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Research paper

A New Length-Biased Xrama Distribution: Properties and Application to Cancer Data

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Abstract

This research introduces the Length-Biased Xrama distribution, an extension of the Xrama model. The study analyzes the length-biased distribution's characteristics, comparing it to the original Xrama distribution, and investigates its statistical properties like moments and reliability. Furthermore, the paper explores order statistics and likelihood ratio tests and utilizes Maximum Likelihood Estimation (MLE) to determine the distribution's parameters. The findings, supported by real-world cancer data applications, suggest that the Length-Biased Xrama distribution provides a superior fit compared to competing distributions.

Keywords: *Length-Biased Xrama Distribution; Statistical Properties; Reliability; Likelihood Ratio Tests.*

1. Introduction

Weighted distributions are statistical models used to adjust for biases in data collection, particularly when samples are drawn without a proper sampling frame. These distributions modify the probabilities of events as observed and recorded, accounting for the method of ascertainment. They have been applied in various fields, including reliability and survival analysis, meta-analysis, analysis of family data, ecology, and forestry. Weighted distributions, introduced by Fisher in 1934 and generalized by Rao in 1965, address how data collection methods influence observed distributions. These distributions adjust probabilities of recorded events to account for potential biases introduced during data collection.

Length-biased distributions are a specific type of weighted distribution. Patil and Rao (1978) provided examples of length-biased versions of common distributions. Later, the length-biased Rayleigh distribution was studied by Kayid et al. (2013).

In recent times, several researchers have examined and reviewed different length-biased probability models, discussing their applications in various fields. Al-Omari and Alsmairam (2019) investigated the length-biased Suja distribution and its applications. Ekhosuehi et al. (2020) introduced the Weibull Length-Biased Exponential (WLBE) distribution, a flexible three-parameter lifetime model that incorporates length-biased sampling within a Weibull framework. The authors investigated its statistical properties and demonstrated its applicability through real-world datasets. Klinjan and Aryuyuen (2021) introduced the length-biased power Garima distribution, developed its statistical properties, and demonstrated its flexibility in modeling lifetime data through real-world applications. Akanbi and Oyebanjo (2021) introduced the length-biased Gumbel distribution and applied it to wind speed data. Ben Ghorbal (2022) studied the length-biased exponential distribution and explained its main properties, like the PDF, CDF, moments, and hazard function. The paper also used maximum likelihood estimation to find the parameters and showed how the model works well using real data and simulations. Finally, Benchettah et al. (2023) introduced a new model called the composite length-biased exponential–Pareto distribution. They explained its properties, estimated its parameters using MLE, and applied it to real insurance data. The model gave a better fit compared to other distributions. And Sakthivel and Pandiyan (2024) proposed a stochastic model for the Length-Biased Loai distribution, focusing on its key properties and uses.

This study introduces and investigates the Length-Biased Xrama distribution, a novel one-parameter lifetime model. The original Xrama distribution, recently developed by Harrison O. Etaga et al. (2023), serves as the foundation for this extension. In this work, we derive the probability density function (PDF) and cumulative distribution function (CDF) of the Length-Biased Xrama distribution and examine its key statistical properties. Parameter estimation is carried out using the Maximum Likelihood Estimation (MLE) method. The proposed model demonstrates superior flexibility and performance in modeling real-world cancer survival data, providing a better fit compared to several existing distributions.

2. Length-biased X-ray distribution (LBXD)

The Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of the Length-Biased Xrama distribution are formulated in this section.

The Xrama distribution is a new one-parameter lifetime distribution. The Xrama distribution's probability density function (pdf) is given by

$$f(x) = \frac{\theta^4}{(\theta^3+6)^2}(\theta^3 + 6x^3 + 12)e^{-\theta x} \; x > 0, \theta > 0 \tag{1}$$

The Xrama distribution's cumulative distribution function (cdf) is given by

$$F(x) = 1 - \left\{1 + \frac{1}{(\theta^3+6)^2}(6\theta^3x^3 + 18\theta^2x^2 + 36\theta x\right\} e^{-\theta x} \; x > 0, \theta > 0 \tag{2}$$

Let x be a random variable with a probability density function f(x), and let w(x) be a non-negative weight function. A new probability density function is then defined

$$f_w(x) = \frac{w(x)f(x)}{E[w(x)]} \; ; \; x > 0$$

Where $w(x)$ be the non-negative weight function and $E[w(x)] = \int w(x)f(x)dx < \infty$.
For different weighted models, we have different choices of the weight function $w(x)$. When $w(x) = x^c$, the resulting distribution is termed a weighted distribution. In this paper, we have to find the length-biased version of the Xrama distribution, so we will take c = 1 in the weights x^c To get the length-biased Xrama distribution and its PDF is given by:

$$f_l(x) = \frac{xf(x)}{E(X)} \; ; \; x > 0 \tag{3}$$

Where

$$E(X) = \int_0^\infty x\, f(x)dx$$

$$E(X) = \int_0^\infty x\, f(x;\theta)dx$$

$$E(X) = \int_0^\infty x.\frac{\theta^4}{(\theta^3+6)^2}(\theta^3 + 6x^3 + 12)e^{-\theta x}\,dx$$

$$E(X) = \frac{\theta^4}{(\theta^3+6)^2}\int_0^\infty x(\theta^3 + 6x^3 + 12)e^{-\theta x}\,dx$$

$$E(X) = \frac{\theta^4}{(\theta^3+6)^2}\int_0^\infty x\,\theta^3\,e^{-\theta x}\,dx + \int_0^\infty 6x^4\,e^{-\theta x}\,dx + \int_0^\infty 12\,x\,e^{-\theta x}\,dx$$

$$E(X) = \frac{\theta^4}{(\theta^3+6)^2}\left[\theta^3\int_0^\infty x^{2-1}\,e^{-\theta x}\,dx + 6\int_0^\infty x^{5-1}\,e^{-\theta x}\,dx + 12\int_0^\infty x^{2-1}\,e^{-\theta x}\,dx\right]$$

