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Research Paper

Bayesian estimation of the parameters of the bivariate generalized exponential distribution using accelerated life testing under censoring data

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Abstract

In this paper, the Bayesian estimation for the unknown parameters for the bivariate generalized exponential (BVGE) distribution under Bivariate censoring type-I samples with constant stress accelerated life testing (CSALT) are discussed. The scale parameter of the lifetime distribution at constant stress levels is assumed to be an inverse power law function of the stress level. The parameters are estimated by Bayesian approach using Markov Chain Monte Carlo (MCMC) method based on Gibbs sampling. Then, the numerical studies are introduced to illustrate the approach study using samples which have been generated from the BVGE distribution.

Keywords: *Accelerated Life Testing; Bivariate Generalized Exponential Distribution; Constant Stress; Bivariate Censoring Type-I Samples; Bayesian Estimation; Markov Chain Monte Carlo Method.*

1. Introduction

Many times the lifetime data of interest is bivariate in nature. Any study on twins or on failure data recorded twice at the same system naturally leads to bivariate data. Paired data can be also consisted of blindness in the left /right eye, failure time of the left /right kidney or age at death of parent /child in a genetic study. For example, Houggard et al. [14] studied data on life length of Danish twins and Lin et al. [23] considered data of colon cancer and the time from treatment to death (see Attia et al. [10]).
The BVGE distribution provides a very good fit; Meintanis [25] analyzed one data and concluded that bivariate Marshall and Olkin exponential distribution provided a very good fit. Also Kundu and Gupta [19] re-analyzed the same data set and they are observed that the proposed BVGE distribution provides a much better fit than the Marshal and Olkin bivariate exponential model.
The BVGE distribution is occurring in practice, it presents in some models as follows:
Maintenance Model: Suppose a system has two components and it is assumed that each component has been maintained independently and also there is an overall maintenance. Due to component maintenance, suppose the lifetime of the individual is increased by U_i amount and because of the overall maintenance, the lifetime of each component is increased by U_3 amount. Therefore, the increased lifetimes of the two components are $X_1 = max(U_1\,,\,U_3)$ and $X_2 = max(U_2\,,\,U_3)$ respectively.
Stress Model: Suppose a system has two components. Each component is subject to individual independent stress say U_1 and U_2 respectively. The system has an overall stress U_3 which has been transmitted to both the components equally, independent of their individual stresses. Therefore, the observed stress at the two components are $X_1 = max(U_1\,,\,U_3)$ and $X_2 = max(U_2\,,\,U_3)$ respectively.
Shock model: Suppose there are three independent sources of shocks; say 1,2 and 3. Suppose these shocks are affecting a system with two components, say 1 and 2. It is assumed that the shock from source 1 reaches the system and destroys component 1 immediately, the shock from source 2 reaches the system and destroys component 2 immediately, while if the shock from source 3 hits the system it destroys both components immediately. Let U_i denote the inter-interval times

between the shocks in source i, $i = 1, 2, 3$, which follow the distribution GE. If X_1, X_2 denote the survival times of the components, then (X_1, X_2) follows the BVGE model.

Competing risks model: Assume a system has two components, labeled 1 and 2, and the survival time of component i is denoted by X_i , $i = 1,2.$ It is considered that there are three independent causes of failures, which may affect the system. Only component 1 can fail due to cause 1, and similarly only component 2 can fail due to cause 2, while both the components can fail at the same time due to cause 3. Let U_i be the life time of cause i, $i = 1, 2, 3.$ If U_1, U_2 and U_3 follow a GE distribution, then (X_1, X_2) follow the BVGE model (see Marshall and Olkin [24] and Lakshmi and Durgadevi [22]).

Kundu and Gupta (2009) defined a BVGE distribution and they used the method of maximum likelihood to estimate the four unknown parameters of the BVGE distribution from complete samples. Ashour et al. [5] provided the joint and marginal moments and the joint and marginal moment generating function for the BVGE distribution. Ashour et al. [6] estimated the unknown parameters of the BVGE distribution from censoring type-I samples with random censoring samples using the method of maximum likelihood. Lakshmi and Durgadevi [22] presented a study in the application part for the BVGE distribution. Assar and Abd El-Maseh [7] presented maximum likelihood estimation for the unknown parameters of the BVGE distribution under type-I censoring samples with accelerated life testing.

Accelerated life testing (ALT) is becoming so important and widely used in many fields, such as: in manufacturing industries to assess or demonstrate component and subsystem reliability, in rapidly changing technologies, higher customer expectations for better reliability and the need for rapid device development (see Attia et al. [8]).

There are cases where the reliability of component is high and failure data of the component when operating at normal conditions may not be attainable during its expected life. In such cases, accelerated life testing induces failures, and the failure data at accelerated conditions are used to estimate the reliability at normal operating conditions (see Elsayed [12]).

