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Research paper

Two fixed point theorems in generalized metric spaces

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Abstract

In this paper, we prove there exists a coupled fixed point for a set- valued contraction mapping defined on X× X , where X is incomplete ordered G-metric. Also, we prove the existence of a unique fixed point for single valued mapping with respect to implicit condition defined on a complete G- metric.

Keywords: *G-Metric Spaces; Fixed Points; Coupled Fixed Points; Implicit Conditions.*

1. Introduction

Nadler [1] initiated the study of fixed points for multi-valued contraction mappings and generalized the well-known Banach's fixed point principle. And then, many authors studied many fixed point results for multi-valued contraction mappings, see [2-3].
Mustafa and Sims [4] introduced the G-metric spaces as a generalization of the notion of metric spaces. Mustafa et al. [5- 6] obtained some fixed point theorems for mappings satisfying different contractive conditions. Abbas and Rhoades [7] initiated the study of common fixed point in G-metric spaces. While, Saadati et al. [8] studied some fixed point theorems in generalized partially ordered G-metric spaces.
Ran and Reurings [9] extended Banach's principle in partially ordered metric space to obtained some fixed point results for set valued mappings where the contraction condition is assumed only for the comparable elements of the partially ordered set.
In[10], Bhaskar and Lakshmikantham introduced the notions of mixed monotone property and coupled fixed point for the contractive mapping F:X×X→X, where X is a partially ordered metric space, and proved some coupled fixed point theorems for a mixed monotone operator. As an application of the coupled fixed point theorems, they determined the existence and uniqueness of the solution of a periodic boundary value problem. Recently, Lakshmikantham and Ćirić [11] have proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces.
In this paper, firstly, we prove coupled fixed point theorem for set-valued mappings by weaken the hypotheses and replace the competence by some other conditions depending on ordering. Secondly, we prove the existence of unique fixed point for single valued mappings in incomplete G- metric space by using implicit condition. Throughout this paper 2^X is the class of all non-empty subsets of X, CB(X) is the class of all closed and bounded subsets of X and "⤳ "denote to set-valued mappings.

2. Preliminaries

We begin with some basic definitions and facts.

Definition 2.1: [4] Let X be a non-empty set and G: X × X × X → [0, +∞) be a function for all x, y, z, a in X satisfying the following conditions:

1) $G(x,y,z) = 0 \Leftrightarrow x = y = z$
2) $0 < G(x,x,y)$ with $x \neq y$
3) $G(x,x,y) \leq G(x,y,z)$ with $y \neq z$
4) $G(x,y,z) = G(p(x, z, y))$, p(x, y, z) is a permutation of x, y, z
5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$.

Then the ordered pair (X, G) is called a generalized metric space or G- metric space.

Example 2.2: [12] Consider X= R, with usual distance $d(x, y) = |x - y|$, for all x, y in X. Define G: $X^3 \to R^+$ by

$$G(x,y,z) = d(x, y) + d(y, z) + d(z, x), \text{ for all } x, y, z \in X.$$

Then, X is a G-metric space.

Definition 2.3: *[4] Let (X, G) be a G- metric space. The sequence $\{x_n\}$ is called*

1) A G-Cauchy if, ∀ε > 0, there is k ∈ N such that for all positive integers n, m, l ≥ k, $G(x_n, x_m, x_l) < \varepsilon$.
2) A G-convergent to x ∈ X if, ∀ε > 0, there is k ∈N such that for all n, m ≥ k, $G(x, x_n, x_m) < \varepsilon$.

Also, (X, G) is said to be complete G-metric space if every G-Cauchy sequence in X is G- convergent in G.

A modification for the Husdorff distance is:
Definition 2.4: [13] Let X be a G-metric space. H is called the Hausdorff G- distance on CB(X), if

$H_G(A, B, C) = \max\{\sup_{x\in A} G(x, B,C), \sup_{x\in B} G(x, C, A), \sup_{x\in C} G(x, A, B)\}$,

Where
$G(x, B, C) = d_G(x, B) + d_G(B, C) + d_G(x, C)$,

$d_G(x, B) = \inf\{d_G(x, y), y \in B\}$,

$d_G(A, B) = \inf\{d_G(a, b), a \in A, b \in B\}$.