Using Gamma function

$$\frac{\Gamma(Z)}{a^Z} = \int_0^\infty t^{z-1}e^{-ta}dt$$

$$E(X) = \frac{\theta^4}{(\theta^3+6)^2}\left[\frac{\theta^3\Gamma(2)}{\theta^2} + \frac{6\Gamma(5)}{\theta^5} + \frac{12\Gamma(2)}{\theta^2}\right]$$

$$E(X) = \frac{\theta^4}{(\theta^3+6)^2}\left[\frac{\theta^6\Gamma(2)+6\Gamma(5)+12\theta^3\Gamma(2)}{\theta^5}\right]$$

$$E(X) = \frac{\theta^6\Gamma(2)+6\Gamma(5)+12\theta^3\Gamma(2)}{\theta(\theta^3+6)^2}$$

$$E(X) = \frac{\theta^6+6(24)+12\theta^3}{\theta(\theta^3+6)^2}$$

$$E(X) = \frac{\theta^6+144+12\theta^3}{\theta(\theta^3+6)^2} \tag{4}$$

By substituting equations (1) and (4) into equation (3), we obtain the probability density function (PDF) of the LBXD

$$f_l(x;\theta) = \frac{xf(x;\theta)}{E(X)}$$

$$f_l(x;\theta) = \frac{x\left(\frac{\theta^4}{(\theta^3+6)^2}(\theta^3+6x^3+12)e^{-\theta x}\right)}{\left(\frac{\theta^6+144+12\theta^3}{\theta(\theta^3+6)^2}\right)}$$

$$f_l(x;\theta) = \frac{x\left(\theta^4(\theta^3+6x^3+12)e^{-\theta x}\right)}{\left(\frac{\theta^6+144+12\theta^3}{\theta}\right)}$$

$$f_l(x;\theta)=\frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3} \quad (5)$$

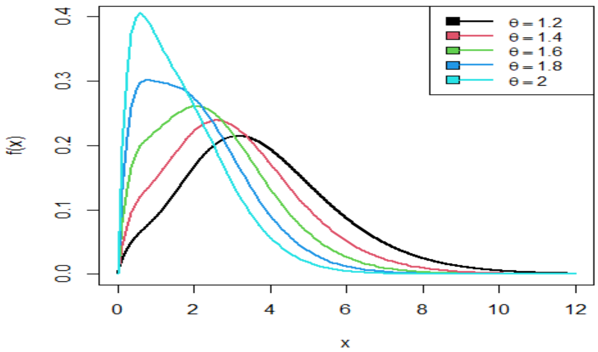

Fig. 1: PDF Plot of Length Biased Xrama Distribution.

We obtain the cumulative distribution function (CDF) of the length-biased Xrama distribution as follows

$$F_l(x;\theta)=\int_0^x f_l(x;\theta)dx$$

$$F_l(x;\theta)=\int_0^x \frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}dx$$

$$F_l(x;\theta)=\frac{\theta^5}{\theta^6+144+12\theta^3}\int_0^x x(\theta^3+6x^3+12)e^{-\theta x}dx$$

The cumulative distribution function of the length-biased Xrama distribution is obtained after simplification

$$F_l(x;\theta)=\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3} \quad (6)$$

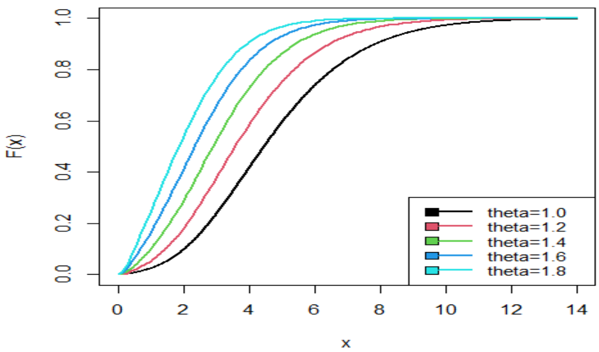

Fig. 2: CDF Plot of Length Biased Xrama Distribution.

3. Reliability analysis

In this section, we will discuss the reliability function, hazard function, reverse hazard function, Mills ratio, and Mean Residual function for the proposed length-biased Xrama distribution.