The most common stress loading is constant stress, the stress is kept at a constant level of stress throughout the life of the test, that is, each unit is run at a constant high stress level until the occurrence of failure or the observation is censoring. Practically, most devices such as lamps, semiconductors, and microelectronics are run at a constant stress (see Nelson [27]).

The inverse power law model is often used to relate product life to pressure like stresses (e.g., voltage) and it is used in this paper (see Elsayed [12]).

There are several mechanisms that can lead to censoring samples. If the test is terminated after a pre-determined time of censoring then it is said to be type-I censoring and if the test is terminated after a pre-determined number of failures then it is said to be type-II censoring. In this paper, it is used the bivariate censoring type-I samples (see Hanagal [13]).

In Bayesian analysis, our information, our belief or our knowledge about the unknown parameters can be incorporating in a measurable form as a prior distribution. The density $\pi(\theta)$ is called the prior distribution of Θ and the $x_1, \cdots, x_n$ be a random sample of size n. The conditional density of Θ given $X_1 = x_1, \cdots, X_n = x_n$, denoted by $H(\theta|x_1, \cdots, x_n)$, is called the posterior distribution of Θ as follows (see Mood et al. [26]):

$$H(\theta|x_1, \cdots, x_n) = \frac{L(\theta)\pi(\theta)}{\int_{\Theta} L(\theta)\pi(\theta).d\theta}, \text{ Where } L(\theta) = \prod_{i=1}^{n} f(x_i; \theta).$$

Ibrahim et al. [16] described that in most models and application, the quantity $m(x) = \int_{\theta} L(\theta|x_1, \cdots, x_n)\, \pi(\theta)\, d\theta$ does not have an analytic closed form, and therefore $H(\theta|x_1, \cdots, x_n)$ does not have a closed form. However, there are several computational method for sampling from $H(\theta|x_1, \cdots, x_n)$. Perhaps one of the most popular of these methods is called the Gibbs sampler.

The Gibbs sampler may be one of the best known MCMC sampling algorithms in the Bayesian computational literature, and it is used in this paper. The Gibbs sampler is Monte Carlo based sampling methods for evaluating high-dimensional posterior integral, and it is very powerful simulation algorithm that allows us to sample from $H(\theta|x_1, \cdots, x_n)$ without knowing $m(x)$.

There are many works presented a Bayesian approach using MCMC method, for example, Aly and Bleed [4] presented Bayesian analysis for generalized Logistic distribution with CSALT under type-II censoring. Attia et al. [11] considered the Bayes estimators for Birnbaum-Saunders distribution with CSALT under type-I censoring. Zhou et al. [31] introduced Bayesian estimation for Log-binomial model using MCMC via slice sampling to simulate from the posterior distributions. There are many authors discussed Bayesian approach for bivariate distributions, such as, Kim and Park [17], Achcar [2], and Kundu and Gupta [20].

This paper is organized as follows: The general model for BVGE distribution is described in Section (2). Section (3) presents accelerated test model. Section (4) introduces Bayesian estimation for the unknown parameters. Section (5) presents a numerical example and OpenBUGS software is used for implementing MCMC simulation. Section (6) presents a conclusion.

2. The general model for BVGE distribution

The BVGE distribution has the probability density function (pdf) with the shape parameters $\alpha_1, \alpha_2, \alpha_3 > 0$ and scale parameter $\lambda > 0$ as follows:

$$f(x,y) = \begin{cases} f_1(x,y) \; if \; 0 < x < y < \infty, \\ f_2(x,y) \; if \; 0 < y < x < \infty, \\ f_3(x) \; if \; 0 < x = y < \infty, \end{cases} \quad (1)$$

Where

$$f_1(x,y) = f_{GE}(x;\alpha_1+\alpha_3,\lambda) f_{GE}(y;\alpha_2,\lambda)$$

$$= (\alpha_1+\alpha_3)\alpha_2\lambda^2(1-e^{-\lambda x})^{\alpha_1+\alpha_3-1}(1-e^{-\lambda y})^{\alpha_2-1}e^{-\lambda(x+y)},$$

$$f_2(x,y) = f_{GE}(x;\alpha_1,\lambda) f_{GE}(y;\alpha_2+\alpha_3,\lambda)$$

$$= (\alpha_2+\alpha_3)\alpha_1\lambda^2(1-e^{-\lambda x})^{\alpha_1-1}(1-e^{-\lambda y})^{\alpha_2+\alpha_3-1}e^{-\lambda(x+y)},$$

And

$$f_3(x) = \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{GE}(x;\alpha_1+\alpha_2+\alpha_3,\lambda)$$

$$= \alpha_3\lambda(1-e^{-\lambda x})^{\alpha_1+\alpha_2+\alpha_3-1}e^{-\lambda x}.$$

Where

$f_{GE}(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}$ is the pdf of the GE distribution and the BVGE distribution will be denoted by BVGE $(\alpha_1, \alpha_2, \alpha_3, \lambda)$.
Note that: Kundu and Gupta [19] provided a special case of the PDF for BVGE distribution with $\lambda = 1$ and denoted that the results are true for general λ also.

