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Research paper

Bivariate Conditional Wigner Semicircle Distribution Modeling of Wind Speed and Direction

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Abstract

Wind direction and speed are increasingly significant for societal and human advancement, as it is essential to comprehending and forecasting various events. Therefore, a model that captures the distinct features of wind direction and speed is developed. The conditional and marginal Wigner semicircle distributions were combined to create a bivariate conditional Wigner semicircle distribution. The joint characteristics and estimation of its parameters were determined. Real-world wind speed and direction data, where the conditional reliance and circularity of the variables are suspected, were used to test the model. Based on the goodness-of-fit test and root mean square error, the outcome demonstrates that the model and the data are compatible. The model is suggested for use in situations where consideration must be given to variables with a conditional structure and cyclicity.

Keywords: *Wigner-Semicircle; Conditional; Marginal; Distribution; Moment; Bivariate; Circular-Variable*

1. Introduction

Globally, energy consumption has increased geometrically with the rapid development of society and the economy. To meet this increased demand, various energy sources are built by industries and countries, such as wind energy (Ogulata, [1]; Eskin et al. [2]). Wind speed and direction play a critical role in renewable energy (e.g, wind turbines), meteorology (prediction of weather patterns and influences local climate conditions), the environment (disperses pollutants and seeds), aviation (i.e., air transport safety), and oceanography. Wind speed and direction are treated as coupled variables to assess variability, predict trends, or understand site-specific wind characteristics. Also, the variables are interconnected because they jointly define the movement and energy of air masses in the atmosphere (e.g., drive ocean currents, which regulate global temperatures and climate).In the literature, wind speed is taken as a random variable and it is described by a probability density function (pdf) for evaluating wind energy potential and wind stochastic characteristics (Hu et al. [3]; Aslam [4]; Wang and Liu [5]). More so, probability density functions such as Gamma, Raleigh, Inverse Gaussian distribution, lognormal and Weibull distribution, two-parameter Weibull distribution and three-parameter Weibull distribution model (Safari and Gasore [6]; Safari [7]; Pobočíková, and Sedliačková [8]; Kantar and Usta [9]; Pishgar-Komleh et al. [10]; Wais [11]; Pobočíkováa et al. [12]; Aries et al. [3]; Akdag and Dinler [14]; Deep et al. [15]; Chen et al.[16]). Also, some other models in cases when wind speed is bimodal or multi-modal are mixture distribution, Weibull-Weibull mixture, the Gamma-Weibull mixture, the truncated Normal-Weibull mixture, etc. (Carta and Mentado [17]; Carta and Ramirez [18]; Kiss and Janosi [19]; Akpinar and Akpinar [20]; Akdag et al. [21]; Zhang et al. [22]; Mazzeo et al. [23]; Ouarda and Charron [24]; Mahbudi et al. [25]). Models like von Mises (voM) distribution (Carta et al. [26]; Vega and Rodrigue [27]; Mandia et al. [28]), uniform distribution, wrapped-normal distribution, wrapped-Cauchy distribution (Masseran et al. [29]; Soukissian [30]; Horn et al. [31]), as well as the bivariate conditional Weibull distribution (Gongsin and Saporu [32]), are used to assess wind direction as a linear variable. Bivariate circular-linear joint distribution (Cheng et al. [33], Velarde et al. [34]; Vanem et al. [35]; Vanem et al. [36]) and vine copulas approach (Han et al. [37]; Li et al. [38]; Wang et al. [39]; Wang et al. [40]) was utilized in investigating environmental conditions. Bivariate distribution models for evaluating wind direction and speed data at several locations were compared by Erdem and Shi [41]. However, in the literature, the problem of cyclicity of circular variables in a multivariate distribution model is often neglected, and circular variable is modeled as linear variables. Thus, this study seeks to demonstrate the bivariate conditional Wigner semicircle distribution's adaptability by highlighting its capacity to capture intricate interdependencies and cyclicity of variables, proving it to be a versatile and powerful tool in diverse scientific and engineering applications. The derived formulas were implemented using Python software.

