International Journal of Advanced Statistics and Probability, *2 (2) (2014) 102-107*
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www.sciencepubco.com/index.php/IJASP
doi: 10.14419/ijasp.v2i2.3397
Research Paper

Parameter estimation for multiple weibull populations under joint type-II censoring

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Abstract

In this paper, we introduce the maximum likelihood estimation for k Weibull populations under joint type II censored scheme and different special cases have been obtained. The asymptotic variance covariance matrix and approximate confidence region based on the asymptotic normality of the maximum likelihood estimators have been obtained. A numerical example is considered to illustrate the proposed estimators.

Keywords: *Approximate Inference; Coverage Probabilities; Joint Type II Censored Scheme; Maximum Likelihood Estimation; Weibull Distribution.*

1. Introduction

Censoring schemes are used to reduce the costs of experiments and to accelerate the performing of the design. There are various types of censored data to be dealt with in the analysis of lifetime experiments see [Lawless [6]]. Almost all of these types of data are concerned with the one-sample problems. But, there are situations in which the experimenter plans to compare different populations. In such problems, the joint censoring scheme has been suggested in the literature. As mentioned by Balakrishnan and Rasouli [3] and Rasouli and Balakrishnan [8], a joint censoring scheme is quite useful in conducting comparative lifetime test of products coming from different units within the same facility.
The joint censoring scheme is of practical significance in conducting comparative life tests of products from different lines within the same facility. Suppose products are being manufactured by different lines within the same facility, and that k independent samples of sizes $n_h, 1 \le h \le k$ are selected from these k lines and placed simultaneously on a life-testing experiment. In order to reduce the cost of the experiment as well as the experimental time, the experimenter may choose to terminate the experiment after a certain number (say, r) of failures has been observed altogether. In this situation, one may be interested in either point or interval estimation of the mean lifetimes of units produced by these k lines.
Let us suppose that $(X_1,...,X_N)$ are N jointly distributed random variables, with $\{X_1,...,X_N\} = \{X_{11},...,X_{1n_1}; X_{21},...,X_{2n_2}; X_{k1},...,X_{kn_k}\}$, with $N = \sum_{h=1}^{k} n_h$. Suppose $X_{11}, X_{12},...,X_{1n_1}$ are the lifetimes of n_1 specimens from production line A_1, and are independent and identically distributed (iid) variables from a population with cdf $F_1(x)$ and pdf $f_1(x)$. Similarly, $X_{21}, X_{22},...,X_{2n_2}$ are the lifetimes of n_2 specimens from production line A_2, and are assumed to be a sample from pdf $f_2(x)$ and cdf $F_2(x)$, and so on, with $X_{k1}, X_{k1},...,X_{kn_k}$ denoting the lifetimes of n_k specimens from production line A_k being iid variables from pdf $f_k(x)$ and cdf $F_k(x)$. Denote the order statistics of these k random samples by $W_1 \le W_2 \le ... \le W_N$, where N is the total sample size.
Let r denote a pre-fixed total number of failures to be observed. Then, under the joint type-II censoring scheme for the k-samples, the observable data consist of (z, w), where $w = (w_1, w_2,...,w_r)$, $w_i \in \{X_{h_i 1}, X_{h_i 2},...,X_{h_i n_i}\}$ for $1 \le h_1, h_2,...,h_r \le k$,

h_i indicating the production line where w_i is from. Moreover, associated to $(h_1, h_2, ..., h_r)$ let us define $z = (z_1(h), z_2(h), ... z_r(h))$ as follows

$$z_i(h) = \begin{cases} 1, & if \ h = h_i \\ 0, & otherwise. \end{cases}$$

Let $M_r(h) = \sum_{i=1}^{r} z_i(h)$ denote the number of $X_h -$ failures in W for $1 \le h \le k$ and $r = \sum_{h=1}^{k} M_r(h)$. Then the likelihood of (Z, W) is given by Balakrishnan and Feng [4] as

$$L = c_r \prod_{i=1}^{r} \prod_{h=1}^{k} \left(f_h(w_i)\right)^{z_i(h)} \prod_{h=1}^{k} \left(\bar{F}_h(w_r)\right)^{n_h - M_r(h)}, \quad -\infty < w_1 < w_2 < ... < w_r < \infty, \tag{1}$$

where $\bar{F}_h(w_r) = 1 - F_h(w_r)$ and $c_r = \dfrac{\prod_{h=1}^{k} n_h!}{\prod_{h=1}^{k} (n_h - M_r(h))!}$.

In the literature, Balakrishnan and Rasouli [3] developed likelihood inference for the parameters of two exponential populations under joint type-II censoring. They developed inferential methods based on maximum likelihood estimates (MLE) and compared their performance with those based on some other approaches such as Bootstrap. Shafay et al. [10] derived the Bayesian inference for the unknown parameters of two exponential populations under joint type II censoring they developed with the use of squared-error, linear-exponential and general entropy loss functions. The problem of predicting the future failure times, both point and interval prediction, based on the observed joint type-II censored data is obtained; see also Rasouli and Balakrishnan [8] for a generalization of their results to progressive type-II censoring for the parameters of two exponential populations. Balakrishnan and Feng [4] generalized Balakrishnan and Rasouli [3], Rasouli and Balakrishnan [8] and Shafay et al. [10] works by considered a jointly type II censored sample arising from h independent exponential populations. Ashour and Abo-Kasem [1] derived Bayesian and non-Bayesian estimators for two generalized exponential populations under joint type II censored scheme. Finally Ashour and Abo-Kasem [2] obtained MLEs for two Weibull populations under joint type II censored scheme.
In this paper, we discuss the maximum likelihood estimation for k Weibull populations under joint type II censored scheme in section 2. The asymptotic variance covariance matrix and approximate confidence region based on the asymptotic normality of the maximum likelihood estimators have been obtained in section 3. The performance analysis of the obtained estimators is carried out by conducting a simulation study in section 4. Finally, in section 5, we use a numerical example to illustrate all the methods of inference developed here.