3. Accelerated test model

The assumptions of accelerated life test are assumed to be as follows:

i) There are k levels of high stress V_j , $j = 1, \dots, k$ in the experiment and V_u is the stress under usual conditions, where $V_u < V_1 < \dots < V_k$.
ii) There are N bivariate observations (x_{ij}, y_{ij}) under study and it divided into n_j bivariate observations for each level of stress V_j, where $i = 1,2, \dots, n_j$ and $j = 1, 2, \dots, k$.
iii) Each n_j bivariate observations (x_{ij}, y_{ij}) in the experiment is run at a pre-specified constant stress V_j .
iv) The i-th pair of the components with life-time (x_{ij}, y_{ij}) have a censoring time t_{ij} and the experiment is terminated at a pre-specified censoring time t_{ij}, $i = 1,2, \dots, n_j$ and $j = 1, 2, \dots, k$.
v) It is assumed that the stress V_j , $j = 1, 2, \dots, k$ affects only the scale parameter λ_j of the BVGE distribution through a certain acceleration function.
vi) By using equation (1), the BVGE distribution under CSALT with scale parameter λ_j has the joint pdf as follows:

$$f(x_{ij}, y_{ij}) = \begin{cases} f_1(x_{ij}, y_{ij}) \; if \; 0 < x_{ij} < y_{ij} < \infty, \\ f_2(x_{ij}, y_{ij}) \; if \; 0 < y_{ij} < x_{ij} < \infty, \\ f_3(x_{ij}) \; if \; 0 < x_{ij} = y_{ij} < \infty, \end{cases} \quad (2)$$

Where

$$f_1(x_{ij}, y_{ij}) = f_{GE}(x_{ij};\alpha_1+\alpha_3,\lambda_j) f_{GE}(y_{ij};\alpha_2,\lambda_j)$$

$$= (\alpha_1+\alpha_3)\alpha_2\lambda_j^2 \left(1-e^{-\lambda_j x_{ij}}\right)^{\alpha_1+\alpha_3-1}\left(1-e^{-\lambda_j y_{ij}}\right)^{\alpha_2-1}e^{-\lambda_j(x_{ij}+y_{ij})},$$

$$f_2(x_{ij}, y_{ij}) = f_{GE}(x_{ij}; \alpha_1, \lambda_j) f_{GE}(y_{ij}; \alpha_2 + \alpha_3, \lambda_j)$$

$$= (\alpha_2 + \alpha_3)\alpha_1 \lambda_j^2 \left(1 - e^{-\lambda_j x_{ij}}\right)^{\alpha_1 - 1} \left(1 - e^{-\lambda_j y_{ij}}\right)^{\alpha_2 + \alpha_3 - 1} e^{-\lambda_j (x_{ij} + y_{ij})},$$

And

$$f_3(x_{ij}) = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_{GE}(x_{ij}; \alpha_1 + \alpha_2 + \alpha_3, \lambda_j)$$

$$= \alpha_3 \lambda_j \left(1 - e^{-\lambda_j x_{ij}}\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} e^{-\lambda_j x_{ij}}.$$

vii) The scale parameter λ_j of the underlying life time distribution (2) have an inverse power law function on stress levels where $\lambda_j = CS_j^P$. (See Singpurwalla [28], Abdel-Ghaly et al. [1], Attia et al. [8] and Attia et al. [9]).

4. Bayesian estimation

In order to obtain the Bayesian estimators of the unknown parameters, it is necessary to obtain the likelihood function for the model. Considering the assumptions in section (2), the likelihood function under bivariate censoring type-I samples with CSALT of the sample size N bivariate observations is given by:

$$L(\alpha_1, \alpha_2, \alpha_3, C, P) = \prod_{j=1}^{k} \{ [\prod_{i=1}^{m_{1j}} f_{A_1}(x_{ij}, y_{ij}) \overline{G_A}(t_{ij})] [\prod_{i=1}^{m_{2j}} f_{A_2}(x_{ij}, y_{ij}) \overline{G_A}(t_{ij})] [\prod_{i=1}^{m_{3j}} f_{A_3}(x_{ij}, y_{ij}) \overline{G_A}(t_{ij})]$$

$$\cdot [\prod_{i=1}^{m_{4j}} f_{A_4}(x_{ij}, t_{ij}) g_A(t_{ij})] [\prod_{i=1}^{m_{5j}} f_{A_5}(t_{ij}, y_{ij}) g_A(t_{ij})] [\prod_{i=1}^{m_{6j}} \overline{F_A}(t_{ij}, t_{ij}) g_A(t_{ij})] \}. \quad (3)$$