2. Methodology

2.1. Wigner semicircle distribution

The Wigner semicircle distribution, a fundamental concept in random matrix theory, is characterized by its elegant and simple form, making it invaluable in fields such as physics, finance, and network theory. Introduced by Eugene Wigner, this distribution has gained prominence for its ability to describe the asymptotic behavior of eigenvalues in large random matrices. While the univariate Wigner semicircle distribution has been well-documented, with thorough reviews by researchers such as Mehta [42], recent attention has turned to its multivariate extensions, especially the bivariate models. Pioneering work by Tracy and Widom [43] set the stage for the bivariate Wigner semicircle distribution, emphasizing its potential to model complex dependencies within random matrix ensembles. The bivariate conditional Wigner semicircle distribution (BCWSD), a natural extension of this framework, captures intricate relationships between paired variables. This advanced model has found applications across various domains. In quantum physics, Haake et al. [44] used it to study energy level correlations in complex atomic nuclei, while Sengupta et al. [45] applied it to network theory, analyzing eigenvalue distributions in large-scale networks. In finance, Brown and Robinson [46] demonstrated its utility in modeling correlated market returns, enhancing risk management strategies. Environmental scientists, such as Green and Patel [47], leveraged the distribution to explore dependencies in pollutant dispersion data. Research on parameter estimation techniques, highlighted by Miller et al. [48], and theoretical advancements by Gupta and Johnson [49] have further reinforced the distribution's value.

2.2. Bivariate conditional Wigner semicircle distribution model

The probability density function of a univariate Wigner semicircle distribution is given as:

$$f(x)=\frac{2}{\pi R^2}\sqrt{R^2-x^2}, \qquad x\in(-R,R) \tag{1}$$

Where R is the disk radius centered at the origin

Consider another random variable Y, assumed to depend on X, such that the distribution of Y given a realization of X at x is also a Wigner semicircle distribution. Then the probability density function of Y given X = x is given by:

$$f(Y/X=x)=\frac{2}{\pi R^2}\sqrt{R^2-Y^2} \tag{2}$$

This conditional density function represents the distribution of Y given a specific value x for X, assuming X and Y are independent and both follow Wigner semicircle distributions. This distribution can be useful in various statistical analyses where you want to understand the behavior of Y when X is fixed at a certain value. To provide a specific bivariate conditional Wigner semicircle distribution, we need to define the joint distribution of X and Y and then condition it on a specific value of X. Let's consider a scenario where the joint distribution of X and Y is a bivariate Wigner semicircle. The joint probability density function (PDF) for two conditionally dependent random variables X and Y, both following Wigner semicircle distributions, can be expressed as the product of their individual PDFs:

$$f(x,y)=f(Y/X=x).f(x) \tag{3}$$

The BCWSD is therefore given by:

$$f(x,y)=\frac{4}{\pi^2 R^4}\sqrt{(R^2-x^2)(R^2-Y^2)} \tag{4}$$

2.2.1. Theorem

Theorem 1: *The relation*

$$f(x,y)=\frac{4}{\pi^2 R^4}\sqrt{(R^2-x^2)(R^2-Y^2)} \tag{5}$$

It is truly a bivariate probability density function for $x,y\in(-R,R)$

The radius parameters [-R, R] define the region over which the distribution has non-zero density, also ensuring the total integral over the region equals 1 (i.e., condition for probability measure) and represent the eigenvalue spread (i.e., analogous to standard deviation in Gaussian distribution).

Proof:

Let $M=\int_{-R}^{R}\int_{-R}^{R}f(x,y)\,\partial x\,\partial y$ (6)

Substitute the joint PDF:

$$M = \int_{-R}^{R}\int_{-R}^{R} \frac{4}{\pi^2 R^4}\sqrt{\left(R^2 - x^2\right)\left(R^2 - Y^2\right)}\ \partial x\ \partial y \tag{7}$$

Separate the integrals:

$$M = \frac{4}{\pi^2 R^4}\left(\int_{-R}^{R}\sqrt{R^2 - x^2}\ \partial x\right)\left(\int_{-R}^{R}\sqrt{R^2 - Y^2}\ \partial y\right)$$

