.7815962	E-Bayes
70	32	0.5175017	0.5174989 0.517501 0.517503	0.5175161 0.5175145 0.5175133	0.5193227	E-Bayes
	35	0.5869058	0.5869056 0.5869058 0.586906	0.5869118 0.5869112 0.5869108	0.5875779	E-Bayes

Table 8: Averaged Values of MSEs for Estimates of the Parameter $h(t)$ Based on LLF

n	r	$\hat{h}_{BL}$	$\hat{h}_{EBL}$	$\hat{h}_{HBL}$	$\hat{h}_{eBL}$	Best estimator
5	2	0.1763238	0.1592111 0.1729069 0.2923221	0.1571097 0.1683651 0.1788406	0.1716604	Hierarchical
	3	0.1897742	0.1585437 0.1820938 0.4584379	0.1523755 0.1690574 0.1903974	0.1628875	Hierarchical
10	4	0.1626394	0.1519278 0.161177 0.3430544	0.1500465 0.1577888 0.1645925	0.1570101	Hierarchical
	5	0.1929771	0.1799092 0.1910202 0.4555219	0.1771111 0.1856013 0.1946963	0.1864011	Hierarchical
15	7	0.2109233	0.2035517 0.2101355 0.4591252	0.2019998 0.2071128 0.2122008	0.2108287	Hierarchical
	12	1.3658455	1.3655501 1.3658569 1.4517256	1.3660657 1.3662818 1.3663222	1.4051697	E-Bayes
20	13	0.7815429	0.7808360 0.7815032 0.8878343	0.7808360 0.7815032 0.7817255	0.8112455	Hierarchical = E-Bayes
	16	1.4826256	1.4825934 1.4826228 1.5017362	1.482692 1.4826932 1.4826956	1.4939107	E-Bayes
25	18	1.1256065	1.1255801 1.1256044 1.1415574	1.1256476 1.1256518 1.125656	1.1345492	E-Bayes
	22	2.1960561	3.4861319 2.1960560 3.7932213	2.1960631 2.1960626 2.1960622	2.196718	E-Bayes
50	25	0.5720357	0.5720181 0.5720345 0.5773542	0.5720563 0.5720583 0.5720598	0.5770112	E-Bayes
	30	0.7808452	0.7873262 0.7808452 0.7861364	0.7808516 0.7808513 0.7808506	0.7815812	Bayes = E-Bayes
70	32	0.5173379	0.5173338 0.5173372 0.5226816	0.517352 0.5173504 0.5173493	0.5192121	E-Bayes
	35	0.5868673	0.5954512 0.5868673 0.5961556	0.5868733 0.5868727 0.5868723	0.5875585	Bayes = E-Bayes

7. Conclusion remarks

- Among four estimates of θ based on SELF shown in Table 1, we can deducted that hierarchical Bayes estimates are the best estimators in most cases of small samples sizes [5], [2], [10], [4], while the E-Bayes are the best estimators in [5], [3] and the Bayes estimators are the best in [10], [5].Also, the E-Bayes estimates have smallest MSE in nearly all cases of moderate and large sample sizes except for [20], [13] where the Bayes estimates are the best. Generally, the overall performance of the four techniques for estimating θ can be ordered due to number of having smaller MSE as follows:

$$\hat{\theta}_{EBS} > \hat{\theta}_{BS} = \hat{\theta}_{HBS} > \hat{\theta}_{eBS}$$

- Among four estimates of θ based on QLF shown in Table 2, we can deducted that hierarchical Bayes estimates are the best estimators in most cases of small samples sizes [10], [4], [10], [5], while the E-Bayes are the best estimators in [5], [2] and the empirical Bayes estimators are the best in [5], [3]. In addition, the E-Bayes estimates have smallest MSE in most cases of moderate sample sizes except for [15], [7] where the hierarchical Bayes estimates are the best, [20], [13] where the Bayes estimates are the best and [25], [22] where the Bayes and E-Bayes estimates are equivalent. In large samples, the E-Bayes estimates are the best in all cases. Generally, the overall performance of the four methods for estimating θ can be ordered due to number of having smaller MSE as follows:

$$\hat{\theta}_{EBQ} > \hat{\theta}_{HBQ} > \hat{\theta}_{BQ} > \hat{\theta}_{eBQ}$$

- Among four estimates of θ based on ELF shown in Table 3, we can deducted that hierarchical Bayes estimates have smallest MSE in most cases of small samples sizes [5], [2], [10], [4], while the Bayes are the best estimators in [5], [3] and the E-Bayes estimators are the best in [10], [5]. Furthermore, the E-Bayes estimates are the best in most cases of moderate sample sizes except for [15], [7] where the hierarchical Bayes estimates are the best and [20], [13] where the Bayes estimates are the best. In large samples, the E-Bayes have smallest MSE in nearly all cases except for [50], [30] where the empirical Bayes estimates are the best. Generally, the overall performance of the four methods for estimating θ can be ordered due to number of having smaller MSE as follows:

$$\hat{\theta}_{EBE} > \hat{\theta}_{HBE} > \hat{\theta}_{BE} > \hat{\theta}_{eBE}$$