Where

$$g_A(t_{ij}) = CS_j^P e^{-CS_j^P t_{ij}}; t_{ij} > 0, C, P > 0.$$

$$\overline{G_A}(t_{ij}) = P[T_{ij} > max(x_{ij}, y_{ij})]$$

$$= exp[-CS_j^P max(x_{ij}, y_{ij})],$$

$$f_{A_1}(x_{ij}, y_{ij}) = f_{GE}(x_{ij}; \alpha_1 + \alpha_3, C, P) f_{GE}(y_{ij}; \alpha_2, C, P)$$

$$= (\alpha_1 + \alpha_3)\alpha_2 C^2 S_j^{2P} \left(1 - e^{-CS_j^P x_{ij}}\right)^{\alpha_1 + \alpha_3 - 1} \left(1 - e^{-CS_j^P y_{ij}}\right)^{\alpha_2 - 1} e^{-CS_j^P (x_{ij} + y_{ij})},$$

$$f_{A_2}(x_{ij}, y_{ij}) = f_{GE}(x_{ij}; \alpha_1, C, P) f_{GE}(y_{ij}; \alpha_2 + \alpha_3, C, P)$$

$$= (\alpha_2 + \alpha_3)\alpha_1 C^2 S_j^{2P} \left(1 - e^{-CS_j^P x_{ij}}\right)^{\alpha_1 - 1} \left(1 - e^{-CS_j^P y_{ij}}\right)^{\alpha_2 + \alpha_3 - 1} e^{-CS_j^P (x_{ij} + y_{ij})},$$

$$f_{A_3}(x_{ij}) = \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} f_{GE}(x_{ij}; \alpha_1 + \alpha_2 + \alpha_3, C, P)$$

$$= \alpha_3 CS_j^P \left(1 - e^{-CS_j^P x_{ij}}\right)^{\alpha_1 + \alpha_2 + \alpha_3 - 1} e^{-CS_j^P x_{ij}},$$

$$f_{A_4}(x_{ij}, t_{ij}) = lim_{\delta x_{ij} \to 0} \frac{P(x_{ij} < X_{ij} < x_{ij} + \delta x_{ij} | Y_{ij} > t_{ij}) P(Y_{ij} > t_{ij})}{\delta x_{ij}}$$

$$= CS_j^P (\alpha_1 + \alpha_3) e^{-CS_j^P x_{ij}} \left(1 - e^{-CS_j^P x_{ij}}\right)^{\alpha_1 + \alpha_3 - 1} \left[1 - \left(1 - e^{-CS_j^P t_{ij}}\right)^{\alpha_2}\right], \quad 0 < x_{ij} < t_{ij} \text{ and } C, P > 0,$$

$$f_{A_5}(t_{ij}, y_{ij}) = lim_{\delta y_{ij} \to 0} \frac{P(y_{ij} < Y_{ij} < y_{ij} + \delta y_{ij} | X_{ij} > t_{ij}) P(X_{ij} > t_{ij})}{\delta y_{ij}}$$

$$= CS_j^P (\alpha_2 + \alpha_3) e^{-CS_j^P y_{ij}} \left(1 - e^{-CS_j^P y_{ij}}\right)^{\alpha_2 + \alpha_3 - 1} \left[1 - \left(1 - e^{-CS_j^P t_{ij}}\right)^{\alpha_1}\right], \quad 0 < y_{ij} < t_{ij} \text{ And } C, P > 0,$$

$$\bar{F}(t_{ij},t_{ij}) = P[X_{ij} > t_{ij}, Y_{ij} > t_{ij}]$$

$$= 1 - \left(1 - e^{-CS_j^P t_{ij}}\right)^{\alpha_1+\alpha_2+\alpha_3},$$

And $f_A(x_{ij}, y_{ij})$ and $g_A(t_{ij})$ are the joint pdf with accelerated life testing of (X_{ij}, Y_{ij}) and T_{ij} respectively, and $\overline{F_A}(t_{ij}, t_{ij})$ and $\overline{G_A}(t_{ij})$ are survivor function with accelerated life testing of (T_{ij}, T_{ij}) and T_{ij} respectively.
Note that: Hanagal [13] presented the likelihood function for the bivariate exponential distributions based on type-I censoring samples.
Assumed that $P, \alpha_1, \alpha_2, \alpha_3$ are unknown parameters and C are known parameter, then the likelihood function (3) reduced to:

$$L(x,y|\alpha_1,\alpha_2,\alpha_3,P) =$$
$$B\alpha_1^{\vartheta_2}\alpha_2^{\vartheta_1}\alpha_3^{\vartheta_3}(\alpha_1+\alpha_3)^{\vartheta_3}(\alpha_2+\alpha_3)^{\vartheta_4}\cdot\prod_{j=1}^{k}\{S_j^{(U_{1j}+U_{2j})P}e^{-CS_j^P U_{3j}}\prod_{i=1}^{m_{1j}}\left[A(x_{ij})^{\alpha_1+\alpha_3-1}A(y_{ij})^{\alpha_2-1}\right]$$

$$\cdot\prod_{i=1}^{m_{2j}}\left[A(x_{ij})^{\alpha_1-1}A(y_{ij})^{\alpha_2+\alpha_3-1}\right]\prod_{i=1}^{m_{3j}}\left[A(x_{ij})^{\alpha_1+\alpha_2+\alpha_3-1}\right]\prod_{i=1}^{m_{4j}}\left[A(x_{ij})^{\alpha_1+\alpha_3-1}(1-A(t_{ij})^{\alpha_2})\right]$$

$$\cdot\prod_{i=1}^{m_{5j}}[A(y_{ij})^{\alpha_2+\alpha_3-1}(1-A(t_{ij})^{\alpha_1})]\prod_{i=1}^{m_{6j}}\left[1-A(t_{ij})^{\alpha_1+\alpha_2+\alpha_3}\right]\}. \quad (4)$$