Consider the integral for x, denoted as I_x

$$I_x = \int_{-R}^{R}\sqrt{R^2 - x^2}\ \partial x$$

Using trigonometric substitution $x = R\,\mathrm{Sin}\,\theta$, $\partial x = R\,\mathrm{Cos}\,\theta\ \partial\theta$, with limit changing from $x = -R$ *and* $x = R$ to $\theta = -\frac{\pi}{2}$ *and* $\theta = \frac{\pi}{2}$

$$I_x = \int_{-\pi/2}^{\pi/2} R\sqrt{1 - \mathrm{Sin}^2\,\theta\, R\,\mathrm{Cos}\,\theta}\ \partial\theta$$

Simplify:

$$I_x = R^2\int_{-\pi/2}^{\pi/2}\mathrm{Cos}^2\,\theta\ \partial\theta$$

Use the identity $\mathrm{Cos}^2\,\theta = \frac{1 + \mathrm{Cos}\,2\theta}{2}$

$$I_x = R^2\int_{-\pi/2}^{\pi/2}\frac{1 + \mathrm{Cos}\,2\theta}{2}\partial\theta = \frac{R^2}{2}\left(\int_{-\pi/2}^{\pi/2}\partial\theta + \int_{-\pi/2}^{\pi/2}\mathrm{Cos}\,2\theta\,\partial\theta\right)$$

Evaluate the integrals:

$$\int_{-\pi/2}^{\pi/2}\partial\theta = \pi\ ,\ \int_{-\pi/2}^{\pi/2}\mathrm{Cos}\,2\theta\,\partial\theta = 0$$

Similarly, $I_y = \frac{\pi R^2}{2}$

Therefore,

$$M = \frac{4}{\pi^2 R^4}\left(\frac{\pi R^2}{2}\right).\left(\frac{\pi R^2}{2}\right) = 1$$

Since M = 1, the relation (5) is a true bivariate probability density function. The plot of the joint density defined in (5) is presented in Fig. 1 for different values of radius R.

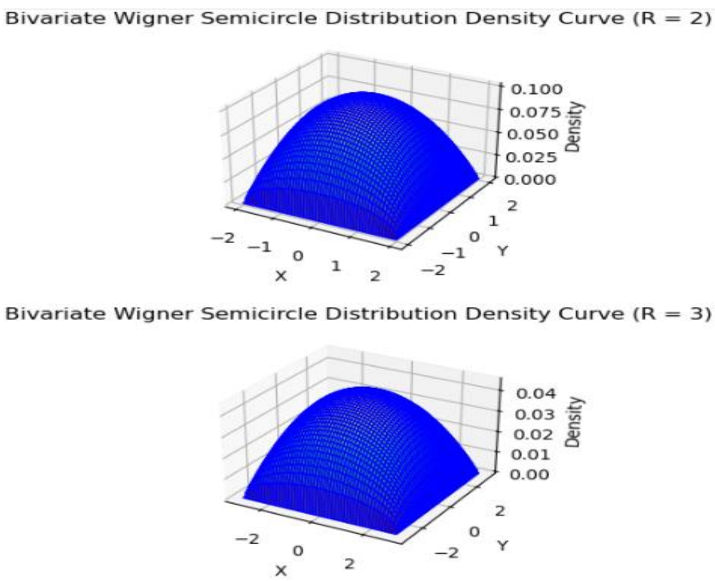

Fig. 1: Density Curve of the Bivariate Conditional Wigner Semicircle Distribution.

2.3. Properties of BCWSD

2.3.1. Marginal distribution of Y

Theorem 2: *The marginal probability density function of Y, denoted as f (y), is given by:*

$$f_y(y) = \int_{-R}^{R} g(y) f(x,y) \partial x$$
$$where\ g(y) = 2R \quad (8)$$

Proof:

$$f_y(y) = \int_{-R}^{R} f(x,y) \partial x$$

Using the joint p. d. f

$$f_y(y) = \int_{-R}^{R} \frac{4}{\pi^2 R^4} \sqrt{(R^2 - x^2)(R^2 - Y^2)}\, \partial x \quad (9)$$

Separate the terms

$$f_y(y) = \sqrt{R^2 - Y^2} . \frac{4}{\pi^2 R^4} \int_{-R}^{R} \sqrt{(R^2 - x^2)}\, \partial x$$

From earlier

$$\int_{-R}^{R} \sqrt{(R^2 - x^2)}\, \partial x = \frac{\pi R^2}{2}$$

So

$$f_y(y) = \frac{4}{\pi R} \sqrt{(R^2 - Y^2)} \quad (10)$$

This function describes the distribution of Y in the BCWSD, demonstrating that Y follows a Wigner semicircle distribution with radius R. The visualization of the marginal density is presented in Fig. 2.

