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Research paper

Nonparametric Prediction Intervals of generalized order statistics from two independent sequences

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Abstract

This paper, discusses the problem of predicting future a generalized order statistic of an iid sequence sample was drawn from an arbitrary unknown distribution, based on observed also generalized order statistics from the same population. The coverage probabilities of these prediction intervals are exact and free of the parent distribution $F()$. Prediction formulas of ordinary order statistics and upper record values are extracted as special cases from the productive results. Finally, numerical computations on several models of ordered random variables are given to illustrate the proposed procedures.

Keywords: *Prediction intervals; Generalized order statistics; Coverage probability; Confidence intervals; Two-sample prediction.*

1. Introduction

The prediction subject of unobserved data has received a considerable attention in the literatures during the last two decades. Several applications of the prediction problems can be found in medical, engineering, stock market studies. Prediction subject have been categorized generally into two types: one-sample prediction and two-sample prediction. One-sample prediction case is based on a sequence of observations, experimenter seeks to predict the future random variables from the same sequence. In two-sample prediction type, using the available observations of past information sample, experimenter interested in (with some level of confidence) predicting some statistics in a future unobserved sample from the same underlying distribution $F()$. The past sample and the future sample are iid. Also, the predictor can be either point or interval. Such an interval is said to be a prediction interval (PIs) for the statistic of interest. Also, the prediction interval can be parametric (if it depends on the distribution parameters) or nonparametric (distribution-free). Often the observed data may not appropriate with certain distribution. Therefore, the results may include some error resulting from the mistakes in determining the suitable distribution. Distribution-free predictive inference is a statistical procedure to learn from data in the absence of prior knowledge and using only few modeling assumptions. The proposed prediction intervals are distribution-free, i.e. the corresponding coverage probabilities are known exactly without any assumption about the parent distribution other than being continuous. An exact expression for the prediction coefficient of these intervals is derived.

Many contexts have taken place in the distribution-free PIs direction using several assumptions [1]-[10]. But all these articles shared in one to one prediction way, i.e. predict a future certain type of samples based on another one. For the purpose of generalization, Mohie El-Din and Emam [11] discussed the predicting of future generalized order statistics, as well as outer and inner PIs based on ordinary order statistics. For more generalization, this article discusses the predicting of future generalized order statistics based on generalized order statistics. The study was conducted over all assumptions of generalized order statistics ($gOSs$) [12] and [13]. Paper is organized as follows: In section 2, some preliminaries are given. In Section 3, distribution-free PIs for a gOS from a future Y-sequence of iid random variables, based on also $gOSs$ from the X-sequence are derived. Section 4, includes numerical computations. Finally, conclusions are given in section 5.

This paper aims to construct nonparametric PIs for future $gOSs$ based on $gOSs$, therefore, all the previous researches in the field of nonparametric PIs for future Y-sample based on another X-sample (which can be obtained as a special model of $gOSs$ as table 1 considerations) is a special case of this paper. Some numerical results of $oOSs$ and $Krecord$ (for some $K \in N$) as special models of $gOSs$ case I; and $oOSs$, $nonI$, $Seque$, $Pfief$, $PCOs$ with two stages ($m_r = 0$ if $r \neq r_1$, $m_{r_1} = n_1$, $k = \nu - n_1 - n + 1$, $\gamma_r = \nu - n_1 - r + 1$ if $r \leq r_1$, $\gamma_r = \nu - n_1 - r + 1$ if $r > r_1$, see, [12] p.48), $Trunc$ and $k_n rec$ as special models of case II.

2. Preliminaries

The concept of $gOSs$ was introduced by Kamps [12] and was developed by Kamps and Cramer [13]. This concept was introduced as a unified approach to several models of ordered random variables such as, ordinary order statistics ($oOSs$), order statistics with non-integral sample size ($nonI$), K-records model ($Krecord$), upper

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record values (*record*), sequential order statistics (*Seque*), Pfeifer's record model ($Pfeif$), progressively Type-II right-censored order statistics sample (*PCOs*), ordering via truncation ($Trunc$) and k_n-record values ($k_n rec$). Table 1, shows these models as special cases of $gOSs$. Let $n \in N$, $\widetilde{m} = (m_1,\ldots,m_{n-1}) \in R_{n-1}$, if $n \geq 2$ ($\widetilde{m} \in R$, arbitrary if $n=1$) $k \geq 1$, be given constants such that for all $1 \leq i \leq n-1, \gamma_i = k+n-i+M_i > 0$, where $M_i = \sum_{j=i}^{n-1} m_j$. We may classify $gOSs$ into the following two cases:

Case I: If $m_1 = m_2 = \ldots = m_{n-1} = m$, the probability density function (pdf) of $X_{r,n,m,k}$, as introduced by Kamps [12], can be written as

$$f_{X_{r,n,m,k}}(x) = \frac{c_{r-1}}{(r-1)!}\overline{F}^{\gamma_r - 1}(x) f(x) g_m^{r-1}(F(x)), \tag{1}$$

where $c_{r-1} = \prod_{j=1}^{r}\gamma_j, \gamma_j = k+(n-j)(m+1)$,
$g_m(z) = h_m(z) - h_m(0),\ 0<z<1$, such that

$$h_m(z) = \begin{cases} \frac{-(1-z)^{m+1}}{m+1}, & m \neq -1, \\ -\ln(1-z), & m=-1. \end{cases} \tag{2}$$

The survival function of $X_{r,n,m,k}$, is given by:

$$\overline{F}_{X_{i,n,m,k}}(x) = c_{i-1}(1-F(x))^{\gamma_i}\sum_{\nu=0}^{i-1}\frac{g_m^{\nu}(F(x))}{\nu!\, c_{i-\nu-1}}, 1 \leq i \leq n. \tag{3}$$

Case II: If $\gamma_i \neq \gamma_j$, $i,j = 1,2,\ldots,n-1$ and $i \neq j$, the pdf of $X_{r,n,\widetilde{m},k}$ which is given in Eq.(1) as be introduced by Kamps and Cramer [13], is given by:

$$f_{X_{r,n,\widetilde{m},k}}(x) = c_{r-1}\sum_{i=1}^{r} a_i(r)\left(\overline{F}(x)\right)^{\gamma_i - 1} f(x), \tag{4}$$

where $a_i(r) = \prod_{j=1, j\neq i}^{r}\frac{1}{\gamma_j - \gamma_i}$, $1 \leq r \leq n$ and $\gamma_i = k+n-i+M_i > 0$. And the survival function of $X_{r,n,\widetilde{m},k}$, is given by:

$$\overline{F}_{X_{i,n,\widetilde{m},k}}(x) = c_{i-1}\sum_{\nu=1}^{i}\frac{a_\nu(i)}{\gamma_\nu}(1-F(x))^{\gamma_\nu}. \tag{5}$$

Let c_{i-1}, m, $a_i(r)$ and γ_i denote the past scheme X constants, and c^*_{r-1}, m^*, $a^*_i(r)$ and γ^*_r follow the future unobserved scheme Y. The prediction coefficients ϕ not only depend on subscripts, but also depend on observed sample sizes n and n^* in addition the constants k and k^*, i.e. $\phi(.) = \phi(.\ ,n,k;.\ ,n^*,k^*)$.

Table 1: Some special cases of $gOSs$, with λ_i, $\alpha_i \in \mathbf{R}^+$, $1 \leq i \leq n-1$.