Where B is the constant of proportionality,

$$A(w) = 1 - e^{-CS_j^P z}, w = x_{ij}, y_{ij}, t_{ij},$$

$$\vartheta_r = \sum_{j=1}^{k} m_{rj}, r = 1,2,3,$$

$$\vartheta_4 = \sum_{j=1}^{k}(m_{1j} + m_{4j}),$$

$$\vartheta_5 = \sum_{j=1}^{k}(m_{2j} + m_{5j}),$$

$$U_{1j} = 2(m_{1j} + m_{2j} + m_{4j} + m_{5j}),$$

$$U_{2j} = m_{3j} + m_{6j},$$

$$U_{3j} = \sum_{i=1}^{m_{1j}}[(x_{ij}+y_{ij}) + max(x_{ij},y_{ij})] + \sum_{i=1}^{m_{2j}}[(x_{ij}+y_{ij}) + max(x_{ij},y_{ij})] + \sum_{i=1}^{m_{3j}}[x_{ij} + max(x_{ij},y_{ij})] +$$
$$\sum_{i=1}^{m_{4j}}(x_{ij}+t_{ij}) + \sum_{i=1}^{m_{5j}}(y_{ij}+t_{ij}) + \sum_{i=1}^{m_{6j}}[t_{ij}],$$

For all $i = 1, 2, \ldots, n_j$ and $j = 1, 2, \ldots, k$.

Now, the Bayesian estimation of the unknown parameters will be presented. Under the assumption that C is assumed to be known parameter. We assume the prior distributions for $P, \alpha_1, \alpha_2, \alpha_3$ as gamma prior as follows (see Kundu and Gupta [20]):

$$\pi_1(P) \propto P^{b-1} e^{-aP}, P > 0,$$

$$\pi_2(\alpha_1) \propto {\alpha_1}^{m-1} e^{-o\alpha_1}, \alpha_1 > 0,$$

$$\pi_3(\alpha_2) \propto {\alpha_2}^{k-1} e^{-l\alpha_2}, \alpha_2 > 0,$$

$$\pi_4(\alpha_3) \propto {\alpha_3}^{w-1} e^{-u\alpha_3}, \alpha_3 > 0.$$

All the hyper parameters a, b, m, k, l, w, o and u are assumed to be known and non-negative.
Suppose $\{(x_{11}, y_{11}), \ldots, (x_{Nk}, y_{Nk})\}$ is a random sample from BVGE $(\alpha_1, \alpha_2, \alpha_3, C, P)$ where N is the sample size and k is levels of stress , then based on the likelihood function of the observed data equation (4), the joint posterior density function of $\alpha_1, \alpha_2, \alpha_3$ and P can be written as (see Kundu and Gupta [18] and Walker [30]):

$$H(\alpha_1,\alpha_2,\alpha_3,P|X,Y)=\frac{L(X,Y|\alpha_1,\alpha_2,\alpha_3,P)\pi(\alpha_1,\alpha_2,\alpha_3,P)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(X,Y|\alpha_1,\alpha_2,\alpha_3,P)\pi(\alpha_1,\alpha_2,\alpha_3,P)d\alpha_3 d\alpha_2 d\alpha_1 dP}, \tag{5}$$

The posterior in equation (5) distribution is proportional to:

$$H(\alpha_1,\alpha_2,\alpha_3,P|X,Y)\propto L(X,Y|\alpha_1,\alpha_2,\alpha_3,P)\pi(\alpha_1,\alpha_2,\alpha_3,P)$$

$$H(\alpha_1,\alpha_2,\alpha_3,P|X,Y)\propto BP^{\beta_1-1}\alpha_1{}^{\beta_3+\vartheta_2-1}\alpha_2{}^{\beta_5+\vartheta_1-1}\alpha_3{}^{\beta_7+\vartheta_3-1}(\alpha_1+\alpha_3)^{\vartheta_4}(\alpha_2+\alpha_3)^{\vartheta_5}e^{-(\beta_2 P+\beta_4\alpha_1+\beta_6\alpha_2+\beta_8\alpha_3)}$$