Lemma 2.5: *[13] If A, B ∈ CB(X) and a ∈ A, then for each ε > 0, there exists b ∈ B such that*

$$G(a, b, b) \le H_G(A, B, B) + \varepsilon$$

Lemma 2.6: *Let A ∈ CB(X) and B ∈ K(X) then for any a ∈ A, there is b∈ B such that:*

$$G(a, b, b) \le H_G(A, B, B).$$

Lemma 2.7: *Let X be a G-metric space, if A, B and C ∈ CB(X) with $H_G(A,B,C) \le \epsilon$, then for each a ∈A there exist elements b ∈ B, c ∈ C such that $G(a,b,c) \le \epsilon$.*

Lemma 2.8: *Let X be a G-metric space, and $\{A_n\}$ be sequence in CB(X) and $\lim_{n\to\infty} H_G(A_n, A, A) = 0$ for A ∈ CB(X). If $x_n \in A_n$ and $\lim_{n\to\infty} G(x_n, x, x) = 0$, then x ∈ A.*

Proof: It is enough to prove that $G(x_n, A_n, A_n) \to G(x, A, A)$.

Definition 2.9: *[14] Let (X , ≼) be a partially ordered set . Then x , y ∈ X are called comparable if x ≼ y or y ≼ x holds .*

Definition 2.10: *[13] The point x in X is called a fixed point of the multivalued mapping T: X⇝2^X if x ∈ Tx and x is fixed point of a single mapping T: X⟶ X if x = Tx.*

Definition 2.11: *An element (x, y) ∈ X× X is called a coupled fixed point of mapping T: X×X ⇝ 2^X if x ∈ T(x, y) and y ∈ T(y, x) .*

The following lemmas are needed:

3. Coupled fixed point

Theorem 3.1: Let X is ordered G-metric space and let T: X × X ⇝ K (X) satisfying:

1) There exists k ∈ (0,1) with

$$H_G(T(x, y), T(u, v), T(w, z)) \le \frac{k}{2} G((x, y), (u, v), (w, z))$$

For all (w, z) ≼ (u, v) ≼(x, y).

2) Let $u_1 \in T(x_1, y_1)$, $v_1 \in T(y_1, x_1)$ If $x_2 \preccurlyeq x_3$, $y_3 \preccurlyeq y_2$, x_i , $y_i \in X$ (i = 2,3) then for all $u_2 \in T(x_2, y_2)$ there exists $u_3 \in T(x_3, y_3)$ with $u_2 \preccurlyeq u_3$,and for all $v_1 \in T(y_1, x_1)$ there exists $v_2 \in T(y_2, x_2)$ with $v_3 \preccurlyeq v_2$ provided $G((u_1, v_1), (u_2, v_2), (u_3, v_3)) < 1$
3) There exist $x_0, y_0 \in X$, and some $x_1 \in T(x_0, y_0)$, $y_1 \in T(y_0,x_0)$, $x_2 \in T(x_1, y_1)$, $y_1 \in T(y_0,x_0)$ with $x_0 \preccurlyeq x_1 \preccurlyeq x_2$, $y_2 \preccurlyeq y_1 \preccurlyeq y_0$ such that $G((x_0, y_0), (x_1, y_1), (x_2, y_2)) < 1-k$, wherek ∈ (0,1).
4) If a non-decreasing sequence $x_n \to x$ in X then $x_n \preccurlyeq x$, for all n and if a non-increasing sequence $y_n \to y$ in X , then $y \preccurlyeq y_n$, for all n

Then T has a coupled fixed point.

Proof: Let $x_0, y_0 \in X$ then by hypotheses (3) there exists $x_1 \in T(x_0, y_0)$, $y_1 \in T(y_0, x_0)$, $x_2 \in T(x_1, y_1)$, $y_2 \in T(y_1,x_1)$ with $x_0 \preccurlyeq x_1 \preccurlyeq x_2$, $y_2 \preccurlyeq y_1 \preccurlyeq y_0$ such that

$$G((x_0, y_0), (x_1,y_1), (x_2, y_2)) < 1- k. \quad \ldots \quad (3.1)$$

Since $(x_0,y_0) \preccurlyeq (x_1, y_1) \preccurlyeq (x_2, y_2)$.By using assumption (1) and (3.1), we have

$$H_G(T(x_0, y_0), T(x_1, y_1), T(x_2, y_2)) \le \frac{k}{2} G((x_0, y_0), (x_1, y_1), (x_2, y_2))$$

$$< \frac{k}{2}(1- k)$$

And similarly

$$H_G(T(y_0, x_0), T(y_1, x_1), T(y_2, x_2)) < \frac{k}{2}(1- k)$$

Using assumption (2) and lemma (2.7), there exist $x_3 \in T(x_2, y_2)$, $y_3 \in T(y_2, x_2)$ with $x_2 \preccurlyeq x_3$ and $y_3 \preccurlyeq y_2$ such that