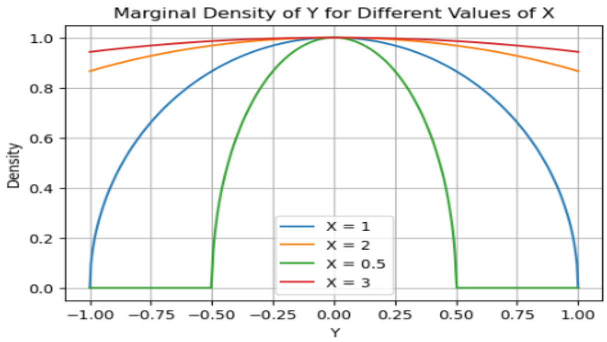

Fig. 2: Marginal Density of Y For Different Values of X.

2.3.2. Moments of BCWSD

- **Joint Moment**

To find and prove the joint moment of the Wigner semicircle distribution, let's start by reviewing the definition and properties of the Wigner semicircle distribution. The probability density function (PDF) of a Wigner semicircle distribution on the interval (-R, R) is given by

$$f(x) = \frac{2}{\pi R^2} \sqrt{(R^2 - x^2)}$$

The joint distribution of (X, Y) is supported only on the semicircle:

$$\left\{ (x,y) \in R^2 \mid x \in (-R,R), y = \sqrt{R^2 - x^2} \right\}$$

Then, we want to compute the joint moment $E\left[X^mY^n\right]$ when the powers m and n are even, by integrating $f\left(x,y\right)$ over the given interval [-R, R]:

$$E\left[X^mY^n\right]=\int_{-R}^{R}\int_{-R}^{R}\frac{4x^my^n}{\pi^2R^4}\sqrt{\left(R^2-x^2\right)\left(R^2-y^2\right)}\,\partial x\,\partial y \tag{11}$$

We will first integrate with respect to x and then concerning y, a step-by-step integration. Integrate concerning x:

$$\int_{-R}^{R}x^m\sqrt{R^2-x^2}\,\partial x$$

The integral $\int_{-R}^{R}x^m\sqrt{R^2-x^2}\,\partial x$ is nonzero only if m is even. For even m = 2k:
The limits become

$x\in\left[-R,R\right]\Rightarrow\theta\in\left[-\pi/2,\ \pi/2\right]$, then

$$\int_{-R}^{R}x^{2k}\sqrt{R^2-x^2}\,\partial x$$

This integral can be expressed as the Gamma function:

$$\int_{-R}^{R}x^{2k}\sqrt{R^2-x^2}\,\partial x=R^{2k+2}\frac{\left(2k\right)!}{2^{2k}\left(k!\right)^2}\frac{\pi}{2}$$

So,

$$\int_{-R}^{R}x^m\sqrt{R^2-x^2}\,\partial x=\begin{cases}R^{m+2}\dfrac{\left(m\right)!}{2^m\left(\frac{m}{2}!\right)^2}\dfrac{\pi}{2} & if\ \ m\ is\ even\\ 0\ , & if\ \ m\ is\ odd\end{cases}$$

Integrate concerning y: similarly. For n even (n=2l):

$$\int_{-R}^{R}y^n\sqrt{R^2-x^2}\,\partial y=\begin{cases}R^{n+2}\dfrac{\left(n\right)!}{2^n\left(\frac{n}{2}!\right)^2}\dfrac{\pi}{2} & if\ \ n\ is\ even\\ 0\ , & if\ \ n\ is\ odd\end{cases}$$