Model	m_r	k	γ_r
$oOSs$	0	1	$n-r+1$
$nonI$	0	$\alpha-n+1$	$\alpha-r+1$
$Krecord$	-1	$k \in \mathcal{N}$	k
$record$	-1	1	1
$Seque$	$(n-r+1)\alpha_r -$ $(n-r)\alpha_{r+1}-1$	α_n	$(n-r+1)\alpha_r$
$Pfeif$	$\lambda_r - \lambda_{r+1} - 1$	λ_n	λ_r
$PCOs$	$R_r \in \mathcal{N}_0$	R_m+1	$m-r+1+M_i$
$Trunc$	$\alpha_r k_r - \alpha_{r+1}k_{r+1}-1$	$\alpha_n k_n$	$\alpha_r k_r$
$k_n rec$	$\lambda_r k_r - \lambda_{r+1}k_{r+1}-1$	$\lambda_n k_n$	$\lambda_r k_r$

$Lemma$1. Based on X-sample observations, suppose we are interested in obtaining $(1-\alpha)100\%$ distribution-free upper pound PIs for Y_r from a future Y-sample of the form $(-\infty, X_i)$, $0 \leq \alpha \leq 1$, such that, the coverage probability $P(X_i \geq Y_r) = 1-\alpha$. We refer to the interval $(-\infty, X_i)$ as $(1-\alpha)100\%$ PI for Y_r. Upon the non-parametric prediction procedure by assuming that Y_r is continuous random variable. Then, we get

$$\phi(i;r) = P(X_i \geq Y_r) = \int_{-\infty}^{\infty} P(X_i \geq y) f_{Y_r}(y)dy. \tag{6}$$

Distribution-free prediction attempt transforming integration base from the random variable to the survival function, to distribute $\overline{F}()$ randomly as standard uniform $(0,1)$ random variable. Therefore, the sampling distribution doesn't appear in final results. The coverage probabilities of these PIs intervals are exact and are free of the parent distribution $F()$, and depend only on the prefixed subscripts of the samples. Therefor, $\phi(i;r) = 1-\alpha$ represents the prediction coefficient which does not depend on the parameters of the parent distribution F, it depends only on the random variable's positions (the indices i and r) in addition modeling prefixes. Here, X_i are the upper bounds of the prediction interval for Y_r. Under the assumptions of $lemma$1, assume we are interested in obtaining $(1-\alpha)100\%$ distribution-free two sided PIs for Y_r from a future Y-sample of the form (X_i, X_j), $1 \leq i < j$, such that, the coverage probability $P\left(X_i \leq Y_r \leq X_j\right) = 1-\alpha$. We refer to the interval (X_i, X_j) as $(1-\alpha)100\%$ PI for Y_r. Then, we get

$$\begin{aligned} p\left(X_i \leq Y_r \leq X_j\right) &= P\left(X_j \geq Y_r\right) - P\left(X_i \geq Y_r\right) \\ &= \phi(j;r) - \phi(i;r). \end{aligned} \tag{7}$$

Such that, $p\left(X_i \leq Y_r \leq X_j\right) = 1-\alpha$ presents the coverage probability for Y_r. Thus, we have (X_i, X_j) is a prediction interval for Y_r, Here, X_i and X_j are the lower and upper bounds of the prediction interval for Y_r, respectively.

3. PIs for individual $gOSs$

In this section, we obtain one and two-sided distribution-free PIs for a future r^{th} $gOSs$ $Y_{r,n^*,\widetilde{m^*},k^*}$, $1 \leq i \leq n^*$ based on the endpoints of observed $gOSs$. Let $Y_{1,n^*,\widetilde{m^*},k^*}, \ldots, Y_{n^*,n^*,\widetilde{m^*},k^*}$ be n^* $gOSs$ based on the continuous cumulative distribution function (cdf) F with density function (pdf) f from a future Y-sequence of i.i.d. random variables, under assumption $m^*_1 = m^*_2 = \ldots = m^*_{n-1} = m^*$, or $\gamma^*_i \neq \gamma^*_j$, $i,j = 1,2,\ldots,n^*-1$ and $i \neq j$, and let $X_{\iota,n,\widetilde{m},k}$ be ι^{th} $gOSs$ from another observed random sample, under assumption $m_1 = m_2 = \ldots = m_{n-1} = m$, or $\gamma_i \neq \gamma_j$, $i,j = 1,2,\ldots,n^*-1$ and $i \neq j$, and further let the underling distribution of the two samples be the same. We are interested here in obtaining one and two-sided distribution-free PIs for a future r^{th} $gOSs$ $Y_{r,n^*,\widetilde{m^*},k^*}$, $1 \leq r \leq n^*$ based on the endpoints of observed $gOSs$. The coverage probabilities of this PIs are exact and do not depend on the sampling distribution.

$Theorem$1. Let $\{X_{i,n,m,k}, 1 \leq i \leq n\}$ under assumption $m_1 = m_2 = \ldots = m_{n-1} = m$, and $\{Y_{r,n^*,m^*,k^*}, 1 \leq r \leq n^*\}$ under assumption $m^*_1 = m^*_2 = \ldots = m^*_{n-1} = m^*$, be two independent $gOSs$ from continuous cdf F. then $(-\infty, X_{i,n,m,k})$, $1 \leq i \leq n$, is an upper prediction bound for the future Y_{r,n^*,m^*,k^*}, with the corresponding prediction coefficient, being free of F, given by:

$$\phi_1(i,m;r,m^*) = \sum_{\nu=0}^{i-1} C_\nu(i;r) \times \begin{cases} \sum_{\lambda=0}^{\nu}\frac{b_\lambda^\nu(m)B\left(r,\frac{\gamma_i+\gamma^*_r+\lambda(m+1)}{m^*+1}\right)}{(m^*+1)^r}, & m \neq -1; m^* \neq -1, \\ \sum_{\lambda=0}^{\nu}\frac{(r-1)!\,b_\lambda^\nu(m)}{(\gamma_i+k^*+\lambda(m+1))^r}, & m \neq -1; m^* = -1, \\ \sum_{\eta=0}^{r-1}\frac{\nu!\,b_\eta^{r-1}(m^*)}{(k+\gamma^*_r+\eta(m^*+1))^{\nu+1}}, & m = -1; m^* \neq -1, \\ \frac{(\nu+r-1)!}{(k+k^*)^{\nu+r}}, & m=-1; m^*=-1, \end{cases} \tag{8}$$

where $C_\nu(i;r) = \frac{c_{i-1}c^*_{r-1}}{(r-1)!\,\nu!\,c_{i-\nu-1}}$, $b_\lambda^\nu(m) = \frac{(-1)^\lambda\binom{\nu}{\lambda}}{(m+1)^\nu}$ and $B(.,.)$ is a beta constant.

Table 2: Some values of $p\left(X_{i,20,m,k} \leq Y_{r,n^*,m^*,k^*} \leq X_{j,20,m,k}\right)$ for some i, j and r.

Sample				Y					
X	r	i	j	$Y_{r:25}$	*record*	2*record*	3*record*	4*record*	5*record*
$X_{t:20}$	1	1	6	0.4889	0.1923	0.3276	0.4228	0.4892	0.5351
			12	0.5000	0.4231	0.6268	0.7219	0.7619	0.7732
			18	0.5000	0.6538	0.8234	0.8562	0.8482	0.8278
		3	6	0.1062	0.1154	0.18803	0.2320	0.2566	0.2682
			18	0.1173	0.5769	0.6838	0.6654	0.6156	0.5609
	3	1	6	0.7593	0.0037	0.0222	0.0574	0.1055	0.1617
			12	0.8808	0.0306	0.1441	0.2983	0.4508	0.5812
			18	0.8827	0.1315	0.4334	0.6783	0.8267	0.9065
		3	18	0.5000	0.1310	0.4298	0.6680	0.8060	0.8721
	6	1	18	0.9886	0.0028	0.0503	0.1835	0.3661	0.5443
			22	0.9889	0.0226	0.2249	0.5212	0.7460	0.8755
			25	0.9889	0.2096	0.6889	0.9063	0.9734	0.9923
		5	18	0.6371	0.0028	0.0503	0.1832	0.3647	0.5401
			22	0.6374	0.0226	0.2248	0.5209	0.7447	0.8719
	10	10	22	0.4998	0.0002	0.0167	0.1118	0.2950	0.5007
			24	0.5000	0.0027	0.0948	0.3489	0.6156	0.7979
			25	0.5000	0.0207	0.2692	0.6079	0.8240	0.9259
		15	24	0.0782	0.0027	0.0947	0.3473	0.6063	0.7676
			25	0.0782	0.0207	0.2691	0.6063	0.8146	0.8957
record	5	1	4	0.1921	0.3320	0.6097	0.6489	0.6160	0.5674
			8	0.1923	0.7749	0.8496	0.7599	0.6717	0.5980
			20	0.1923	0.9680	0.8683	0.7627	0.6723	0.5981
		2	20	0.0235	0.8899	0.6488	0.4661	0.3446	0.2632
	10	1	4	0.3820	0.0452	0.3051	0.5279	0.6400	0.6804
			8	0.3846	0.3136	0.8108	0.9035	0.8817	0.8350
			12	0.3846	0.6672	0.9615	0.9420	0.8924	0.8385
			18	0.3846	0.9380	0.9823	0.9437	0.8926	0.8385
		2	18	0.0931	0.9331	0.9244	0.8029	0.6779	0.5693
		3	18	0.0170	0.9197	0.8185	0.6093	0.4417	0.3226
	15	1	8	0.5770	0.0669	0.5378	0.8251	0.9087	0.9141
			12	0.5770	0.2786	0.8772	0.9712	0.9625	0.9347
			18	0.5770	0.7017	0.9913	0.9865	0.9648	0.9351
		2	18	0.2238	0.7014	0.9799	0.9364	0.8593	0.7728
		3	18	0.0661	0.7005	0.9495	0.8361	0.6904	0.5565
	20	1	8	0.6912	0.0002	0.0262	0.1338	0.2850	0.4251
			12	0.7692	0.0748	0.6772	0.9324	0.9758	0.9711
			18	0.7692	0.3714	0.9613	0.9952	0.9884	0.9739
		2	18	0.4451	0.3714	0.9593	0.9794	0.9423	0.8870
		3	18	0.2042	0.5000	0.9752	0.9390	0.8455	0.7348