3.1. Reliability function

The survival function or the reliability function of the LBXD is given by

$$S_l(x;\theta)=1-F_l(x;\theta)$$

$$S_l(x;\theta)=1-\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3} \quad (7)$$

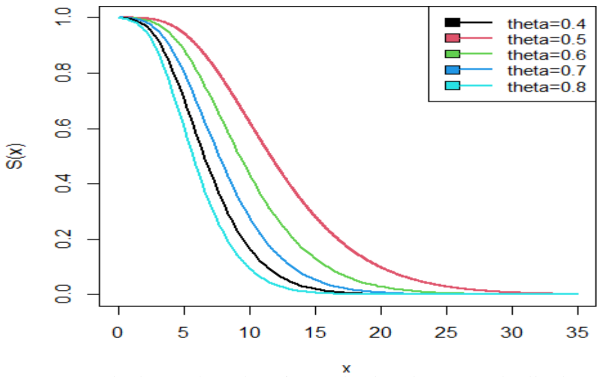

Fig. 3: Survival Function Plot of Length Biased Xrama Distribution.

3.2. Hazard function

The force of mortality, instantaneous failure rate, or hazard rate are other names for the hazard function, which is provided by

$$h(x) = \frac{f_l(x;\theta)}{s_l(x;\theta)}$$

$$h(x) = \frac{\left(\frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right)}{\left(1-\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right)}$$

$$h(x) = \frac{(\theta^5 x(\theta^3+6x^3+12)e^{-\theta x})}{(\theta^6+144+12\theta^3)-(\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x))} \quad (8)$$

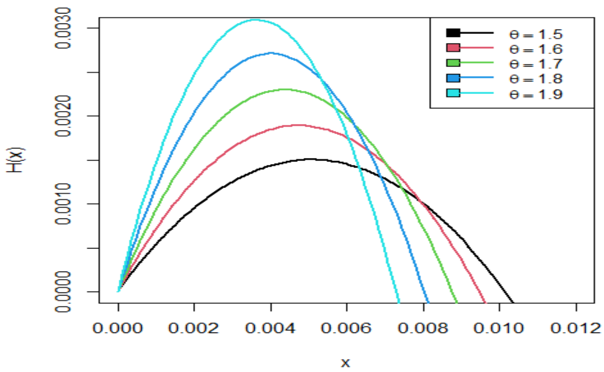

Fig. 4: Hazard Function Plot of Length Biased Xrama Distribution.

3.3. Reverse hazard function

The length-biased Xrama distribution's reverse hazard function can be obtained by

$$h_r(x) = \frac{f_l(x;\theta)}{F_l(x;\theta)}$$

$$h_r(x) = \frac{\left(\frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right)}{\left(\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right)}$$

$$h_r(x) = \frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)} \quad (9)$$

3.4. Mills ratio

$$\text{Mills Ratio} = \frac{1}{h_r(x)} = \frac{1}{\frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}}$$

$$\text{Mills Ratio} = \frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}} \quad (10)$$

3.5. Mean residual function

The LBXD has the following mean residual function

$$M_l(x) = \frac{1}{S_l(x;\theta)}\int_x^\infty x\, f_l(x;\theta)dx$$

$$=\frac{1}{\left(1-\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right)}\int_x^\infty x\left(\frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right)dx-x$$

$$=\frac{1}{\left(1-\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right)}\times\frac{\theta^5}{\theta^6+144+12\theta^3}\left[\int_x^\infty x^2(\theta^3+6x^3+12)e^{-\theta x}\right]dx-x$$

$$=\frac{1}{\left(1-\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right)}\times\frac{\theta^5}{\theta^6+144+12\theta^3}\left[\int_x^\infty x^2\theta^3 e^{-\theta x}\,dx+\int_x^\infty 6\,x^5e^{-\theta x}\,dx+\int_x^\infty 12\,x^2\,e^{-\theta x}\,dx\right]-x$$

Put $\theta x = t\; x=\frac{t}{\theta}\; dx=\frac{dt}{\theta}$

When $x\to 0, t\to 0$ and $x\to x, t\to\theta x$

$$=\frac{1}{\left(1-\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right)}\times\frac{\theta^5}{\theta^6+144+12\theta^3}\left[\theta^3\int_{\theta x}^\infty\left(\frac{t}{\theta}\right)^2 e^{-t}\frac{dt}{\theta}+6\int_{\theta x}^\infty\left(\frac{t}{\theta}\right)^5 e^{-t}\frac{dt}{\theta}+12\int_{\theta x}^\infty\left(\frac{t}{\theta}\right)^2 e^{-t}\frac{dt}{\theta}\right]-x$$

After evaluating the integral, we obtain the following expression for $M_l(x)$

$$M_l(x)=\left[\frac{\theta^6\Gamma(3,\theta x)+6\Gamma 6,\theta x+12\,\theta^3\Gamma(3,\theta x)}{\theta(\theta^6+144+12\theta^3)-(\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x))}\right]-x \quad (11)$$

4. Statistical properties

In this section, we derived the structural properties of length biased Xrama distribution