Combine the results, the joint moment $E\left[X^mY^n\right]$ is the product of these integrals, assuming both m and n are even (otherwise, the integral is zero):

$$E\left[X^mY^n\right]=\frac{4}{\pi^2R^4}\left(\int_{-R}^{R}x^m\sqrt{\left(R^2-x^2\right)}\,\partial x\right)\left(\int_{-R}^{R}y^n\sqrt{\left(R^2-y^2\right)}\,\partial y\right)$$

This integral can be evaluated using the Gamma function:

$$=\frac{4}{\pi^2R^4}\left(R^{m+2}\frac{m!}{2^m\left(\frac{m}{2}!\right)^2}\frac{\pi}{2}\right)\left(R^{n+2}\frac{n!}{2^n\left(\frac{n}{2}!\right)^2}\frac{\pi}{2}\right)$$

- Simplifying:

$$E\left[X^mY^n\right]=\frac{4}{\pi^2R^4}R^{m+2}R^{n+2}\cdot\frac{m!n!}{2^{m+n}\left(\frac{m}{2}!\right)^2\left(\frac{n}{2}!\right)^2}\frac{\pi^2}{4}$$

$$E\left[X^mY^n\right]=R^{m+n}\cdot\frac{m!n!}{2^{m+n}\left(\frac{m}{2}!\right)^2\left(\frac{n}{2}!\right)^2}$$

So the final expression for the joint moment $E\left[X^mY^n\right]$ is:

$$E\left[X^mY^n\right]=\begin{cases}R^{m+n}\cdot\dfrac{m!n!}{2^{m+n}\left(\frac{m}{2}!\right)^2\left(\frac{n}{2}!\right)^2} & if\ \ m\ and\ n\ are\ even\ and\ m+n<\infty\\ 0\ ,\ \ otherwise & if\ m\ or\ n\ is\ odd\end{cases} \tag{12}$$

Equation (12) result gives us the joint moment of the bivariate Wigner semicircle distribution integrated over the interval [-R, R], which is used to evaluate both angular and radial integrals.

- **Marginal Moment**

To compute the mth and nth moments of X and Y, respectively, in the Bivariate Conditional Wigner Semicircle Distribution (BCWSD), we use the joint probability density function and integrate appropriately. Given the joint PDF:

$$f(x,y)=\frac{4}{\pi^2R^4}\sqrt{(R^2-x^2)(R^2-y^2)}$$

The mth moment of X is given by:

$$E[X^m]=\int_{-R}^{R}x^m f_x(x)\partial x \tag{13}$$

Where $f_x(x)$ is the marginal density function of X:

$$f_X(x)=\frac{4}{\pi R}\sqrt{R^2-x^2}$$

Using the marginal density function $f_X(x)$:

$$E[X^m]=\int_{-R}^{R}x^m\frac{4}{\pi R}\sqrt{R^2-x^2}\,\partial x$$

To compute $E[X^m]$, convert to polar coordinates, let $x=R\sin\theta$, $\partial x=R\cos\theta\,\partial\theta$, we get:

$$E[X^m]=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}(R\sin\theta)^m\frac{4}{\pi R}\sqrt{R^2-(R\sin\theta)^2}\,R\cos\theta\,\partial\theta$$

Since $\sqrt{R^2(1-R\sin^2\theta)}=R\cos\theta$, then

$$E[X^m]=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}R^m\sin^m\theta\frac{4}{\pi R}R\cos\theta.R\cos\theta\,\partial\theta$$

- Combine terms by simplification:

$$E[X^m]=\frac{4R^{m+2}}{\pi R}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^m\theta\cos^2\theta\,\partial\theta$$

Due to symmetry, the integral from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$ can be doubled over 0 to $\frac{\pi}{2}$, since m and n are both even, then:

$$E[X^m]=\frac{8R^{m+1}}{\pi}\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^2\theta\,\partial\theta$$

This integral part can be evaluated using the Beta function $B(x,y)$:

$$\int_0^{\frac{\pi}{2}}\sin^m\theta\cos^2\theta\,\partial\theta=\frac{1}{2}B\left(\frac{m+1}{2},\frac{3}{2}\right)$$