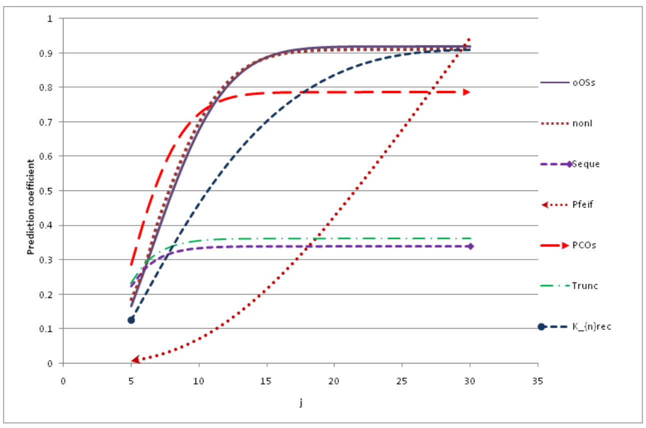

Figure 1: The coverage probabilities of the event $X_{3:30} \leq Y_{5,20,\tilde{m}^*,k^*} \leq X_{j:30}$ under table 3 assumptions.

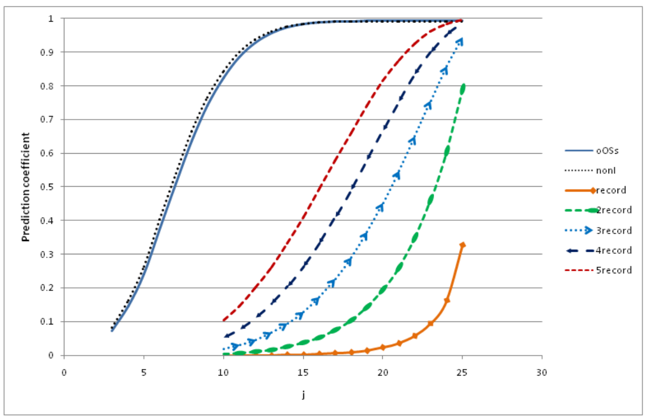

Figure 2: The coverage probabilities of the event $X_{1:25:30} \leq Y_{5,20,\tilde{m}^*,k^*} \leq X_{j:25:30}$ under table 4(a) assumptions.

Table 3: Some values of $p\left(X_{i,n,m,k} \leq Y_{r,20,\widetilde{m^*},k^*} \leq X_{j,n,m,k}\right)$ for some i, j and r.

Sample X	r	i	j	Y: $oOSs$	$nonI$	$Seque$	$Pfeif$	$PCOs$	$trunc$	$K_n rec$
$X_{t:30}$	1	1	10	0.5971	0.5870	0.5971	0.2903	0.4996	0.6086	0.5845
			15	0.6000	0.5893	0.6000	0.4516	0.5000	0.6121	0.7595
			20	0.6000	0.5894	0.6000	0.6129	0.5000	0.6122	0.8567
		2	10	0.3522	0.3401	0.3522	0.2581	0.2454	0.3662	0.4993
		3	10	0.2042	0.1937	0.2042	0.2258	0.1183	0.2167	0.4195
	5	1	10	0.7508	0.7772	0.7921	0.0727	0.9113	0.8039	0.5416
			15	0.9617	0.9671	0.7971	0.2171	0.9715	0.8102	0.7807
			20	0.9913	0.9906	0.7973	0.4289	0.9739	0.8103	0.9151
		3	20	0.9187	0.9088	0.3418	0.4269	0.7881	0.3633	0.8374
		5	20	0.7512	0.7273	0.1167	0.4196	0.5000	0.1299	0.7106
	10	1	15	0.5227	0.5796	0.8321	0.1705	0.3544	0.8453	0.7777
			20	0.8902	0.9170	0.8322	0.3839	0.7466	0.8454	0.9159
			25	0.9944	0.9965	0.8323	0.6518	0.9659	0.8455	0.9759
			30	1.0000	1.0000	0.8328	0.9414	0.9998	0.8455	0.9903
		5	15	0.5165	0.5712	0.1321	0.1672	0.3510	0.1474	0.5863
		7	17	0.6705	0.7120	0.0407	0.2352	0.4998	0.0475	0.5117
	15	1	20	0.2911	0.3746	0.8502	0.3675	0.0118	0.8642	0.9160
			25	0.7795	0.8458	0.8502	0.6413	0.1109	0.8642	0.9764
		5	25	0.7794	0.8457	0.1421	0.6392	0.1109	0.1596	0.7880
		7	27	0.9220	0.9526	0.0439	0.7508	0.2426	0.0517	0.6562
	20	1	30	0.6000	0.8202	–	0.9386	–	0.0000	0.9911
		5	30	0.6000	0.8202	–	0.9370	–	0.0000	0.8038
		8	30	0.6000	0.8202	–	0.9260	–	0.0000	0.5925
$record$	5	1	2	0.2016	0.1949	0.1019	0.2923	0.2016	0.1099	0.2355
			5	0.2381	0.2283	0.1132	0.6224	0.2381	0.1191	0.3191
			10	0.2381	0.2283	0.1132	0.6726	0.2381	0.1191	0.3204
		2	20	0.0365	0.0334	0.0083	0.3819	0.0365	0.0092	0.0845
	10	1	5	0.4750	0.4557	0.1556	0.6445	0.5286	0.1647	0.3253
			20	0.4762	0.4566	0.1556	0.7009	0.5311	0.1647	0.3267
		2	20	0.1486	0.1354	0.0146	0.4072	0.1903	0.0163	0.0872
	15	1	5	0.6985	0.6736	–	0.6518	0.7678	–	0.3267
			6	0.7101	0.6821	–	0.6806	0.8029	–	0.3278
			20	0.7143	0.6849	–	0.7099	0.8242	–	0.3281
		2	20	0.3728	0.3362	–	0.4162	0.5408	–	0.0877
	20	1	5	0.6532	0.7570	–	0.6554	–	–	0.3273
			10	0.9353	0.9108	–	0.7125	–	–	0.3287

Proof. Based on *lemma*1, we can write

$$\begin{aligned}\phi_1(i,m;r,m^*) &= P\left(X_{i,n,m,k} > Y_{r,n^*,m^*,k^*}\right)\\ &= \int_{-\infty}^{\infty} P\left(X_{i,n,m,k} > y\right) f_{Y_{r,n^*,m^*,k^*}}(y)dy. \quad (9)\end{aligned}$$