4.1. Moments

Let X denoted the random variable following LBXD then r^{th} Order moments $E(X^r)$ is obtained as

$$E(X^r)=\mu_r{}'=\int_0^\infty x^r f_l(x;\theta)dx$$

$$E(X^r)=\int_0^\infty x^r\left(\frac{\theta^5x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right)dx\; E(X^r)=\frac{\theta^5}{\theta^6+144+12\theta^3}\int_0^\infty x^r x(\theta^3+6x^3+12)e^{-\theta x}dx$$

$$E(X^r)=\frac{\theta^5}{\theta^6+144+12\theta^3}\int_0^\infty x^{r+1}(\theta^3+6x^3+12)e^{-\theta x}dx$$

$$E(X^r)=\frac{\theta^5}{\theta^6+144+12\theta^3}\left(\theta^3\int_0^\infty x^{(r+2)-1}e^{-\theta x}dx+6\int_0^\infty x^{(r+5)-1}e^{-\theta x}dx+12\int_0^\infty x^{(r+2)-1}e^{-\theta x}dx\right)$$

$$E(X^r)=\mu_r{}'=\frac{\theta^6\Gamma(r+2)+6\Gamma(r+5)+12\theta^3\Gamma(r+2)}{\theta^r(\theta^6+144+12\theta^3)} \quad (12)$$

Putting r =1 in equation (12), we will get the mean of length biased Xrama distribution which is given by

$$E(X)=\mu_1{}'=\frac{2\theta^6+720+24\theta^3}{\theta(\theta^6+144+12\theta^3)} \quad (13)$$

and putting r = 2, we obtain the second moment as

$$E(X^2)=\mu_2{}'=\frac{6\theta^6+4320+72\theta^3}{\theta^2(\theta^6+144+12\theta^3)} \quad (14)$$

Variance = $\mu_2{}'-(\mu_1{}')^2$

$$\text{variance}(\sigma)^2=\frac{(\theta^6+144+12\theta^3)(6\theta^6+4320+72\theta^3)-(2\theta^6+720+24\theta^3)^2}{\theta^2(\theta^6+144+12\theta^3)^2} \quad (15)$$

Standard Deviation

$$S.D(\sigma)=\frac{\sqrt{(\theta^6+144+12\theta^3)(6\theta^6+4320+72\theta^3)-(2\theta^6+720+24\theta^3)^2}}{\theta^2(\theta^6+144+12\theta^3)^2} \quad (16)$$

4.2. Harmonic mean

The Harmonic mean of the length biased Xrama distribution can be obtained as

$$H.M=E\left(\frac{1}{x}\right)$$

$$H.M=\int_0^\infty\frac{1}{x}f_l(x;\theta)dx$$

$$H.M = \int_0^\infty \frac{1}{x}\left(\frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right)dx$$

$$H.M = \frac{\theta(\theta^5+36=12\theta^2)}{\theta^6+144+12\theta^3} \tag{17}$$

4.3. Moment generating function and characteristic function

Let X have a LBXD then the MGF of X is obtained as

$$M_X(t) = E(e^{tx})$$

$$= \int_0^\infty e^{tx} f_l(x;\theta)dx$$

Using Taylor's series

$$M_X(t) = E(e^{tx}) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \cdots\right) f_l(x)dx$$

$$M_X(t) = \int_0^\infty \sum_{j=0}^\infty \frac{t^j}{j!} x^j f(x)dx$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!} \int_0^\infty x^j f(x)dx$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\mu_j'$$

$$M_X(t) = \sum_{j=0}^\infty \frac{t^j}{j!}\left(\frac{\theta^6\Gamma(j+2)+6\Gamma(j+5)+12\theta^3\Gamma(j+2)}{\theta^j(\theta^6+144+12\theta^3)}\right)$$

$$M_X(t) = \frac{1}{(\theta^6+144+12\theta^3)}\sum_{j=0}^\infty \frac{t^j}{j!\theta^j}\theta^6\Gamma(j+2)+6\Gamma(j+5)+12\theta^3\Gamma(j+2) \tag{18}$$

Likewise, the characteristic function of the LBXD can be determined as follows.

$$\phi_X(t) = M_X(it)$$

$$\phi_X(t) = \sum_{j=0}^\infty \frac{it^j}{j!}\mu_j'$$

$$\phi_X(t) = \sum_{j=0}^\infty \frac{it^j}{j!}\left(\frac{\theta^6\Gamma(j+2)+6\Gamma(j+5)+12\theta^3\Gamma(j+2)}{\theta^j(\theta^6+144+12\theta^3)}\right)\phi_X(t) = \frac{1}{(\theta^6+144+12\theta^3)}\sum_{j=0}^\infty \frac{it^j}{j!\theta^j}\theta^6\Gamma(j+2)+6\Gamma(j+5)+12\theta^3\Gamma(j+2) \tag{19}$$

5. Order statistics

In this section, we obtained the distributions of the order statistics based on the length-biased Xrama distribution.
Let $X_{(1)}, X_{(2)}, X_{(3)}, \dots, X_{(n)}$ be the order statistics of the random sample taken from length-biased Xrama. The probability density function of r^{th} order statistics, $X_{(n)}$ is defined as.