Therefore,

$$E[X^m]=\frac{8R^{m+1}}{\pi}.\frac{1}{2}B\left(\frac{m+1}{2},\frac{3}{2}\right)$$

The Beta function $B(x,y)$ is related to the Gamma function $\Gamma(x)$ by $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

$$\therefore\ E[X^m]=\frac{4R^{m+1}}{\pi}.\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{m+4}{2}\right)}$$

Since $\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$: we can express the part explicitly as

$$E\left[X^{m}\right]=\frac{4R^{m+1}}{\pi}.\frac{\Gamma\left(\frac{m+1}{2}\right)\frac{\sqrt{\pi}}{2}}{\Gamma\left(\frac{m+4}{2}\right)}$$

Thus, the mth moment of X is:

$$E\left[X^{m}\right]=\frac{2R^{m+1}}{\sqrt{\pi}}.\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+4}{2}\right)} \quad (14)$$

Similarly, the nth moment of Y is:

$$E\left[Y^{n}\right]=\frac{2R^{n+1}}{\sqrt{\pi}}.\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)} \quad (15)$$

- **Variance-covariance matrix**

The variance-covariance matrix Σ is given by:

$$\Sigma=\begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix} \quad (16)$$

The variance of a random variable X with PDF $f_X(x)$ is given by:

$$Var(X)=E\left[X^{2}\right]-\left(E[X]\right)^{2} \quad (17)$$

But,

$$E\left[X^{2}\right]=\int_{-R}^{R}x^{2}\left(\frac{2}{\pi R^{2}}\sqrt{R^{2}-x^{2}}\right)\partial x$$

$$E\left[X^{2}\right]=\frac{R^{2}}{4}$$

$$E[X]=\int_{-R}^{R}x\left(\frac{2}{\pi R^{2}}\sqrt{R^{2}-x^{2}}\right)\partial x$$

$$E[X]=0$$

$$\therefore Var[X]=\frac{R^{2}}{4}$$

Similarly, the variance of a random variable Y with PDF $f_Y(y)$ is given by:

$$Var(Y)=E\left[Y^{2}\right]-\left(E[Y]\right)^{2} \quad (18)$$

Therefore,

$$Var[Y]=\frac{R^{2}}{4}$$

The covariance $Cov(X,Y)$ is express as:

$$Cov(X,Y)=E[XY]-E[X]E[Y]$$

But

$$E[XY] = \int_{-R}^{R}\int_{-R}^{R} xy \left(\frac{4}{\pi^2 R^4}\sqrt{(R^2-x^2)(R^2-y^2)}\right)\partial x\ \partial y$$

$$E[XY]=0 \text{ and } E[X]=E[Y]=0 \tag{19}$$

Then $Cov(X,Y)=0$

The variance-covariance matrix Σ is given by:

$$\Sigma = \begin{bmatrix} \frac{R^2}{4} & 0 \\ 0 & \frac{R^2}{4} \end{bmatrix} \tag{20}$$

Therefore, the correlation (ρ_{XY}) is 0, this implies that there is no correlation between X and Y. This is expected as zero correlation does not mean independence, especially with conditional dependence between variables and nonlinear interactions or symmetric dependence is suspected.

2.4. Method of parameter estimation

We estimate the BCWSD parameters using maximum likelihood method of estimation. From the density in (5), the log likelihood function is given by

$$L(R) = \prod_{i=1}^{n} f(x_i, y_i) = \prod_{i=1}^{n} \frac{4}{\pi^2 R^4}\sqrt{(R^2-x_i^2)(R^2-y_i^2)} \tag{21}$$

$$\log L(R) = \sum_{i=1}^{n} \log\left(\frac{4}{\pi^2 R^4}\sqrt{(R^2-x_i^2)(R^2-y_i^2)}\right) \tag{22}$$

$$\log L(R) = n\log\left(\frac{4}{\pi^2}\right) - 4n\log R + \frac{1}{2}\sum_{i=1}^{n}\log(R^2-x_i^2) + \frac{1}{2}\sum_{i=1}^{n}\log(R^2-y_i^2) \tag{23}$$