Upon substituting the survival function (3) and constructing pdf of Y_{r,n^*,m^*,k^*} from (1), we can express

$$\phi_1(i,m;r,m^*) = \sum_{\nu=0}^{i-1} C_\nu(i;r) I_\nu(i), \quad (10)$$

such that

$$\begin{aligned}I_\nu(i) &= \int_{-\infty}^{\infty} \overline{F}^{\gamma_i+\gamma_r^*-1}(y) g_m^\nu(F(y)) g_{m^*}^{r-1}(F(y)) f(y)dy\\ &= \int_0^1 y^{\gamma_i+\gamma_r^*-1} g_m^\nu(1-y) g_{m^*}^{r-1}(1-y)dy. \quad (11)\end{aligned}$$

The integration (11) is given for all $m, m^* \in \mathbf{R}$. Thus, based on assumptions of case I, $I_\nu(i)$ is given by:

$$\begin{cases} \int_0^1 y^{\gamma_i+\gamma_r^*-1}\left(\frac{1-y^{m+1}}{m+1}\right)^\nu \left(\frac{1-y^{m^*+1}}{m^*+1}\right)^{r-1} dy, & m \neq -1; m^* \neq -1,\\ \int_0^1 y^{\gamma_i+k^*-1}\left(\frac{1-y^{m+1}}{m+1}\right)^\nu (-\ln y)^{r-1} dy, & m \neq -1; m^* = -1,\\ \int_0^1 y^{k+\gamma_r^*-1}(-\ln y)^\nu \left(\frac{1-y^{m^*+1}}{m^*+1}\right)^{r-1} dy, & m = -1; m^* \neq -1,\\ \int_0^1 y^{k+k^*-1}(-\ln y)^{\nu+r-1} dy \\ \quad = \int_0^\infty z^{\nu+r-1} e^{-(k+k^*)z} = \frac{(\nu+r-1)!}{(k+k^*)^{\nu+r}}, & m = -1; m^* = -1. \end{cases} \quad (12)$$

Now, using the binomial expansion on some brackets and solving (12) as the last one, we obtain the required result. Using *lemma*1 and under assumptions of *theorem*1, the coverage probability of the event $X_{i,n,m,k} \leq Y_{r,n^*,m^*,k^*} \leq X_{j,n,m,k}$, is given by:

$$p\left(X_{i,n,m,k} \leq Y_{r,n^*,m^*,k^*} \leq X_{j,n,m,k}\right) = \phi_1(j,m;r,m^*) - \phi_1(i,m;r,m^*), \quad (13)$$

Table 4(a): Some values of $p\left(X_{i,30,\widetilde{m},k}\leq Y_{r,20,\widetilde{m}^{*},k^{*}}\leq X_{j,30,\widetilde{m},k}\right)$ based on *PCOs* and *Trunk*, for some i,j and r.

X	r	i	j	Y: *oOSs*	*nonI*	*record*	*2record*	*3record*	*4record*	*5record*
PCOs	1	1	10	0.5984	0.5881	0.3610	0.5834	0.7233	0.8132	0.8719
			20	0.6000	0.5894	0.7604	0.9357	0.9810	0.9939	0.9979
	3	1	10	0.9069	0.9074	0.0139	0.0738	0.1699	0.2814	0.3930
		2	10	0.7955	0.7885	0.0139	0.0738	0.1699	0.2814	0.3930
	5	1	10	0.8234	0.8443	0.0002	0.0045	0.0201	0.0535	0.1034
			20	0.9926	0.9914	0.0237	0.1957	0.4522	0.6694	0.8135
		3	20	0.9200	0.9096	0.0237	0.1957	0.4522	0.6694	0.8135
	7	1	25	0.9992	0.9991	0.1209	0.5634	0.8411	0.9477	0.9831
		7	25	0.7632	0.7328	0.1209	0.5634	0.8411	0.9477	0.9831
	11	8	23	0.9578	0.9463	0.0002	0.0184	0.1251	0.3244	0.5393
			25	0.9603	0.9477	0.0105	0.1930	0.5115	0.7549	0.8883
	15	1	24	0.9740	0.9854	0.0000	0.0055	0.0616	0.2094	0.4099
			25	0.9946	0.9973	0.0007	0.0480	0.2319	0.4809	0.6906
	19	1	24	0.5700	0.7320	0.0000	0.0004	0.0114	0.0653	0.1823
			25	0.7995	0.8992	0.0000	0.0103	0.0877	0.2560	0.4605
Trunk	5	3	10	0.6604	0.6369	0.1243	0.4907	0.7492	0.8798	0.9413
		4	10	0.3702	0.3453	0.1244	0.4907	0.7492	0.8798	0.9413
			11	0.3703	0.3453	0.2068	0.6387	0.8585	0.9443	0.9770
	9	2	10	0.9908	0.9900	0.0050	0.1054	0.3256	0.5475	0.7129
		4	10	0.7850	0.7586	0.0050	0.1054	0.3256	0.5475	0.7129
			12	0.7880	0.7608	0.0335	0.3319	0.6570	0.8417	0.9282
		6	14	0.2972	0.2646	0.1328	0.6375	0.8897	0.9669	0.9885
	13	2	15	0.9998	0.9997	0.0419	0.4613	0.8058	0.9380	0.9801
		7	15	0.4443	0.3911	0.0419	0.4613	0.8058	0.9362	0.9741
	19	3	18	0.9993	0.9999	0.0455	0.5649	0.8912	0.9761	0.9946
			19	0.9998	0.9999	0.0999	0.7206	0.9515	0.9920	0.9986

where ϕ_1 is given in (8). Thus, we have a prediction interval $(X_{i,n,m,k},X_{j,n,m,k})$, $1\leq i<j\leq n$, for Y_{r,n^*,m^*,k^*} $(1\leq r\leq n^*)$, whose prediction coefficient given by (13), is free of F.

Some special cases of ϕ_1:

We have the following simpler expressions for some special cases one-sided prediction coefficient ϕ_1 that given by (8), and the corresponding two-sided prediction coefficient that given by (13).

• Distribution-free upper pound PIs for $Y_{r:n^*}$ from a future Y-sample of the form $(-\infty,X_{i:n})$, can be obtained as a special case from $\phi_1\left(i,m;r,m^*\right)$ by setting $m=m^*=0$, $k=k^*=1$, $\gamma_i=n-i+1$, $1\leq i\leq n$ and $\gamma_r^*=n^*-r+1$, $1\leq r\leq n^*$, and has the following form

$$p\left(X_{i:n}\geq Y_{r:n^*}\right)=\phi_1\left(i,0;r,0\right)=\sum_{\nu=0}^{i-1}r\binom{n-i+\nu}{\nu}\binom{n^*}{r}\sum_{\lambda=0}^{\nu}(-1)^{\lambda}\binom{\nu}{\lambda}B(r,n-i+n^*-r+2). \tag{14}$$

Distribution-free two sided PIs for $Y_{r:n^*}$ of the form $(X_{i:n},X_{j:n})$ which was discussed by Mohie El-Din et al [1] appear here as a special case, such that

$$p\left(X_{i:n}\leq Y_{r:n^*}\leq X_{j:n}\right)=P\left(X_{j:n}\geq Y_{r:n^*}\right)-P\left(X_{i:n}\geq Y_{r:n^*}\right). \tag{15}$$

Similarly:

• Prediction interval of future *record* based on *oOSs* which was discussed by Ahmadi and Balakrishnan [5], can be given with *Lemma*1 by:

$$p\left(X_{i:n}\geq U_r\right)=\phi_1\left(i,0;r,-1\right)=\sum_{\nu=0}^{i-1}\binom{n-i+\nu}{\nu}\sum_{\lambda=0}^{\nu}\frac{(-1)^{\lambda}\binom{\nu}{\lambda}}{(n-i+\lambda+2)^{r}}. \tag{16}$$