- Among four estimates of θ based on LLF shown in Table 4, we can deducted that hierarchical Bayes estimates have smallest MSE in most cases of small samples sizes [5], [2], [10], [4], while the Bayes estimates have smallest MSE in [5], [3] and the E-Bayes estimators are the best in [10], [5]. Also, the E-Bayes estimates are the best in most cases of moderate sample sizes except for [15], [7] where the hierarchical Bayes estimates are the best and [20], [13] where the E-Bayes estimates and hierarchical Bayes estimates are equivalent. In large samples, the E-Bayes have smallest MSE in nearly all cases except for [50], [30] where the E-Bayes and Bayes estimates are equivalent. Generally, the overall performance of the four methods for estimating θ can be ordered due to number of having smaller MSE as follows:

$$\hat{\theta}_{EBL} > \hat{\theta}_{HBL} > \hat{\theta}_{BL} > \hat{\theta}_{eBL}$$

- In comparing the various techniques relative to the different loss functions in estimating θ, we can ordered them due to having smallest MSE to be

$$\hat{\theta}_{SELF} > \hat{\theta}_{LLF} > \hat{\theta}_{ELF} > \hat{\theta}_{QLF}$$

- Among four estimates of $h(t)$ based on SELF shown in Table 5, we can deducted that hierarchical Bayes estimates have smallest MSE in nearly all cases of small samples sizes except for [5], [3], while the empirical Bayes estimates are the best. In addition, the E-Bayes estimates are the best in most cases of moderate sample sizes except for [15], [7] where the hierarchical Bayes estimates are the best and [15], [12] where the Bayes estimates are the best. In large samples, the E-Bayes have smallest MSE in all cases. Generally, the overall performance of the four methods for estimating $h(t)$ can be ordered due to number of having smaller MSE as follows:

$$\hat{h}_{EBS} > \hat{h}_{HBS} > \hat{h}_{BS} = \hat{h}_{eBS}$$

- Among four estimates of $h(t)$ based on QLF shown in Table 6, we can deducted that hierarchical Bayes estimates have smallest MSE in nearly all cases of small samples sizes except for [5], [2] where the E-Bayes estimates are the best. Also, the E-Bayes estimates are the best in most cases of moderate sample sizes except for [15], [7] where the hierarchical Bayes estimates are the best and [15], [12] where the empirical Bayes estimates are the best. In large samples, the E-Bayes have smallest MSE in all cases. Generally, the overall performance of the four methods for estimating $h(t)$ can be ordered due to number of having smaller MSE as follows:

$$\hat{h}_{EBQ} > \hat{h}_{HBQ} > \hat{h}_{eBQ} > \hat{h}_{BQ}$$

- Among four estimates of $h(t)$ based on ELF shown in Table 7, we can deducted that hierarchical Bayes estimates are the best in nearly all cases of small samples sizes except for [5], [3] where the E-Bayes estimates are the best. Furthermore, the E-Bayes estimates are the best in nearly all cases of moderate and large sample sizes except for [15], [7] where the hierarchical Bayes estimates are the best. Generally, the overall performance of the four methods for estimating $h(t)$ can be ordered due to number of having smaller MSE as follows:

$$\hat{h}_{EBE} > \hat{h}_{HBE} > \hat{h}_{BE} = \hat{h}_{eBE}$$

- Among four estimates of $h(t)$ based on LLF shown in Table 8, we can deducted that hierarchical Bayes estimates are the best in all cases of small samples sizes. Also, the E-Bayes estimates are the best in nearly all cases of moderate samples sizes except for [15], [7] where the hierarchical Bayes estimates are the best and [20], [13] where the E-Bayes and hierarchical Bayes estimates are equivalent. In the end, The E-Bayes estimates have smallest MSE in most cases except for [50], [30], [70], [35] where the Bayes and E-Bayes and Bayes estimates are equivalent. Generally, the overall performance of the four methods for estimating $h(t)$ can be ordered due to number of having smaller MSE as follows:

$$\hat{h}_{EBL} > \hat{h}_{HBL} > \hat{h}_{BL} = \hat{h}_{eBL}$$

- In comparing the different approaches within the various loss function, we can ordered them due to having smallest MSE to be

$$\hat{h}_{SELF} > \hat{h}_{LLF} > \hat{h}_{ELF} > \hat{h}_{QLF}$$

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