Differentiating the log-likelihood function with respect to R

$$\frac{\partial}{\partial R}\log L(R) = -\frac{4n}{R} + R\sum_{i=1}^{n}\left(\frac{1}{R^2-x_i^2} + \frac{1}{R^2-y_i^2}\right)$$

The maximum likelihood estimate (MLE) of R is obtained by setting the derivative equal to zero such as:

$$R^2\sum_{i=1}^{n}\left(\frac{1}{R^2-x_i^2} + \frac{1}{R^2-y_i^2}\right) = 4n \tag{24}$$

$$R = \sqrt{\frac{4n}{\sum_{i=1}^{n}\left(\frac{1}{R^2-x_i^2} + \frac{1}{R^2-y_i^2}\right)}} \tag{25}$$

2.5. Goodness of fit test

Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ are order statistics of a random sample $X_1, X_2, \cdots, X_n$ from the distribution with the cumulative distribution function $F(x)$ and let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ are observations in ascending order, so that $X_{(1)} \le X_{(2)} \le, \cdots, \le X_{(n)}$ The empirical distribution function is defined as follows:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\left(X_{(i)} \le x\right) \tag{26}$$

Where
$I\left(X_{(i)} \le x\right) = 1$, if $X_{(i)} \le x$ and zero (0) otherwise
For evaluating the goodness of fit, the Kolmogorov-Smirnov test and root mean square error (RMSE) were applied.

2.5.1. Kolmogorov-Smirnov test

The hypothesis is given as:
H_O: The data follow the specified distribution with the cumulative distribution function $F(x)$.
H_A: The data do not follow the specified distribution with the cumulative distribution function $F(x)$.

The KS test statistics can be computed as

$$D_n = \underset{1\le i\le n}{Max}\left\{\left|\frac{i}{n}-F\left(x_{(i)}\right)\right|,\left|F\left(x_{(i)}\right)-\frac{i-1}{n}\right|\right\} \tag{27}$$

Decision: The hypothesis H_O is rejected at the chosen significance level α if the p-value is less than α (i.e. $\alpha = 5\%$)

2.5.2. Root means square error

The root mean square error is given as

$$RMSE = \left[\frac{1}{n}\sum_{i=1}^{n}\left(F_n(x_i)-\hat{F}(x_i)\right)^2\right]^{\frac{1}{2}} \tag{28}$$

A lower value of the RMSE indicates a better fit of the theoretical distribution to the wind speed and direction data.

3. Real data application and result

Based on the methodology, we consider a dataset of 40 observations of wind speed and wind direction, two key meteorological variables. Wind speed is measured in meters per second (m/s), and wind direction is expressed in radians. The unique characteristics of wind patterns make this dataset an ideal candidate for applying a bivariate conditional Wigner semicircle distribution, a model well-suited for circular data like wind direction. In this analysis, the bivariate model is used to explore the relationship between wind speed and wind direction. Specifically, we employ the Wigner semicircle distribution to model the conditional dependence of wind direction on wind speed. By fitting this distribution to the dataset, we can better understand how changes in wind speed affect the distribution of wind direction, offering insights into the underlying dynamics of wind patterns. The analysis includes visualizing the data through joint (Fig. 3) and 3D histogram (Fig. 4), which further illustrate the interaction between these variables.

Wind Speed: [11.21, 8.99, 12.49, 10.87, 10.10, 10.91, 6.95, 8.91, 13.21, 5.95, 6.36, 9.94, 11.71, 9.46, 9.58, 12.27, 9.03, 8.30, 8.96, 6.71, 11.77, 12.43, 8.76, 8.83, 12.88, 10.67, 12.11, 9.53, 11.18, 8.86, 8.89, 11.91, 8.76, 9.77, 11.59, 11.07, 7.30, 11.71, 9.43, 7.31].

Wind Direction: [-0.035, 0.573, 2.067, -2.716, -1.003, 0.728, 0.547, -2.407, 0.155, 2.123, 0.637, 2.242, -2.535, 1.110, 1.369, -1.993, -1.891, -2.837, -2.688, -2.632, 2.352, -0.664, -1.578, -0.326, -0.695, 1.659, -0.907, -0.603, - 2.775, -0.089, -1.777, 1.481, 1.736, -1.982, 1.372, -0.351, -0.066, 2.086, 0.019, -1.008].