$$\cdot\prod_{j=1}^{k}\{e^{-CS_j^P U_{3j}}S_j^{(U_{1j}+U_{2j})P}\prod_{i=1}^{m_{1j}}\left[A(x_{ij})^{\alpha_1+\alpha_3-1}A(y_{ij})^{\alpha_2-1}\right]\prod_{i=1}^{m_{2j}}\left[A(x_{ij})^{\alpha_1-1}A(y_{ij})^{\alpha_2+\alpha_3-1}\right]$$

$$\cdot\prod_{i=1}^{m_{3j}}\left[A(x_{ij})^{\alpha_1+\alpha_2+\alpha_3-1}\right]\prod_{i=1}^{m_{4j}}\left[A(x_{ij})^{\alpha_1+\alpha_3-1}(1-A(t_{ij})^{\alpha_2})\right]$$

$$\cdot\prod_{i=1}^{m_{5j}}[A(y_{ij})^{\alpha_2+\alpha_3-1}(1-A(t_{ij})^{\alpha_1})]\prod_{i=1}^{m_{6j}}\left[1-A(t_{ij})^{\alpha_1+\alpha_2+\alpha_3}\right]\}.$$

The Bayesian estimators of $\alpha_1,\alpha_2,\alpha_3$ and P parameters can be obtained as follows (see Mood et al. [26]):

$\tilde{\alpha}_1=E(\alpha_1|X,Y)$,

$\tilde{\alpha}_2=E(\alpha_2|X,Y)$,

$\tilde{\alpha}_3=E(\alpha_3|X,Y)$,

And

$\tilde{P}=E(P|X,Y)$.

The Bayesian estimators of the parameters $\alpha_1,\alpha_2,\alpha_3$ and P is proportional to (see Mood et al. [26]):

$\tilde{\alpha}_1\propto\int_0^\infty\alpha_1\,H(\alpha_1,\alpha_2,\alpha_3,P|X,Y)\,d\alpha_1$,

$\tilde{\alpha}_2\propto\int_0^\infty\alpha_2\,H(\alpha_1,\alpha_2,\alpha_3,P|X,Y)\,d\alpha_2$,

$\tilde{\alpha}_3\propto\int_0^\infty\alpha_3\,H(\alpha_1,\alpha_2,\alpha_3,P|X,Y)\,d\alpha_3$,

And

$\tilde{P}\propto\int_0^\infty P\,H(\alpha_1,\alpha_2,\alpha_3,P|X,Y)\,dP$.

It is not possible to compute analytically the solution for these equations. So that Markov Chain Monte Carlo approach used to approximate these equations. In most models and application the posterior predictive distribution does not have an analytic closed form and therefore the posterior distribution does not have a closed form. The Gibbs sampler may be one of the best known MCMC sampling algorithms in the Bayesian computational literature, and it is used in this thesis. The Gibbs sampler is Monte Carlo based sampling methods for evaluating high-dimensional posterior integral, and it is very powerful simulation algorithm that allows us to sample from the posterior distribution without knowing the posterior predictive distribution (see Ibrahim et al. [16]). In this paper, we use OpenBUGS software, a specialized software package for implementing MCMC simulation and Gibbs sampling.

5. Simulation study

To illustrate the theoretical results, a numerical example will be given to obtain the Bayesian estimation of unknown parameters $\alpha_1,\alpha_2,\alpha_3$ and P using OpenBUGS software. The simulation procedures are described through the follows steps:

i) Consider three accelerated stress levels V_1= 0.75, V_2= 1.5, V_3= 2.25 and assume that usual stress is V_u= 0.5.

ii) Assume that the experiment is terminated at a pre-specified censoring time t_{ij}, i = 1, 2... n_j and j = 1, 2, 3.

iii) Generating random samples under usual stress of size N=30, 60, 90 from the BVGE distribution with parameters $\lambda=0.6$, $\alpha_1=2$, $\alpha_2=1.5$ and $\alpha_3=1$ as follows:

- Generate N from uniform (0, 1).

- Generate $\{v_{11},\dots,v_{1N}\}$ from GE(α_1, λ), similarly, $\{v_{21},\dots,v_{2N}\}$ from GE(α_2, λ), $\{v_{31},\dots,v_{3N}\}$ from GE(α_3, λ).
- Obtain $x_{i1} = max\ \{v_{1i}, v_{3i}\}$ and $y_{i1} = max\ \{v_{2i}, v_{3i}\}$, for $i = 1, 2, n_j$ and $j = 1, 2, 3$.