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)[F_X(x)]^{r-1}[1-F_X(x)]^{n-r} \tag{20}$$

By combining equations (5) and (6) with equation (20), the probability density function of order statistics $X_{(r)}$ The length-biased Xrama distribution is obtained by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\left(\frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right) \times \left[\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right]^{r-1} \times \left[1-\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right]^{n-r}$$

Thus, the PDF of higher order statistics, $X_{(n)}$ of the LXD can be derived as

$$f_{X(n)}(x) = n\left(\frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right) \times \left[\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right]^{n-1}$$

The pdf of the first order statistic $X_{(1)}$ of the LBXD

$$f_{X(1)}(x) = n\left(\frac{\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right) \times \left[1-\frac{\theta^6\gamma(2,\theta x)+6\gamma(5,\theta x)+\theta^3 12\gamma(2,\theta x)}{\theta^6+144+12\theta^3}\right]^{n-1}$$

6. Likelihood ratio test

Let $X_1, X_2, \dots, X_n$ be a random sample from the LBXD. To test the hypothesis

$H_0: f(x) = f(x;\theta)$ against $H_1: f(x) = f_l(x;\theta)$

To test whether the random sample of size n comes from the Xrama distribution or LBXD, the following test statistic is used

$$\Delta = \frac{L_1}{L_2} = \prod_{i=1}^{n} \frac{f_l(x_i;\theta)}{f(x_i;\theta)}$$

$$\Delta = \prod_{i=1}^{n} \frac{\left(\frac{\theta^5 x_i(\theta^3+6x_i{}^3+12)e^{-\theta x_i}}{\theta^6+144+12\theta^3}\right)}{\left(\frac{\theta^4}{(\theta^3+6)^2}(\theta^3+6x_i{}^3+12)e^{-\theta x_i}\right)}$$

$$\Delta = \prod_{i=1}^{n} \frac{\theta x_i(\theta^3-6)^2}{\theta^6+144+12\theta^3}$$

$$\Delta = \left(\frac{\theta(\theta^3-6)^2}{\theta^6+144+12\theta^3}\right)^n \prod_{i=1}^{n} x_i$$

We reject the null hypothesis if

$$\Delta = \left(\frac{\theta(\theta^3-6)^2}{\theta^6+144+12\theta^3}\right)^n \prod_{i=1}^{n} x_i > k$$

Equivalently, we also reject the null hypothesis

$$\Delta^* = \prod_{i=1}^{n} x_i > k\left(\frac{\theta(\theta^3-6)^2}{\theta^6+144+12\theta^3}\right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i > k^* \text{ where } k^* = k\left(\frac{\theta(\theta^3-6)^2}{\theta^6+144+12\theta^3}\right)^n$$

For a large sample size n, $2\log\Delta$ follows a chi-square distribution with one degree of freedom. The p-value is computed from this distribution, and the null hypothesis is rejected when the corresponding probability value is given by $p(\Delta^* > \beta^*)$, where $\beta^* = \prod_{i=1}^{n} x_i$ is less than the specified level of significance, can $\prod_{i=1}^{n} x_i$ Is the observed value of the statistic Δ^*.

7. Bonferroni and Lorenz curves

In this section, we have derived the Bonferroni and Lorenz curves from the length-biased Xrama distribution. The Bonferroni and Lorenz curves are a powerful tool in the analysis of distributions and have applications in many fields, such as economics, insurance, income, reliability, and medicine. The Bonferroni and Lorenz curves for a X, where X is the random variable of a unit, and $f(x)$ is the probability density function of x. $f(x)dx$ will be represented by the probability that a unit selected at random is defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x\ f_l(x_i;\theta)dx$$

And

$$L(p) = \frac{1}{\mu}\int_0^q x\ f_l(x_i;\theta)dx$$

Where $q = F^{-1}(p); q\epsilon[0,1]$
And $\mu = E(X)$
Thus, the Bonferroni and Lorenz curves of our distribution are determined by

$$\mu = \frac{2\theta^6+720+24\theta^3}{\theta(\theta^6+144+12\theta^3)}$$

$$B(p) = \frac{1}{p\left(\frac{2\theta^6+720+24\theta^3}{\theta(\theta^6+144+12\theta^3)}\right)}\int_0^q x\left(\frac{\theta^5 x\,(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right)dx$$

$$B(p) = \frac{\theta}{p(2\theta^6+720+24\theta^3)} \times \left(\int_0^q x(\theta^5 x(\theta^3+6x^3+12)e^{-\theta x}dx\right)$$

$$B(p) = \frac{\theta}{p(2\theta^6+720+24\theta^3)} \times \theta^5 \int_0^q x^2(\theta^3+6x^3+12)e^{-\theta x}dx$$

$$B(p) = \frac{\theta}{p(2\theta^6+720+24\theta^3)} \times \theta^5\left[\int_0^q x^2\theta^3 e^{-\theta x}dx + 6\int_0^q x^5 e^{-\theta x}dx + 12\int_0^q x^2 e^{-\theta x}dx\right]$$

Put $\theta x = t, x = \frac{t}{\theta}, dx = \frac{dt}{\theta}$

When $x \to 0, t \to 0,$ and $x \to q, t \to \theta q$

$$B(p) = \frac{\theta}{p(2\theta^6+720+24\theta^3)} \times \theta^5\left[\int_0^{\theta q}\left(\frac{t}{\theta}\right)^2 \theta^3 e^{-t}\frac{dt}{\theta} + 6\int_0^{\theta q}\left(\frac{t}{\theta}\right)^5 e^{-t}\frac{dt}{\theta} + 12\int_0^{\theta q}\left(\frac{t}{\theta}\right)^2 e^{-t}\frac{dt}{\theta}\right]$$