$$G(x_1, x_2, x_3) \le \frac{k}{2}(1- k) \quad \ldots \quad (3.2)$$

and

$$G(y_1, y_2, y_3) \le \frac{k}{2}(1- k) \quad \ldots \quad (3.3)$$

From (3.2) and (3.3)

$$G((x_1, y_1), (x_2, y_2), (x_3, y_3)) \le k(1- k) \quad \ldots \quad (3.4)$$

Again by assumption (1) and (3.4), we have

$$H_G(T(x_1, y_1), T(x_2, y_2), T(x_3, y_3)) \le \frac{k^2}{2}(1- k)$$

and

$$H_G(T(y_1, x_1), T(y_2, x_2), T(y_3, x_3)) \le \frac{k^2}{2}(1- k)$$

Further from lemma (2.7) and assumption (2), there exist $x_4 \in T(x_3, y_3)$, $y_4 \in T(y_3, x_3)$ with $x_3 \preccurlyeq x_4$, $y_4 \preccurlyeq y_3$ such that

$$G(x_2, x_3, x_4) \le \frac{k^2}{2}(1- k).$$

and

$$G(y_2, y_3, y_4) \le \frac{k^2}{2}(1- k).$$

It follows that

$$G((x_2, y_2), (x_3, y_3), (x_4, y_4)) \le k^2(1- k).$$

Continue in this way, we obtain $x_{n+2} \in T(x_{n+1}, y_{n+1})$, $y_{n+2} \in T(y_{n+1}, x_{n+1})$ with $x_{n+1} \preccurlyeq x_{n+2}$, $y_{n+2} \preccurlyeq y_{n+1}$ such that

$$G(x_n, x_{n+1}, x_{n+2}) \le \frac{k^n}{2}(1- k)$$

and

$$G(y_n, y_{n+1}, y_{n+2}) \le \frac{k^n}{2}(1- k)$$

thus

$$G((x_n, y_n), (x_{n+1}, y_{n+1}), (x_{n+2}, y_{n+2})) \le k^n(1- k) \ldots \quad (3.5)$$

Next, we show that $\{x_n\}$ is a G-Cauchy sequence in X. Let m > n. Then

$G(x_n, x_{n+1}, x_m) \le G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+1}, x_m) \le G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_m)$

[by using definition (2.1-3)]

$$\le G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+2}, x_m)$$

[by using definition (2.1-5)]

$$\le G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + G(x_{n+2}, x_{n+3}, x_m)$$

[by using definition (2.1-3)]

$$\le G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_m) + G(x_n, x_{n+1}, x_{n+1}) + \ldots + G(x_{m-2}, x_{m-1}, x_m)$$

$\leq [\, k^n + k^{n+1} + \ldots + k^{m-2}\,] \frac{(1-k)}{2}$

$= k^n [1 + k + \ldots + k^{m-n-2}\,] \frac{(1-k)}{2}$

$= k^n \left[\frac{(1-k^{m-n-1})}{1-k}\right] \frac{(1-k)}{2}$

$= \frac{k^n}{2} (1 - k^{m-n-1}) < \frac{k^n}{2}$,

Since $k \in (0, 1)$, $1 - k^{m-n-1} < 1$, we obtain $G(x_n, x_{n+1}, x_m) \to 0$ as $n \to \infty$. This implies that $\{x_n\}$ is a G-Cauchy sequence and hence converges to some point x in the complete in X.

Similarly, we can show that $\{y_n\}$ is also a G-Cauchy sequence in X.By completeness of X, there exist x, y ∈ X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Finally, we show that x ∈ T(x, y) and y ∈ T(y, x).

Since $\{x_n\}$ is a non-decreasing sequence and $\{y_n\}$ is a non-increasing sequence in X such that $x_n \to x$ and $y_n \to y$, therefore we have $x_n \preccurlyeq x$, $y \preccurlyeq y_n$ for all n.

From hypothesis (1) it follows that

$H_G(T(x_n, y_n), T(x, y), T(x, y)) \leq k\, G((x_n, y_n), (x, y), (x, y)) \to 0.$

Now, since $x_{n+1} \in T(x_n, y_n)$ and $\lim_{n\to\infty} G(x_{n+1}, x, x) = 0$, it then by using lemma (2.8) follows that x ∈ T(x, y). Again by assumption (1)

$H_G(T(y_n, x_n), T(y, x), T(y, x)) \leq k\, G((y_n, x_n), (y, x), (y, x)) \to 0.$

Since $y_{n+1} \in T(y_n, x_n)$ and $\lim_{n\to\infty} G(y_{n+1}, y, y) = 0$, it then by using lemma (2.8) y ∈ T(y, x). Hence (x, y) is a coupled fixed point of T. ■

Corollary 3.2: Let X be an ordered G-metric space and let T: X ×X ⤳ K(X) satisfying:

1) There exists k ∈ (0,1) with

$$H_G(T(x, y), T(u, v), T(u, v)) \leq \frac{k}{2} G((x,y), (u,v), (u,v))$$

for all $(u, v) \preccurlyeq (x, y)$.