• Prediction interval of future *oOSs* based on *record* which was discussed by Ahmadi and Balakrishnan [5], is given by:

$$p\left(U_i\leq Y_{r:n^*}\leq U_j\right)=\phi_1\left(j,-1;r,0\right)-\phi_1\left(i,-1;r,0\right)=\sum_{\nu=i}^{j-1}r\binom{n^*}{r}\sum_{\lambda=0}^{r-1}\frac{(-1)^{\lambda}\binom{r-1}{\lambda}}{(n^*-r+\lambda+2)^{\nu+1}}. \tag{17}$$

• Prediction interval of future *record* based on *record* also, which was discussed by Raqab and Balakrishnan [8], is given by:

$$p\left(U_i\leq U_r^*\leq U_j\right)=\phi_1\left(j,-1;r,-1\right)-\phi_1\left(i,-1;r,-1\right)=\sum_{\nu=i}^{j-1}\binom{\nu+r-1}{\nu}\frac{1}{2^{\nu+r}}. \tag{18}$$

Table 4(b): Some values of $p\left(X_{i,30,\widetilde{m},k} \leq Y_{r,20,\widetilde{m}^*,k^*} \leq X_{j,30,\widetilde{m},k}\right)$ based on *Seque* and *Pfief*, for some i, j and r.

X	r	i	j	Y: $Y_{r:20}$	nonI	record	2record	3record	4record	5record
Seque	1	1	5	0.7205	0.7151	0.2335	0.4027	0.5278	0.7220	0.6940
			7	0.7478	0.7398	0.4027	0.6271	0.7585	0.8387	0.8894
			9	0.7499	0.7416	0.5776	0.8036	0.9016	0.9477	0.9708
		2	5	0.4143	0.4045	0.2335	0.4027	0.5279	0.6221	0.6941
			7	0.4417	0.4292	0.4027	0.6271	0.7585	0.8387	0.8894
	2	1	7	0.9280	0.9255	0.1090	0.2902	0.4555	0.5872	0.6873
			9	0.9392	0.9351	0.2356	0.5186	0.7071	0.8211	0.8889
			11	0.9398	0.9356	0.4066	0.7315	0.8791	0.9435	0.9723
		3	11	0.4765	0.4584	0.4066	0.7315	0.8791	0.9435	0.9723
	4	1	9	0.9883	0.9895	0.0218	0.1358	0.3038	0.4687	0.6058
			13	0.9969	0.9964	0.1786	0.5654	0.8027	0.9126	0.9606
		3	13	0.8294	0.8144	0.0721	0.3241	0.5720	0.7432	0.8481
	11	5	16	0.9713	0.9632	0.0054	0.1521	0.4603	0.7146	0.8612
			19	0.9716	0.9634	0.0589	0.5197	0.8466	0.9570	0.9880
	16	8	20	0.9491	0.9252	0.0093	0.2879	0.6933	0.8961	0.9672
		10	22	0.7381	0.6668	0.0550	0.5910	0.9048	0.9809	0.9961
	20	12	22	0.9159	0.8409	0.0102	0.3469	0.7688	0.9364	0.9799
Pfeif	5	1	10	0.2381	0.2283	0.2002	0.6024	0.8250	0.9228	0.9647
			15	0.2381	0.2283	0.2558	0.6968	0.8939	0.9629	0.9865
		2	15	0.0649	0.0598	0.2558	0.6968	0.8939	0.9629	0.9865
	7	1	10	0.3331	0.3194	0.0629	0.3633	0.6435	0.8121	0.9015
		5	10	0.0084	0.0069	0.0629	0.3633	0.6435	0.8121	0.9015
	9	1	10	0.4271	0.4099	0.0176	0.1956	0.4576	0.6678	0.8039
		7	11	0.0064	0.0050	0.0192	0.2096	0.4825	0.6940	0.8260
	11	1	11	0.5196	0.4994	0.0051	0.1058	0.3235	0.5443	0.7111
		5	17	0.0560	0.0463	0.0076	0.1496	0.4248	0.6665	0.8214
	13	1	20	0.6174	0.5927	0.0023	0.0851	0.3154	0.5682	0.7548
			30	0.6188	0.5935	0.0034	0.1181	0.4052	0.6775	0.8466
	15	5	20	0.2088	0.1750	0.0060	0.0408	0.2025	0.4327	0.6393
			25	0.2141	0.1782	0.0007	0.0498	0.2388	0.4916	0.7014
	17	1	30	0.7885	0.7668	0.0002	0.0277	0.1729	0.4111	0.6370
		8	30	0.2156	0.1674	0.0002	0.0277	0.1729	0.4111	0.6370
	19	1	30	0.7497	0.7979	0.0001	0.0128	0.1061	0.3003	0.5223
		8	30	0.3475	0.3060	0.0001	0.0128	0.1061	0.3003	0.5223

- Prediction interval of future *gOSs* case I based on *oOSs*, is given by:

$$p\left(X_{i:n} \geq Y_{r,n^*,m^*,k^*}\right) = \sum_{\nu=0}^{i-1} \frac{\binom{n-i+\nu}{\nu} c_{r-1}}{(r-1)!} \sum_{\lambda=0}^{\nu} (-1)^{\lambda} \binom{\nu}{\lambda} \begin{cases} \frac{B(r, \frac{n-i+1+\gamma_r^*+\lambda}{m^*+1})}{(m^*+1)^r}, & m^* \neq -1, \\ \frac{(r-1)!}{(n-i+1+k^*+\lambda)^r}, & m^* = -1. \end{cases} \tag{19}$$

- Prediction interval of future *gOSs* case I based on *record*, is given by:

$$p\left(U_i \geq Y_{r,n^*,m^*,k^*}\right) = \sum_{\nu=0}^{i-1} \frac{1}{(r-1)!\nu!} \begin{cases} c_{r-1}^* \sum_{\eta=0}^{r-1} \frac{\nu! b_\eta^{r-1}(m^*)}{(\gamma_r^*+\eta(m^*+1)+1)^{\nu+1}}, & m^* \neq -1, \\ \frac{(\nu+r-1)!(k^*)^r}{(k^*+1)^{\nu+r}}, & m^* = -1. \end{cases} \tag{20}$$

- Prediction interval of future *oOSs* based on *gOSs* case I, is given by:

$$p\left(X_{i,n,m,k} \geq Y_{r:n^*}\right) = \sum_{\nu=0}^{i-1} \frac{n^*!}{(n^*-r)!(r-1)!\nu!} \begin{cases} \frac{c_{i-1}}{c_{i-\nu-1}} \sum_{\lambda=0}^{\nu} b_\lambda^\nu(m) B(r, \gamma_i + n^* - r + 1 + \lambda(m+1)), & m \neq -1, \\ k^\nu \sum_{\eta=0}^{r-1} \frac{\nu!(-1)^\eta \binom{r-1}{\eta}}{(k+n^*-r+1+\eta)^{\nu+1}}, & m = -1. \end{cases} \tag{21}$$

- Prediction interval of future *record* based on *gOSs* case I, given by:

$$p\left(X_{i,n,m,k} \geq U_r^*\right) = \sum_{\nu=0}^{i-1} \frac{1}{(r-1)!\nu!} \begin{cases} \frac{c_{i-1}}{c_{i-\nu-1}} \sum_{\lambda=0}^{\nu} \frac{(r-1)! b_\lambda^\nu(m)}{(\gamma_i+1+\lambda(m+1))^r}, & m \neq -1, \\ \frac{(\nu+r-1)! k^\nu}{(k+1)^{\nu+r}}, & m = -1. \end{cases} \tag{22}$$

*Theorem*2. Let $\{X_{i,n,m,k}, 1 \leq i \leq n\}$ under assumption $m_1 = m_2 = \ldots = m_{n-1} = m$, and $\{Y_{r,n^*,\widetilde{m}^*,k^*}, 1 \leq r \leq n^*\}$ under assumption $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n^*-1$ and $i \neq j$, be two independent *gOSs* from continuous cdf F. then $(-\infty, X_{i,n,m,k})$, $1 \leq i \leq n$, is distribution-free one-sided

Table 5: Some values of $p\left(X_{i,30,\widetilde{m},k} \le Y_{r,20,\widetilde{m}^*,k^*} \le X_{j,30,\widetilde{m},k}\right)$ for some i,j and r.