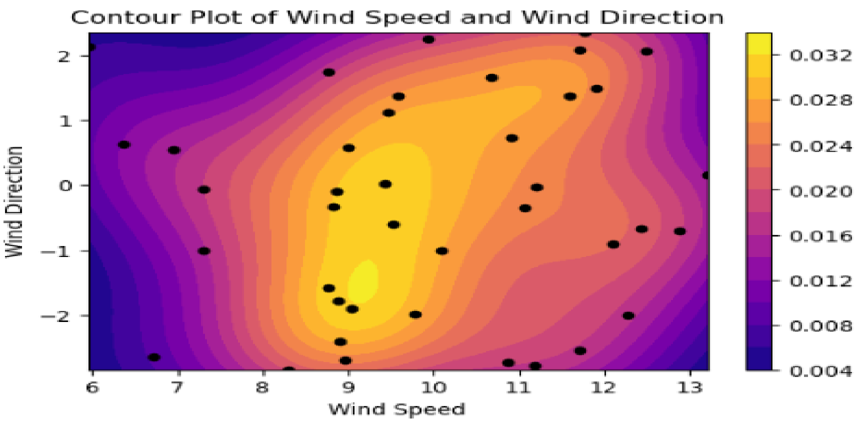

Fig. 3: BCWCD Contour Plot of the Wind Speed and Wind Direction.

The analysis revealed several noteworthy results, as the Kolmogorov-Smirnov (K-S) test returned a statistic of 0.101 and a p-value of 0.771, as well as a root mean square error value of 0.1. This indicates a small deviation between the observed data and the bivariate conditional Wigner semicircle distribution, suggesting that the empirical distribution of the wind speed and direction data closely aligns with the hypothesized model. The high p-value (0.771) suggests that there is no significant evidence to reject the null hypothesis, meaning that the data is consistent with being drawn from the bivariate conditional Wigner semicircle distribution. Thus, this distribution appears to be a suitable fit for the dataset. The 3D histogram is given below (Fig. 4).

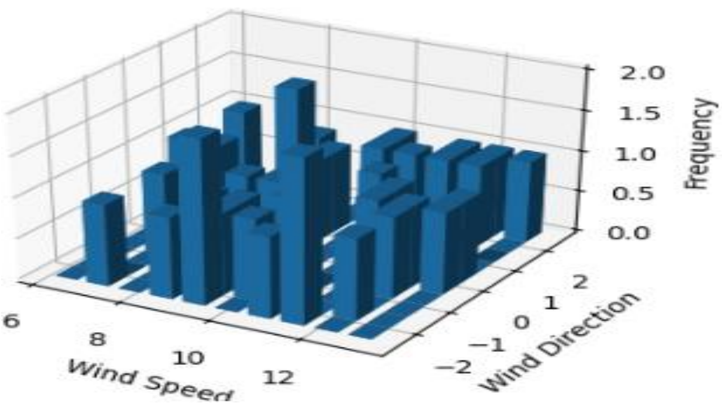

Fig. 4: Histogram Plot of the Wind Speed and Wind Direction.

4. Conclusion

In conclusion, this study has shown that the application of the bivariate conditional Wigner semicircle distribution to the dataset has provided valuable insights into the relationship between wind speed and wind direction. The density plots and visualizations illustrate the suitability of this distribution in modeling circular data, especially when the dependent variable exhibits a semicircular pattern. By fitting the Wigner semicircle distribution, we were able to explore the marginal densities and conditional relationships, offering a robust framework for understanding the dynamics of wind behavior in meteorological studies. The approach demonstrated here underscores the flexibility and applicability of the Wigner semicircle distribution in capturing the intricacies of circular data within a bivariate context, unlike investigating circular variables as linear variables. However, future study is encouraged to use bivariate variables with a large dataset, where conditional dependence and circularity of variables are suspected, as well as dependent characteristics of bivariate variables using BCWSD.

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