2) If $x_1 \preccurlyeq x_2$, $y_2 \preccurlyeq y_1$, x_i , $y_i \in X$ (i = 1,2) then for all $u_1 \in T(x_1, y_1)$ there exists $u_2 \in T(x_2, y_2)$ with $u_1 \preccurlyeq u_2$, and for all $v_1 \in T(y_1, x_1)$ there exists $v_2 \in T(y_2, x_2)$ with $v_2 \preccurlyeq v_1$ provided $G((u_1, v_1), (u_2, v_2), (u_2, v_2)) < 1$
3) There exists x_0, $y_0 \in X$, and some $x_1 \in T(x_0, y_0)$, $y_1 \in T(y_0, x_0)$ with $x_0 \preccurlyeq x_1$, $y_1 \preccurlyeq y_0$ such that $G((x_0, y_0), (x_1, y_1), (x_1, y_1)) < 1 - k$, where k ∈ (0,1).
4) If a non-decreasing sequence $x_n \to x$ in X , then $x_n \preccurlyeq x$, for all n and if a non-increasing sequence $y_n \to y$ in X , then $y \preccurlyeq y_n$, for all n

Then T has a coupled fixed point.

As a consequence the above theorem also true for single valued mappings.

4. Implicit condition

For a nonempty subset S of the G-metric space X , the diameter of S is defined as

$$\delta_G(S) = \sup \{G(x, y, z) : x, y, z \in S\}.$$

If $\{y_n\}$ is a bounded sequence in G-metric space X, let $j_n = \delta_G(\{y_n, y_{n+1}, y_{n+2}, \ldots\})$ for n ∈ N. Then $j_n < \infty$ .for all n ∈ N, and $\{j_n\}$ is non-increasing and $j_n \geq 0$ for all n ∈ N, and so there exists an j ≥ 0 such that $\lim_{n\to\infty} j_n = j$.

Theorem 4.1: *Let X be a complete bounded G-metric space and T : X⟶ X be a mapping such that for all x, y, z ∈ X ,*

$$\phi(G(x,y,z), G(x,Tx,z), G(y,Ty,z), G(x,Ty,z), G(y,Tx,z), G(Tx,Ty,Tz)) \geq 0 \quad \ldots.. \quad (4.1)$$

and ϕ : $R^5_+ \times R+ \to R+$ be upper - semi continuous and be non-decreasing on R^5_+ $\phi((u, u, u, u, u), v) \geq 0$ implies $v \leq \psi(u)$,where ψ : R+ ⟶ R+ is a non-decreasing upper semi- continuous function with $\psi(0) = 0$ and $\psi(t) < t$ for t > 0,Then T has a unique fixed point p in X and T is continuous at p .

Proof: Suppose that x_0 in X and $x_{n+1} = Tx_n$. Then the orbit $\{x_n\}$ is bounded.

Let

$$j_n = \delta_G(\{x_n, x_{n+1}, x_{n+2}, \ldots\}),\ n \in N.$$

from, the above remark $\lim_{n\to\infty} j_n = j$ for some j ≥ 0. If $x_{n+1} = x_n$ for some n ∈ N , then T has a fixed point, say p ∈ X .

Assume that $x_{n+1} \neq x_n$ for each n ∈ N . Let k ∈ N be fixed Taking $x = x_{n-1}$, $y = x_{n+m-1}$ and $z = x_{n+m+v-1}$ in (4.1) where n ≥ k and m , v ∈ N, we have

$\phi(G(x_{n-1}, x_{n+m-1}, x_{n+m+v-1}), G(x_{n-1}, Tx_{n-1}, x_{n+m+v-1}), G(x_{n+m-1}, Tx_{n+m-1}, x_{n+m+v-1}), G(x_{n-1}, Tx_{n+m-1}, x_{n+m+v-1}), G(x_{n+m-1}, Tx_{n-1}, x_{n+m+v-1}), G(Tx_{n-1}, Tx_{n+m-1}, Tx_{n+m+v-1}))$

$= \phi(G(x_{n-1}, x_{n+m-1}, x_{n+m+v-1}), G(x_{n-1}, x_n, x_{n+m+v-1}), G(x_{n+m-1}, x_{n+m}, x_{n+m+v-1}), G(x_{n-1}, x_{n+m}, x_{n+m+v-1}), G(x_{n+m-1}, x_n, x_{n+m+v-1}), G(x_n, x_{n+m}, x_{n+m+v})) \geq 0$