				Y						
X	r	i	j	*oOSs*	*nonI*	*Seque*	*Pfeif*	*PCOs*	*trunc*	$K_n rec$
Seque	1	1	10	0.6000	0.6012	0.6000	0.8347	0.5000	0.6122	0.9021
			30	0.6000	0.6012	0.6000	0.9677	0.5000	0.6122	0.9091
		2	10	0.2522	0.2534	0.2522	0.7723	0.1629	0.2650	0.7462
	5	1	10	0.9926	0.9927	0.7973	0.7803	0.9739	0.8103	0.9779
		2	10	0.9070	0.9081	0.3931	0.7771	0.7850	0.4169	0.8962
		3	10	0.6769	0.6795	0.1358	0.7598	0.4573	0.1379	0.7274
	10	1	10	0.9921	0.9918	0.8323	0.7611	0.9740	0.8455	0.9806
		2	12	0.9713	0.9974	0.4346	0.9035	0.9966	0.4565	0.9150
		3	12	0.9710	0.9717	0.1523	0.8939	0.9812	0.1662	0.7501
	15	4	20	0.9922	0.9986	0.0429	0.9612	0.9986	0.0490	0.5466
			30	0.9922	0.9986	0.0429	0.9614	0.9998	0.0490	0.5466
		6	30	0.8554	0.8598	0.0016	0.7801	0.9917	0.0019	0.2075
	20	1	15	0.8678	0.8437	–	0.9835	0.0000	0.0000	0.9913
		4	20	0.9979	0.9967	–	0.9643	0.0000	0.0000	0.5478
Pfief	2	1	10	0.8235	0.8242	0.7013	0.1792	0.7435	0.7137	0.5860
			15	0.8380	0.8388	0.7045	0.3542	0.7457	0.7177	0.7980
		2	15	0.6386	0.6402	0.4419	0.3464	0.4870	0.4600	0.7441
	6	1	10	0.6470	0.6697	0.7951	0.0696	0.7781	0.8074	0.5565
		3	18	0.9488	0.8201	0.3383	0.3480	0.9099	0.3602	0.8058
		4	29	0.9139	0.9119	0.1984	0.9390	0.8349	0.2163	0.8467
	12	1	18	0.5800	0.5623	0.8321	0.3089	0.2581	0.8458	0.8852
		3	18	0.5799	0.5502	0.3737	0.3085	0.2581	0.3988	0.8124
		6	24	0.9604	0.9204	0.0697	0.6289	0.6289	0.7679	0.7007
	18	1	25	0.3914	0.3321	–	0.6870	0.0000	0.8711	0.9803
			26	0.5265	0.3223	–	0.7487	0.0000	0.8711	0.9847
			27	0.6755	0.3001	–	0.8111	0.0000	0.8711	0.9875
PCOs	3	1	10	0.8757	0.8763	0.6447	0.1495	0.8757	0.4546	0.6177
			20	0.8813	0.8819	0.6450	0.6433	0.8814	0.6549	0.9560
		2	20	0.6940	0.6953	0.3510	0.6409	0.6940	0.3628	0.9171
	6	4	20	0.6920	0.6917	0.0981	0.0865	0.6982	0.1052	0.4708
		6	20	0.4708	0.4681	0.0186	0.5812	0.5000	0.0207	0.6755
	12	3	15	0.9059	0.9001	0.2210	0.2564	0.8101	0.2328	0.7844
			20	0.9958	0.9732	0.2210	0.5660	0.9913	0.2328	0.8958
	16	1	16	0.7422	0.7222	0.7315	0.3009	0.5000	0.7423	0.8842
			20	0.9746	0.9521	0.7315	0.5584	0.8958	0.7423	0.9640
		2	20	0.9745	0.9517	0.4330	0.5583	0.8958	0.4477	0.9391
	18	2	20	0.9291	0.8880	0.4372	0.5556	0.7429	0.4522	0.9393
		8	22	0.9778	0.9231	0.0031	0.6799	0.9158	0.0037	0.5356

PI for the future $Y_{r,n^*,\widetilde{m}^*,k^*}$, with the corresponding prediction coefficient $\phi_2\left(i,m;r,\widetilde{m}^*\right)$, that does not depend on the sampling distribution F, and is given by:

$$\sum_{\nu=0}^{i-1}(r-1)!C_\nu(i;r)\sum_{\mu=1}^{r}a_\mu^*(r)\begin{cases}\frac{B(\nu+1,\frac{\gamma_i+\gamma_\mu^*}{m+1})}{(m+1)^{\nu+1}}, & m\neq-1,\\ \\ \frac{\nu!}{(k+\gamma_\mu^*)^{\nu+1}}, & m=-1.\end{cases} \tag{23}$$

Proof. Under the assumption that $\{Y_{r,n^*,\widetilde{m}^*,k^*}, 1\le r\le n^*\}$ are continuous r.v.'s, we can write

$$\begin{aligned}\phi_2\left(i,m;r,\widetilde{m}^*\right) &= P\left(X_{i,n,m,k} > Y_{r,n^*,\widetilde{m}^*,k^*}\right)\\ &= \int_{-\infty}^{\infty}P\left(X_{i,n,m,k}>y\right)f_{Y_{r,n^*,\widetilde{m}^*,k^*}}(y)dy.\end{aligned} \tag{24}$$

Using (3) and (4), ϕ_2 can be written as

$$\phi_2\left(i,m;r,\widetilde{m}^*\right) = \sum_{\nu=0}^{i-1}\frac{c_{i-1}c_{r-1}^*}{\nu!c_{i-\nu-1}}\sum_{\mu=1}^{r}a_\mu^*(r)J_{\nu,\mu}(i), \tag{25}$$

such that

$$J_{\nu,\mu}(i)=\int_{-\infty}^{\infty}\overline{F}^{\gamma_i+\gamma_\mu^*-1}(y)g_m^\nu\left(F(y)\right)f(y)dy. \tag{26}$$

Thus, based on $g_m(.)$ and (2), $J_{\nu,\mu}(i)$ is given by:

$$J_{\nu,\mu}(i)=\begin{cases}\int_{-\infty}^{\infty}\overline{F}^{\gamma_i+\gamma^*_\mu-1}(y)\left(\frac{1-\overline{F}^{m+1}(y)}{m+1}\right)^{\nu}f(y)dy, & m\neq -1,\\ \int_{-\infty}^{\infty}\overline{F}^{k+\gamma^*_\mu-1}(y)\left(-\ln\overline{F}(y)\right)^{\nu}f(y)dy, & m=-1.\end{cases} \tag{27}$$

Using the following transformation $\overline{F}(y)=u$, we get

$$J_{\nu,\mu}(i)=\begin{cases}\int_0^1 u^{\gamma_i+\gamma^*_\mu-1}\left(\frac{1-u^{m+1}}{m+1}\right)^{\nu}du, & m\neq -1,\\ \int_0^1 u^{k+\gamma^*_\mu-1}\left(-\ln u\right)^{\nu}du, & m=-1.\end{cases} \tag{28}$$

The prediction coefficient $\phi_2(i,m;r,\widetilde{m}^*)$, given in (23) is obtained directly by solving (28).