After evaluating the integral, the expression for B(p) is given by

$$B(p) = \frac{\theta^6\gamma(3,\theta q)+6\gamma(6,\theta q)+12\theta^3\gamma(3,\theta q)}{p(2\theta^6+720+24\theta^3)}$$

$$L(p) = pB(p)$$

$$L(p) = p \times \left(\frac{\theta^6\gamma(3,\theta q)+6\gamma(6,\theta q)+12\theta^3\gamma(3,\theta q)}{p(2\theta^6+720+24\theta^3)}\right)$$

$$L(p) = \left(\frac{\theta^6\gamma(3,\theta q)+6\gamma(6,\theta q)+12\theta^3\gamma(3,\theta q)}{2\theta^6+720+24\theta^3}\right)$$

8. Stochastic ordering

Stochastic ordering is a useful tool in finance and dependability to evaluate model performance. Let X and Y be two random variables with PDF, CDF, and reliability functions $f(x), f(y), F(x), F(y). S(x) = 1 - F(x)$and $F(y)$

1) Likelihood ratio order ($X \leq_{LR} Y$)if $\frac{f_{X_l}(x)}{f_{Y_l}(x)}$decreases in x
2) Stochastic order ($X \leq_{ST} Y$) if $F_{X_l}(x) \geq F_{Y_l}(x)$ for all x
3) Hazard rate order($X \leq_{HR} Y$) if $h_{X_l}(x) \geq h_{Y_l}(x)$ for all x
4) Mean residual life order($X \leq_{MRL} Y$) if $MRL_{X_l}(x) \geq MRL_{Y_l}(x)$ for all x

Prove that the weighted Xrama distribution gives the strongest ordering (likelihood ratio ordering). Suppose X and Y are independent random variables with probability distribution functions$f_{l_x}(x;\theta)$ and $f_{l_y}(x;\lambda)$.If θ<λ, then

$$\Lambda = \frac{f_{l_x}(x;\theta)}{f_{l_y}(x;\lambda)}$$

$$\Lambda = \frac{\left[\frac{\theta^5 x\,(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right]}{\left[\frac{\lambda^5 x\,(\lambda^3+6x^3+12)e^{-\lambda x}}{\lambda^6+144+12\lambda^3}\right]}$$

$$\Lambda = \left[\frac{\theta^5 x\,(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right] \times \left[\frac{\lambda^6+144+12\lambda^3}{\lambda^5 x\,(\lambda^3+6x^3+12)e^{-\lambda x}}\right]$$

$$\Lambda = \left[\frac{\theta^5 x\,(\theta^3+6x^3+12)e^{-\theta x}}{\lambda^5 x\,(\lambda^3+6x^3+12)e^{-\lambda x}}\right] \times \left[\frac{\lambda^6+144+12\lambda^3}{\theta^6+144+12\theta^3}\right]$$

$$\Lambda = \frac{\theta^5(\lambda^6+144+12\lambda^3)}{\lambda^5(\theta^6+144+12\theta^3)} \times \frac{x\,(\theta^3+6x^3+12)}{x\,(\lambda^3+6x^3+12)}\,e^{-(\theta-\lambda)x}$$

Therefore

$$\log[\Lambda] = \log\left[\frac{\theta^5(\lambda^6+144+12\lambda^3)}{\lambda^5(\theta^6+144+12\theta^3)}\right] \times \log\left[\frac{x\,(\theta^3+6x^3+12)}{x\,(\lambda^3+6x^3+12)}\right] \times \log e^{-(\theta-\lambda)x}$$

$$\log[\Lambda] = \log\left[\frac{\theta^5(\lambda^6+144+12\lambda^3)}{\lambda^5(\theta^6+144+12\theta^3)}\right] + \log\ (\theta^3 x + 6x^4 + 12x) - \log(\lambda^3 x + 6x^4 + 12x) - (\theta - \lambda)x$$

Differentiating with respect to x, we get.

$$\frac{\partial \log[\Lambda]}{\partial x} = \left[\frac{\theta^3+24x^3+12}{\theta^3 x+6x^4+12x}\right] - \left[\frac{\lambda^3+24x^3+12}{\lambda^3+6x^3+12}\right] + (\lambda - \theta)$$

Hence $\frac{\partial \log[\Lambda]}{\partial x} < 0$ if $\theta < \lambda$

9. Entropies

Entropy is a crucial concept across disciplines such as probability and statistics, physics, communication theory, and economics. It quantifies a system's diversity, uncertainty, or randomness. In particular, the entropy of a random variable X measures the variation in uncertainty associated with its possible outcomes.

In this section, we derived three entropy measures: Shannon entropy, Rényi entropy, and Tsallis entropy from the length-biased Xrama distribution.

9.1. Shannon entropy

Shannon entropy of the random variable X, such that length length-biased Xrama distribution is defined as

$$S_\lambda = -\int_0^\infty f(x)\log\big(f(x)\big)dx \quad ;\lambda > 0, \lambda \neq 1$$

$$S_\lambda = -\int_0^\infty f_l(x;\theta)\log\big(f_l(x;\theta)\big)dx$$

$$S_\lambda = -\int_0^\infty \left(\frac{\theta^5 x\,(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right)\log\left(\frac{\theta^5 x\,(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right)dx \tag{21}$$