Thus we have

$$\phi((j_{n-1}, j_{n-1}, j_{n+m-1}, j_{n-1}, j_{n+m-1}), G(x_n, x_{n+m}, x_{n+m+v})) \geq 0.$$

Since ϕ is non-decreasing on R^5_+ and $\{j_n\}$ is non-increasing, we have

$$\phi((j_{k-1}, j_{k-1}, j_{k-1}, j_{k-1}, j_{k-1}), G(x_n, x_{n+m}, x_{n+m+v})) \geq 0,$$

which implies

$$G(x_n, x_{n+m}, x_{n+m+v}) \leq \psi(j_{k-1}).$$

Taking limit sup over n ≥ k, we have $j_k \leq \psi(j_{k-1})$. Letting k⟶∞, we get $j \leq \psi(j)$.

If j > 0, then $j \leq \psi(j) < j$, which is a contradiction. Thus j = 0 and hence $\lim_{n\to\infty} j_n = 0$.

Thus given $\epsilon > 0$, there exists C ∈ N such that $j_c < \epsilon$.

Then we have for n ≥ C and m, v ∈ N , $G(x_n, x_{n+m}, x_{n+m+v}) < \epsilon$.

Therefore, $\{x_n\}$ is a G-Cauchy sequence in X. By the completeness of X, there exists a p ∈ X such that $\lim_{n\to\infty} x_n = p$.

Hence $\lim_{n\to\infty} Tx_n = p$.

Taking $x = x_{n-1}$, $y = x_{n+m-1}$ and z = p in (4.1) , we have

$\phi(G(x_{n-1}, x_{n+m-1}, p), G(x_{n-1}, Tx_{n-1}, p), G(x_{n+m-1}, Tx_{n+m-1}, p), G(x_{n-1}, Tx_{n+m-1}, p), G(x_{n+m-1}, Tx_{n-1}, p), G(Tx_{n-1}, Tx_{n+m-1}, Tp))$

$= \phi(G(x_{n-1}, x_{n+m-1}, p), G(x_{n-1}, x_n, p), G(x_{n+m-1}, x_{n+m}, p), G(x_{n-1}, x_{n+m}, p), G(x_{n+m-1}, x_n, p), G(x_n, x_{n+m}, Tp)) \geq 0.$

Taking limit n⟶∞, we have

$$\phi\, G(p,p,p), G(p,p,p), G(p,p,p), G(p,p,p), G(p,p,p), G(p,p,Tp)) \geq 0$$

Which implies $G(p, p, Tp) \leq \psi(G(p, p, p)) = \psi(0) = 0$. Hence Tp = p.

For the uniqueness, let p and w be fixed points of T.

Taking x = p, y = p and z = w in (4.1), we have

ϕ $(G(p,p,w), G(p,Tp,w), G$ $(p,Tp,w), G(p,Tp,w), G(p,Tp,w)$
$, G(Tp,Tp,Tw))$
$= \phi$ $(G(p,p\ ,w), G(p,p\ ,w), G(p,p\ ,w), G(p,p\ ,w), G(p,p\ ,w),$
$G(p,p,w))\ \geq 0$

Which implies $G(p,p,w) \leq \psi$ $(G(p,p\ ,w)) < G(p,p,w)$ which is a contradiction.
Thus, we have p = w.
Now, we show that T is continuous at p.
Suppose that $\{y_n\}$ be a sequence in X and $\lim_{n\to\infty} y_n$ = p. Taking x = p, y = p and z = y_n in (4.1), we have

ϕ $(G(p,p,y_n)$ $,G(p,Tp,y_n)$ $,G(p,Tp,y_n)$ $,G(p,Tp,y_n)$ $,G(p\ ,Tp,y_n),$
$G(Tp,Tp,Ty_n))$
$= \phi$ $(G(p,p,y_n),G(p,p,y_n),G(p,p,y_n),G(p,p,y_n),G(p,p,y_n)\ ,$
$G(p,p,Ty_n)) \geq 0$

Which implies $G(p,p,Ty_n) \leq \psi$ $(G(p,p,y_n))$.
Taking limit sup, we have

$\lim_{n\to\infty} G(p,p,Ty_n) \leq \lim_{n\to\infty} \psi$ $(G(p,p,y_n)) \leq \psi$ $(0) = 0$.

Hence $\lim_{n\to\infty} Ty_n$= p = Tp and hence T is continuous at p.

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