Thus, generated random samples of size N=30, 60, 90 from a BVGE distribution are presented with $\lambda = 0.6, \alpha_1 = 2$ α_2 =1.5 and α_3 =1 (see kundu and Gupta [21]).

iv) The Kolmogorov-Smirnov (K-S) test is used for assessing that the data set follows the BVGE distribution (see Al-Mutrairi et al. [3]).

v) For $\alpha_1, \alpha_2, \alpha_3$ and P unknown parameters, the value of the parameter C is assumed to be known, and applied Bayesian method to determine the posterior density function of the unknown parameters.

vi) Set (C = 0.5) and the prior of the parameter P is the gamma (β_1, β_2) distribution with parameters β_1 = 0.1 and β_2 = 0.1, and the prior of the parameter α_1 is gamma (β_3, β_4) with parameters β_3 = 0.1 and β_4 = 0.1, and the prior of the parameter α_2 is gamma (β_5, β_6) with parameters β_5 = 0.1 and β_6 = 0.1, and the prior of the parameter α_3 is gamma (β_7, β_8) with parameters β_7 = 0.1 and β_8 = 0.1.

vii) Three chains with different initials [(α_1 = 2, α_2 = 1.5, α_3 = 1, P = 0.25), (α_1 = 2.5, α_2 = 2, α_3 = 1.5, P = 0.5), (α_1 = 3, α_2 = 2.5, α_3 = 2, P = 0.75)] are run simultaneously in one simulation. Each chain continues for 10000 iterations.

viii) The Bayesian estimation will be obtained by using MCMC procedure using OpenBUGS Software (see Spiegelhalter [29]).

The sampling results assuming the unknown parameters P, α_1, α_2 and α_3 are displayed in Table(1). Table (1) shows that the MC error for each node is less than 5% of the sample standard deviation (see Aly and Bleed [4]).

Table 1: Estimates of P, α_1, α_2 and α_3.

N	Parameter	Posterior Mean	S.D	MC error	2.5%	97.5%
30	P	0.014140	0.040380	0.000910	$8.587e^{-17}$	0.139700
	α_1	1.730000	0.342900	0.002422	1.124000	2.467000
	α_2	0.805700	0.153600	0.001060	0.537000	1.133000
	α_3	0.460500	0.089030	0.000620	0.303500	0.649800
60	P	0.008574	0.029590	0.000680	$1.389e^{-17}$	0.098340
	α_1	1.117000	0.150900	0.000990	0.842100	1.434000
	α_2	1.324000	0.183100	0.001228	0.987500	1.706000
	α_3	0.850100	0.113100	0.006910	0.643000	1.084000
90	P	0.005930	0.020790	0.000520	$9.265e^{-17}$	0.067390
	α_1	1.456000	0.164400	0.001138	1.150000	1.796000
	α_2	0.746900	0.086040	0.000630	0.587900	0.924800
	α_3	0.688800	0.075100	0.000450	0.548800	0.844400

The convergence and auto-correlation for $\alpha_1, \alpha_2, \alpha_3$ and P Parameters will be show in some graphs for one samples size, for example the third sample where $N = 90$ as follows:

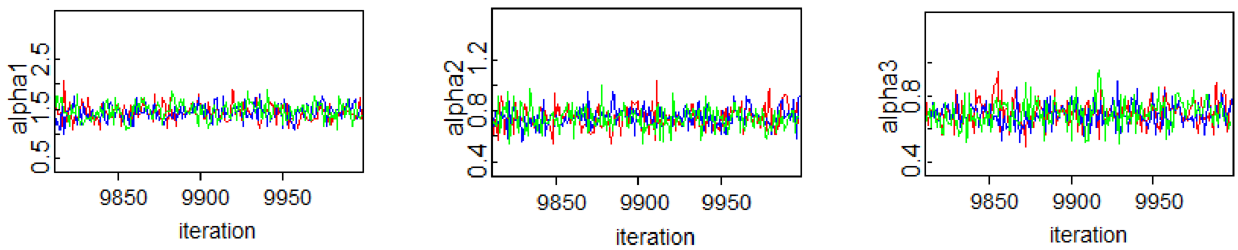

Fig. 1: Dynamic Trace for ($\alpha_1, \alpha_2, \alpha_3$) Parameters.

The dynamic trace for parameter $\alpha_1, \alpha_2, \alpha_3$ respectively in Figure (1) shows that the plots looks like a horizontal band, with no long upward or downward trends and the three chains are well-mixed so this is indicative of convergence.

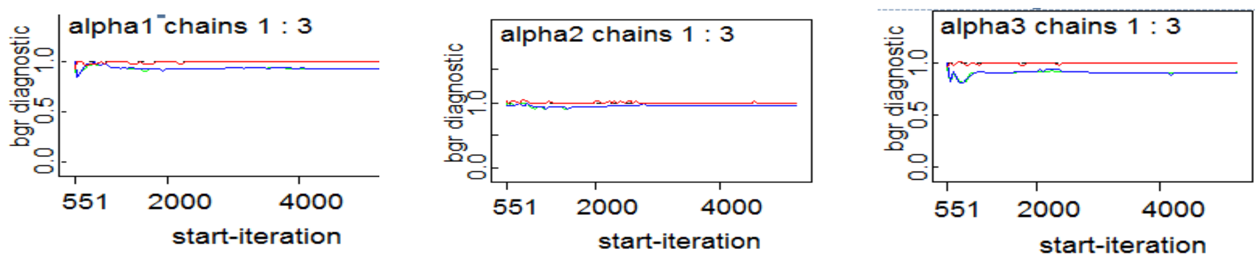

Fig. 2: Gelman-Rubin Convergence For ($\alpha_1, \alpha_2, \alpha_3$) Parameters.