Some special cases of ϕ_2:

- Prediction interval of future *oOSs* based on also *oOSs*, is given by:

$$p\left(X_{i:n}\geq Y_{r:n^*}\right)=\phi_2\left(i,0;r,\widetilde{0}\right)=\sum_{\nu=0}^{i-1}r!\binom{n-i+\nu}{\nu}\binom{n^*}{r}\sum_{\mu=1}^{r}a^*_\mu(r)B(\nu+1,n-i+n^*-\mu+2). \tag{29}$$

- Prediction interval of future *oOSs* based on *record*, is given by:

$$p\left(U_i\geq Y_{r:n^*}\right)=\phi_2\left(i,-1;r,\widetilde{0}\right)=\sum_{\nu=0}^{i-1}r!\binom{n^*}{r}\sum_{\mu=1}^{r}\frac{a^*_\mu(r)}{(n^*-\mu+2)^{\nu+1}}. \tag{30}$$

- Prediction interval of future *gOSs* case II based on *oOSs* which was discussed by Mohie El-Din and Emam [11], is given by:

$$P\left(X_{i:n}\geq Y_{r,n^*,\widetilde{m}^*,k^*}\right)=\phi_2\left(i,0;r,\widetilde{m}^*\right)=\sum_{\nu=0}^{i-1}\frac{\binom{n-i+\nu}{\nu}c^*_{r-1}}{(r-1)!}\sum_{\mu=1}^{r}a^*_\mu(r)B(\nu+1,n-i+1+\gamma^*_\mu). \tag{31}$$

- Prediction interval of future *oOSs* based on *gOSs* case I which was discussed in [11], is given by:

$$P\left(X_{i,n,m,k}>Y_{r:n^*}\right)=\phi_2\left(i,m;r,\widetilde{0}\right)=\sum_{\nu=0}^{i-1}\frac{r\binom{n^*}{r}}{\nu!}\sum_{\mu=1}^{r}a^*_\mu(r)\begin{cases}\frac{c_{i-1}B\left(\nu+1,\frac{\gamma_i+n^*-\mu+1}{m+1}\right)}{c_{i-\nu-1}(m+1)^{\nu+1}}, & m\neq -1,\\ \frac{\nu!k^\nu}{(k+n^*-\mu+1)^{\nu+1}}, & m=-1.\end{cases} \tag{32}$$

- Prediction interval of future *gOSs* case II based on *record*, is given by:

$$P\left(U_i\geq Y_{r,n^*,\widetilde{m}^*,k^*}\right)=\phi_2\left(i,-1;r,\widetilde{m}^*\right)=\sum_{\nu=0}^{i-1}\frac{c^*_{r-1}}{(r-1)!\nu!}\sum_{\mu=1}^{r}a^*_\mu(r)\frac{\nu!}{(\gamma^*_\mu+1)^{\nu+1}}. \tag{33}$$

Under the assumption of *lemma*1, $(X_{i,n,m,k},X_{j,n,m,k})$, $1\leq i<j\leq n$, is a distribution-free PIs for $Y_{r,n^*,\widetilde{m}^*,k^*}$ $(1\leq r\leq n^*)$, whose coverage probability is free of the parent distribution F, given by:

$$p\left(X_{i,n,m,k}\leq Y_{r,n^*,\widetilde{m}^*,k^*}\leq X_{j,n,m,k}\right)=\phi_2\left(j,m;r,\widetilde{m}^*\right)-\phi_2\left(i,m;r,\widetilde{m}^*\right), \tag{34}$$

where ϕ_2 is given in (23).

Proof. By assumption that $\{Y_{r,n^*,\widetilde{m}^*,k^*},1\leq r\leq n^*\}$ are continuous r.v.'s, we have

$$\begin{aligned}\phi_3\left(i,\widetilde{m};r,m^*\right) &= P\left(X_{i,n,\widetilde{m},k}>Y_{r,n^*,m^*,k^*}\right)\\ &= \int_{-\infty}^{\infty}P\left(X_{i,n,\widetilde{m},k}>y\right)f_{Y_{r,n^*,m^*,k^*}}(y)dy. \end{aligned} \tag{36}$$

By using (5) and constructing the pdf of Y_{r,n^*,m^*,k^*} from (1), ϕ_3 take the form

$$\phi_3\left(i,\widetilde{m};r,m^*\right) = C(i;r)\sum_{\nu=1}^{i}\frac{a_\nu(i)}{\gamma_\nu}\zeta_\nu(i), \tag{37}$$

where

$$\zeta_\nu(i)=\begin{cases}\int_{-\infty}^{\infty}(\overline{F}(y))^{\gamma_\nu+\gamma^*_r-1}\left(\frac{1-\overline{F}^{m^*+1}(y)}{m^*+1}\right)^{r-1}f(y)dy, & m^*\neq -1,\\ \int_{-\infty}^{\infty}(\overline{F}(y))^{\gamma_\nu+k^*-1}\left(-\ln(\overline{F}(y))\right)^{r-1}f(y)dy, & m^*=-1.\end{cases} \tag{38}$$

Making the transformation $\overline{F}(y) = u$, we get

$$\zeta_\nu(i) = \begin{cases} \int_0^1 u^{\gamma_\nu+\gamma_r^*-1}\left(\frac{1-u^{m^*+1}}{m^*+1}\right)^{r-1} du, & m^* \neq -1, \\ \int_0^1 u^{\gamma_\nu+k^*-1}\left(-\ln(u)\right)^{r-1} du, & m^* = -1. \end{cases} \tag{39}$$

By solving the integrations (39), easily we obtain the required result.

Some special cases of ϕ_3:

• Prediction coefficient of future *oOSs* based on also *oOSs*, is given by:

$$\phi_3\left(i,\widetilde{0};r,0\right) = r(i!)\binom{n}{i}\binom{n^*}{r}\sum_{\nu=1}^{i}\frac{(-1)^{i-\nu}B(r,n-i+n^*-r+2)}{(\nu-1)!(i-\nu)!(n-\nu+1)}. \tag{40}$$

• Prediction coefficient of future *record* based on *oOSs*, is given by:

$$\phi_3\left(i,\widetilde{0};r,-1\right) = \binom{n}{i}\sum_{\nu=1}^{i}\frac{(-1)^{i-\nu}\nu\binom{i}{\nu}}{(n-\nu+1)(n-\nu+2)^r}. \tag{41}$$

• Prediction coefficient of future *gOSs* case I based on *oOSs*, is given by:

$$\phi_3\left(i,\widetilde{0};r,m^*\right) = \frac{n!}{(n-i)!(r-1)!}\sum_{\nu=1}^{i}\frac{a_\nu(i)}{n-\nu+1}\begin{cases} \frac{B(r,\frac{n-\nu+1+\gamma_r^* c_{r-1}^*}{m^*+1})}{(m^*+1)^r}, & m^* \neq -1, \\ \frac{(r-1)!}{(\frac{n-\nu+1}{k^*}+1)^r}, & m^* = -1. \end{cases} \tag{42}$$

• Prediction coefficient of future *oOSs* based on *gOSs* case II, is given by:

$$\phi_3\left(i,\widetilde{m};r,0\right) = r\binom{n^*}{r}c_{i-1}\sum_{\nu=1}^{i}\frac{a_\nu(i)}{\gamma_\nu}B(r,\gamma_\nu+n^*-r+1). \tag{43}$$

• Prediction coefficient of future *record* based on *gOSs* case II, is given by:

$$\phi_3\left(i,\widetilde{m};r,-1\right) = \frac{c_{i-1}}{(r-1)!}\sum_{\nu=1}^{i}\frac{a_\nu(i)}{\gamma_\nu}\frac{(r-1)!}{(\gamma_\nu+1)^r}. \tag{44}$$

Under assumptions of *theorem*3, then $(X_{i,n,\widetilde{m},k}, X_{j,n,\widetilde{m},k})$, $1 \le i < j \le n$, is a distribution-free PIs for $Y_{r,n^*,m^*,k^*}(1 \le r \le n^*)$, whose coverage probability is free of the parent distribution F, is given by:

$$p\left(X_{i,n,\widetilde{m},k} \le Y_{r,n^*,m^*,k^*} \le X_{j,n,\widetilde{m},k}\right) = \phi_3\left(j,\widetilde{m};r,m^*\right) - \phi_3\left(i,\widetilde{m};r,m^*\right). \tag{45}$$

where ϕ_3 is given in (**??**).