9.2. Renyi entropy

The Renyi entropy is important in ecology and statistics as an index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

$$R_\lambda = \frac{1}{1-\lambda}\log\int_0^\infty [f(x)]^\lambda dx\ ;\lambda > 0, \lambda \neq 1$$

$$R_\lambda = \frac{1}{1-\lambda}\log\int_0^\infty [f_l(x;\theta)]^\lambda\, dx$$

$$R_\lambda = \frac{1}{1-\lambda}\log\int_0^\infty \left[\frac{\theta^5 x\,(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right]^\lambda dx$$

$$R_\lambda = \frac{1}{1-\lambda}\log\left(\frac{\theta^5}{\theta^6+144+12\theta^3}\right)^\lambda \int_0^\infty x^\lambda\, dx(\theta^3+6x^3+12)^\lambda e^{-\lambda\theta x}dx$$

Using Binomial expansion, we get

$$R_\lambda = \frac{1}{1-\lambda}\log\left(\frac{\theta^5}{\theta^6+144+12\theta^3}\right)^\lambda \sum_{i=0}^{\lambda}\sum_{j=0}^{\lambda}\binom{\lambda}{i}\binom{i}{j}\theta^{3(\lambda-i)}6^{(i-j)}12^j\int_0^\infty x^{\lambda+3(i-j)}\, e^{-\lambda\theta x}dx \tag{22}$$

Using the integration of the gamma function in equation (22), we will get the R_λ

$$R_\lambda = \frac{1}{1-\lambda}\log\left(\frac{\theta^5}{\theta^6+144+12\theta^3}\right)^\lambda \sum_{i=0}^{\lambda}\sum_{j=0}^{\lambda}\binom{\lambda}{i}\binom{i}{j}\theta^{3(\lambda-i)}6^{(i-j)}12^j\left(\frac{\Gamma(\lambda+3(i-j)+1)}{(\lambda\theta)^{(\lambda+3(i-j)+1)}}\right) \tag{23}$$

9.3 Tsallis Entropy

The Boltzmann-Gibbs (B-G) statistical properties initiated by Tsallis have received a great deal of attention. This generalization of (B-G) statistics was first proposed by introducing the mathematical expression of Tsallis's entropy (Tsallis, 1988) for continuous random variables, which is defined as

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\int_0^\infty [f_l(x)]^\lambda dx\right] \qquad ;\lambda > 0, \lambda \neq 1$$

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\int_0^\infty \left[\frac{\theta^5 x\,(\theta^3+6x^3+12)e^{-\theta x}}{\theta^6+144+12\theta^3}\right]^\lambda dx\right]$$

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\left(\frac{\theta^5}{\theta^6+144+12\theta^3}\right)^\lambda \int_0^\infty x^\lambda(\theta^3+6x^3+12)^\lambda e^{-\lambda\theta x}dx\right]$$

Using binomial expansion, we get

$$T_\lambda = \frac{1}{\lambda-1}\left[1-\left(\frac{\theta^5}{\theta^6+144+12\theta^3}\right)^\lambda \sum_{i=0}^{\lambda}\sum_{j=0}^{\lambda}\binom{\lambda}{i}\binom{i}{j}\theta^{3(\lambda-i)}6^{(i-j)}12^j\int_0^\infty x^{\lambda+3(i-j)}\, e^{-\lambda\theta x}dx\right] \tag{24}$$

Using the integration of the gamma function in equation (24), we will get the T_λ

$$T_\lambda = \frac{1}{\lambda-1}\left[[1]-\left(\frac{\theta^5}{\theta^6+144+12\theta^3}\right)^\lambda \sum_{i=0}^{\lambda}\sum_{j=0}^{\lambda}\binom{\lambda}{i}\binom{i}{j}\theta^{3(\lambda-i)}6^{(i-j)}12^j\left(\frac{\Gamma(\lambda+3(i-j)+1)}{(\lambda\theta)^{(\lambda+3(i-j)+1)}}\right)\right] \tag{25}$$

10. Estimations of parameter

This section provides the length-biased Xrama distribution parameter's Fisher's information matrix and maximum likelihood estimates.

10.1. MLE and Fisher's information matrix

Assume $x_{(1)}, x_{(2)}, x_{(3)}, \dots, x_{(n)}$ is a random sample of size n from the length-biased Xrama distribution with parameter and the likelihood function, which is defined as

$$L(x;\ \theta) = \prod_{i=1}^{n} f_l(x_i;\theta)$$

$$L(x;\ \theta) = \prod_{i=1}^{n}\left[\frac{\theta^5 x_i\,(\theta^3+6{x_i}^3+12)e^{-\theta x_i}}{\theta^6+144+12\theta^3}\right]$$

$$L(x;\ \theta) = \left[\frac{\theta^5}{\theta^6+144+12\theta^3}\right]^n \prod_{i=1}^{n}[x_i\,(\theta^3+6x^3+12)]\, e^{-\theta x_i}$$

Then, the log-likelihood function is

$\log L = n \log \theta^5 - n \log[\theta^6 + 144 + 12\theta^3] + \sum_{i=1}^n \log[x_i\ (\theta^3 + 6x^3 + 12)] - \theta \sum_{i=1}^n x_i$

$$\log L = n\, 5 \log \theta - n \log[\theta^6 + 144 + 12\theta^3] + \sum_{i=1}^n \log[x_i\ (\theta^3 + 6x^3 + 12)] - \theta \sum_{i=1}^n x_i \quad (26)$$

Deriving (26) partially concerning θ we have.