In Figure (2) the plots for parameter $\alpha_1, \alpha_2, \alpha_3$ respectively shows that all chains have approximately the same average width of the 80% intervals within the individual runs and these approximately equal to the width of the central 80% interval of the pooled runs. The ratio for the width of the central 80% interval of the pooled runs and the average width of the 80% intervals within the individual runs (R = pooled / within) is Gelman-Rubin convergence which is normalized to equal one and the Gelman-Rubin convergence is equal to one so this is indicative of convergence.

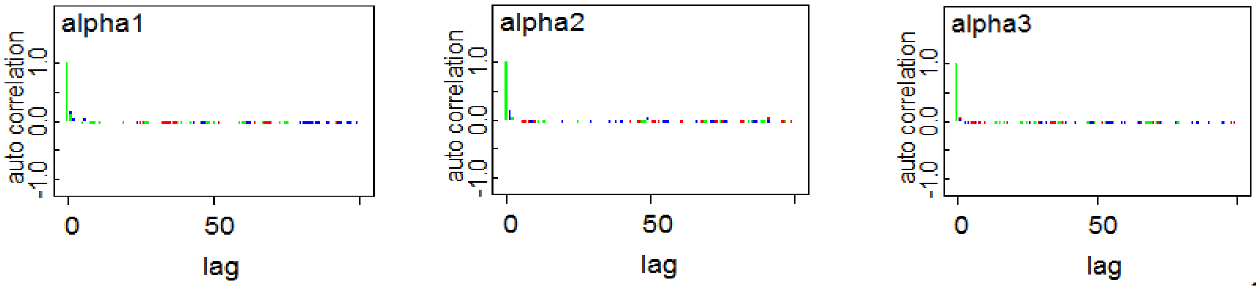

Fig. 3: Auto-Correlation for $(\alpha_1, \alpha_2, \alpha_3)$ Parameters.

The auto-correlation for Parameter $\alpha_1, \alpha_2, \alpha_3$ respectively in Figure (3) shows that the correlation is almost negligible that means that the samples are independent.

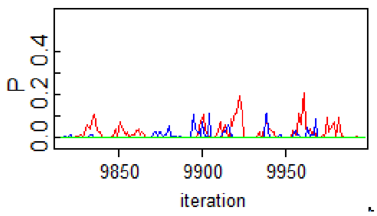

Fig. 4: Dynamic Trace for Parameter (P)

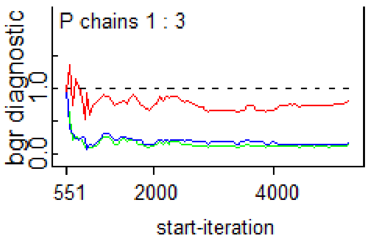

Fig. 5: Gelman-Rubin Convergence for Parameter (P)

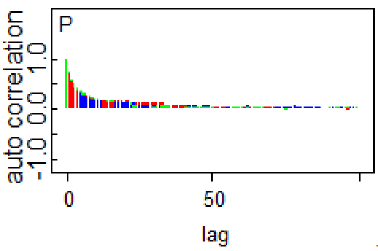

Fig. 6: Auto-Correlation for Parameter (P)

Also for parameter P the dynamic trace in Figure (4) shows that the plot has long upward trends and the three chains are poorly-mixed so this is indicative of failure to convergence. In Figure (5) the plot for parameter P shows that all chains have approximately the same average width of the 80% intervals within the individual runs and these different of the width of the central 80% interval of the pooled runs. The ratio for Gelman-Rubin convergence is normalized to equal one and the Gelman-Rubin convergence is not equal one so this is indicative of failure to convergence. The auto-correlation for parameter P in Figure (6) shows that the chains are hardly auto correlated at all. That is means that the samples are approximately dependent.

6. Conclusion

This paper presented Bayesian estimation for the unknown parameters $\alpha_1, \alpha_2, \alpha_3$ and P for BVGE lifetime distribution and inverse power law acceleration model under bivariate censoring type-I samples with CSALT.

Bayesian estimates are calculated numerically for the unknown parameters. Bayesian analysis was conducted to estimate posterior mean, standard deviation, MC error, 95% asymptotic confidence intervals, dynamic trace, Gelman-rubin convergence and Auto-correlation for unknown Parameters $(\alpha_1, \alpha_2, \alpha_3, P)$.

Finally, for different samples the OpenBUGS technique is used to obtain the numerical results for the proposed model. Then, we conclude that for the unknown parameters $\alpha_1, \alpha_2, \alpha_3$ and P the MC error is less than 5% of the sample standard deviation and the initial values and the point estimation exist in the confidence intervals. Also For the parameters α_1, α_2 and α_3 there are convergence and the samples are independent. For the parameter P there is failure to convergence and the samples are approximately dependent.

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