*Theorem*4. Let $\{X_{i,n,\widetilde{m},k}, 1 \le i \le n\}$ under assumption $\gamma_i \ne \gamma_j$, $i,j = 1,2,\ldots,n-1$ and $i \ne j$ and $\{Y_{r,n^*,\widetilde{m}^*,k^*}, 1 \le r \le n^*\}$ under assumption $\gamma_i^* \ne \gamma_j^*$, $i,j = 1,2,\ldots,n^*-1$ and $i \ne j$ be two independent *gOSs* from continuous cdf F. then $(-\infty, X_{i,n,\widetilde{m},k})$, $1 \le i \le n$, is distribution-free one-sided PI for the future $Y_{r,n^*,\widetilde{m}^*,k^*}$, with the corresponding prediction coefficient $\phi_4(i,\mu;r)$, that does not depend on the sampling distribution F, and is given by:

$$\phi_4(i,\widetilde{m};r,\widetilde{m}^*) = c_{i-1}c_{r-1}^* \sum_{\nu=1}^{i} \frac{a_\nu(i)}{\gamma_\nu} \sum_{\mu=1}^{r} \frac{a_\mu^*(r)}{\gamma_\nu+\gamma_\mu^*}. \quad (46)$$

Proof. Using assumptions, we found that

$$\begin{aligned}\phi_4(i,\widetilde{m};r,\widetilde{m}^*) &= P\left(X_{i,n,\widetilde{m},k} > Y_{r,n^*,\widetilde{m}^*,k^*}\right)\\ &= \int_{-\infty}^{\infty} P\left(X_{i,n,\widetilde{m},k} > y\right) f_{Y_{r,n^*,\widetilde{m}^*,k^*}}(y)dy. \quad (47)\end{aligned}$$

Using (4) and (5), ϕ_4 has the following form

$$\begin{aligned}\phi_4(i,\widetilde{m};r,\widetilde{m}^*) &= \int_{-\infty}^{\infty} c_{i-1}\sum_{\nu=1}^{i}\frac{a_\nu(i)}{\gamma_\nu}(1-F(y))^{\gamma_\nu} c_{r-1}^* f(y)\sum_{\mu=1}^{r} a_\mu^*(r)(1-F(y))^{\gamma_\mu^*-1}dy\\ &= c_{i-1}\sum_{\nu=1}^{i}\frac{a_\nu(i)}{\gamma_\nu}c_{r-1}^*\sum_{\mu=1}^{r}a_\mu^*(r)\int_{-\infty}^{\infty}(1-F(y))^{\gamma_\nu+\gamma_\mu^*-1}f(y)dy. \quad (48)\end{aligned}$$

Using the transformation $\overline{F}(y) = u$, and solving the integration, easily we obtain the required result. Based on ϕ_4 that given in (46) and under *theorem*4 assumptions, then $(X_{i,n,\widetilde{m},k}, X_{j,n,\widetilde{m},k})$, $1 \le i < j \le n$, is a distribution-free PIs for $Y_{r,n^*,\widetilde{m}^*,k^*}(1 \le r \le n^*)$, whose coverage probability is free of the parent distribution F, given by:

$$p\left(X_{i,n,\widetilde{m},k} \le Y_{r,n^*,\widetilde{m}^*,k^*} \le X_{j,n,\widetilde{m},k}\right) = \phi_4(j,\widetilde{m};r,\widetilde{m}^*) - \phi_4(i,\widetilde{m};r,\widetilde{m}^*), \quad (49)$$

4. Numerical Results

In section 3, distribution-free PIs for future *gOSs* based on also *gOSs* is constructed. To illustrate the productive prediction coefficient for some choices of i, j and r, and by using some different choices of γ_i and γ_r^* in table 1 to gaining the special schemes of the *gOSs*. Table 2 presents some values of the coverage probability $p\left(X_{i,20,m,k} \le Y_{r,n^*,m^*,k^*} \le X_{j,20,m,k}\right)$ of future, *oOSs* (by setting $\gamma_r^* = n^* - r + 1$) with $n^* = 25$ and *Krecord* (by setting $\gamma_r^* = K$) with $K = 1,2,3,4$ and 5, based on *oOSs* and *record* consecutively, such that $p\left(X_{i,n,m,k} \le Y_{r,n^*,m^*,k^*} \le X_{j,n,m,k}\right)$ does not depend on the parent distribution F, given by (13). Some values of the coverage probability $p\left(X_{i,n,m,k} \le Y_{r,20,\widetilde{m}^*,k^*} \le X_{j,n,m,k}\right)$ of future *oOSs*, *nonI*(by setting $\gamma_r = (n^*+0.9) - r + 1$), *Seque* (using $\alpha_r = r^2$), *Pfeif* (by setting $\gamma_r = r^2$), *PCOs* (using $r_1 = 5, n_1 = 5$), *Trunc* (using $\alpha_r = r^2, k_r = n^* - r$) and $K_n rec$ (using $\beta_r = r^2, k_r = r+2$), based on *oOSs* with $n = 30$ and *record* are presented respectively, in table 3, with the coverage probability that given by (34).

Table 4(a) presents some values of $p\left(X_{i,30,\widetilde{m},k} \le Y_{r,n^*,m^*,k^*} \le X_{j,30,\widetilde{m},k}\right)$ of future *oOSs* and *Krecord*, $K = 1,2,3,4$ and 5 based on *PCOs* (using $r_1 = 5, n_1 = 5$) and *Trunc* (using $\alpha_r = r^{-1}, k_r = n^* - r$), respectively. Under the same assumption, table 4(b) holds based on *Seque* (using $\alpha_r = 2r^{-1}$) and *Pfeif* (by setting $\gamma_r = r^2$), respectively. Table 5 presents some values of $p\left(X_{i,30,\widetilde{m},k} \le Y_{r,20,\widetilde{m}^*,k^*} \le X_{j,30,\widetilde{m},k}\right)$ based on *Seque* (using $\alpha_r = r^{-1}$), *Pfeif* (by setting $\gamma_r = n - r$) and *PCOs* (using $r_1 = 5, n_1 = 5$), respectively, the prediction coefficient of future *oOSs*, *nonI* (using $\alpha_r = (n^* - 0.1) - r + 1)$), *Seque* (using $\alpha_r = r^2$), *Pfeif* (by setting $\gamma_r = r^2$), *PCOs* (using $r_1 = 5, n_1 = 5$), *Trunc* (using $\alpha_r = r^2, k_r = n^* - r$) and $K_n rec$ (using $\beta_r = r^2, k_r = r+2$), respectively.

Under table 3 assumptions, figure 1 plots $p\left(X_{3,30,0,1} \le Y_{5,20,\widetilde{m}^*,k^*} \le X_{j,30,0,1}\right)$, which presents the coverage probability of future 5^{th} *gOSs* case II based on *oOSs*. Therefore, $(X_{3:30}, X_{j:30})$, $1 \le j \le 30$, is a distribution-free PIs for $Y_{5,20,\widetilde{m}^*,k^*}$. Under table 4(a) assumptions, figure 2 plots the coverage probability of future 5^{th} *gOSs* case I based on *PCOs*. Then, $(X_{1:25:30}, X_{j:25:30})$, $1 \le j \le 30$, is a distribution-free PIs for $Y_{5,20,m^*,k^*}$.

5. Conclusions

The prediction of unobserved statistics arises naturally in several real life situations. In This paper, nonparametric PIs for some statistics in a future unobserved *gOSs* based on *gOSs* from the same underlying distribution $F()$ are constructed. The proposed procedure can be extended to construct the outer and inner prediction intervals for future *gOSs* based on *gOSs*. The following conclusions are noted here:

◦ The prediction coefficient are decreasing with i and increasing with j, as it was expected.

◦ All prediction coefficients under the same assumptions are equivalent, for example

$$\phi_1(i,0;r;0) = \phi_2(i,0;r;\widetilde{0}) = \phi_3(i,\widetilde{0};r;0) = \phi_4(i,\widetilde{0};r;\widetilde{0}) = p\left(X_{i:n} \ge Y_{r:n^*}\right).$$

◦ The generality of our work enabled us to compare the values of different future sampling schemes at the same time, and choose the best one corresponding with the practical work.

◦ Under the same assumptions, the prediction coefficients of future *Krecord* are increasing with K.

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