$$\frac{\partial \log L}{\partial \theta} = \frac{5n}{\theta} + \sum_{i=1}^n \frac{3\theta^2 x_i}{x_i\ (\theta^3+6x_i{}^3+12)} - \sum_{i=1}^n x_i = 0 \quad (27)$$

Equations (27) provide the MLE of the parameters for the LBXD. However, because the problem cannot be solved analytically, we solved it numerically using R programming and a data collection.
We apply asymptotic normality results to get the confidence interval. If $\hat{\lambda} = (\hat{\theta})$ represents the MLE of $\lambda = (\theta)$We can state the following results:$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N_2(0, I^{-1}(\lambda))$
Where I(λ) is Fisher's information matrix. i.e.,

$$I(\lambda) = \frac{1}{n}\left[E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]\right]$$

Where

$$E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right] = \left[\frac{-5n}{\theta^2} + \frac{6\theta x_i(\theta^3+6x_i{}^3+12) - 3\theta^2 x_i(x_i(3\theta^2))}{\left(x_i\ (\theta^3+6x_i{}^3+12)\right)^2}\right]$$

Since λ is unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ And this can be used to obtain an asymptotic confidence interval for θ.

11. Application

Data set 1:
The data under consideration are the life times of 20 Leukemia patients who were treated by a certain drug [12]. The data are
1.013, 1.034, 1.109, 1.169, 1.226, 1.509, 1.533, 1.563, 1.716, 1.929, 1.965, 2.061, 2.344, 2.546, 2.626, 2.778, 2.951, 3.413, 4.118, 5.136.
Data set 2:
The dataset on the remission periods (in months) of 36 bladder cancer patients described in [8] and the data are
0.08, 0.2, 0.4, 0.5,0.51, 0.81, 0.87, 0.9, 1.05, 1.19, 1.26, 1.35, 1.4, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.02, 3.25, 3.31, 3.36, 3.36.
To compare the goodness of fit of the fitted distribution, the following criteria are used: Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Akaike Information Criteria Corrected (AICC), and -2log L.
AIC, BIC, AICC, and -2log L can be evaluated by using the formula as follows.

$$AIC = 2K - 2\log L,\ BIC = k \log n - 2 \log L \text{ and } AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$

Where = number of parameters, n sample size, and -2log L is the maximized value of the loglikelihood function.

Table 1: MLEs AIC, BIC, AICC, and -2log L of the Fitted Distribution for the Given Data Set 1

Distribution	ML Estimates	-2 log L	AIC	BIC	AICC
Length-biased Xrama distribution	$\hat{\theta}$ = 1.821052 (0.148514)	56.5523	58.5523	59.5480	58.7745
Xrama distribution	$\hat{\theta}$ =1.232234(0.116270)	69.9132	71.9132	72.9089	72.1354
Exponential distribution	$\hat{\theta}$ =0.457259(0.102246)	71.3003	73.3003	74.2960	73.5225
Lindley distribution	$\hat{\theta}$ =0.722691(0.119653)	66.5649	68.5649	69.5606	68.7871
Length-biased exponential distribution	$\hat{\theta}$ =1.093475(0.172893)	60.0976	62.0976	63.0933	62.3198
Length-biased Lindley distribution	$\hat{\theta}$ =1.200277(0.158721)	58.1778	60.1778	61.1735	60.4000

Table 2: MLEs AIC, BIC, AICC, and -2log L of the Fitted Distribution for the Given Data Set 1

Distribution	ML Estimates	-2 log L	AIC	BIC	AICC
Length-biased Xrama distribution	$\hat{\theta}$ =1.85667286(0.11226)	104.7575	106.7575	108.3410	106.8751
Xrama distribution	$\hat{\theta}$ =1.856672(0.1122695)	114.5201	116.5201	118.1036	116.6377
Exponential distribution	$\hat{\theta}$ =0.515465 (0.085910)	119.7135	121.7135	123.2970	121.8311
Lindley distribution	$\hat{\theta}$ =0.801582 (0.099521)	113.3975	115.3975	116.9810	115.5152
Length-biased exponential distribution	$\hat{\theta}$ =0.970000(0.1143150	108.5924	110.5924	112.1760	110.7101
Length-biased Lindley distribution	$\hat{\theta}$ =1.339624(0.132508)	106.9635	108.9635	110.5470	109.0812

Based on the results in Tables 1 and 2, the Length-Biased Xrama Distribution (LBXD) fits the cancer data better than the Xrama, Exponential, Lindley, length-biased Exponential, and length-biased Lindley distributions. It gives lower AIC, BIC, AICC, and −2logL values, which shows it is more accurate. This better fit is useful in cancer studies. It helps researchers understand survival patterns more clearly. LBXD also handles length-biased data well, which makes the analysis more reliable and closer to real-life situations.

12. Conclusion

In the present research, we have introduced a new generalization of the Xrama distribution, termed the Length-Biased Xrama Distribution, which involves a single parameter. Several statistical properties have been studied, including the moments, the moment generating function, the mean, and the variance. Additional characteristics such as order statistics, stochastic ordering, entropy measures, Bonferroni, and Lorenz curves have also been derived. The parameters of the proposed distribution have been estimated using the method of maximum likelihood. Furthermore, the model has been applied to real-world cancer data and compared with other well-known lifetime distributions. The Length-Biased Xrama Distribution shows a better fit to real-life cancer data compared to other well-known distributions.

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