2. Maximum likelihood estimators

Suppose that the k populations are Weibull with density and distribution functions as $f_h(x) = \dfrac{\alpha_h}{\theta_h}\left(\dfrac{x}{\theta_h}\right)^{\alpha_h - 1} \exp\left(-\dfrac{x}{\theta_h}\right)^{\alpha_h}$ And $F_h(x) = 1 - \exp\left(-\dfrac{x}{\theta_h}\right)^{\alpha_h}$, $\alpha_h, \theta_h > 0, x > 0,$ for $1 \le h \le k$, respectively. In this case, the likelihood function in (1) becomes

$$L = C_r \prod_{h=1}^{k} \left(\frac{\alpha_h}{\theta_h}\right)^{m_r(h)} \prod_{i=1}^{r} \prod_{h=1}^{k} \left[\left(\frac{w_i}{\theta_h}\right)^{\alpha_h - 1} \exp\left(-\frac{w_i}{\theta_h}\right)^{\alpha_h}\right]^{z_i(h)} \prod_{h=1}^{k} \left[\exp\left(-\frac{w_r}{\theta_h}\right)^{\alpha_h}\right]^{n_h - m_r(h)} \tag{2}$$

Therefore, to obtain the MLE's of α_h and θ_h we find the first derivatives of the natural logarithm of the likelihood function (2) with respect to α_h and θ_h , we get the following equations

$$\frac{\partial \ln L}{\partial \alpha_h} = \frac{m_r(h)}{\alpha_h} + \sum_{i=1}^{r} z_i(h) \ln\left(\frac{w_i}{\theta_h}\right) - \sum_{i=1}^{r} z_i(h) \left(\frac{w_i}{\theta_h}\right)^{\alpha_h} \ln\left(\frac{w_i}{\theta_h}\right) - (n_h - m_r(h)) \left(\frac{w_r}{\theta_h}\right)^{\alpha_h} \ln\left(\frac{w_r}{\theta_h}\right),$$

$$\frac{\partial \ln L}{\partial \theta_h} = \left(\frac{\alpha_h}{\theta_h}\right)\left[-m_r(h) + \left[\sum_{i=1}^{r} z_i(h) \left(\frac{w_i}{\theta_h}\right)^{\alpha_h} + (n_h - m_r(h)) \left(\frac{w_r}{\theta_h}\right)^{\alpha_h}\right]\right]. \tag{3}$$

By equating (3) to zero, we get the following MLEs of $\hat{\alpha}_h$ and $\hat{\theta}_h$ for $1 \le h \le k$ as

$$\frac{(n_h - m_r(h))(w_r)^{\hat{\alpha}_h} \ln(w_r) + \sum_{i=1}^{r} z_i(h)(w_i)^{\hat{\alpha}_h} \ln(w_i)}{(n_h - m_r(h))(w_r)^{\hat{\alpha}_h} + \sum_{i=1}^{r} z_i(h)(w_i)^{\hat{\alpha}_h}} - \frac{1}{\hat{\alpha}_h} = \frac{1}{m_r(h)} \sum_{i=1}^{r} z_i(h) \ln(w_i) ,$$

which can be solved by using an iterative numerical method, and

$$\hat{\theta}_h = \left[\frac{(n_h - m_r(h))(w_r)^{\hat{\alpha}_h} + \sum_{i=1}^{r} z_i(h)(w_i)^{\hat{\alpha}_h}}{m_r(h)} \right]^{\frac{1}{\hat{\alpha}_h}} \tag{4}$$

Special cases

From equation (2), different special cases can be obtained such as:

1) For $h=2$, we obtain the two Weibull populations under joint type II censored introduced by Ashour and Abo-Kasem [2] with MLEs as

$$\frac{(n_1 - m_r(1))(w_r)^{\hat{\alpha}_1}\ln(w_r) + \sum_{i=1}^{r} z_i\,(w_i)^{\hat{\alpha}_1}\ln(w_i)}{(n_1 - m_r(1))(w_r)^{\hat{\alpha}_1} + \sum_{i=1}^{r} z_i\,(w_i)^{\hat{\alpha}_1}} - \frac{1}{\hat{\alpha}_1} = \frac{1}{m_r(1)}\sum_{i=1}^{r} z_i \ln(w_i),$$

$$\frac{(n_2 - m_r(2))(w_r)^{\hat{\alpha}_2}\ln(w_r) + \sum_{i=1}^{r}(1-z_i)(w_i)^{\hat{\alpha}_2}\ln(w_i)}{(n_2 - m_r(2))(w_r)^{\hat{\alpha}_2} + \sum_{i=1}^{r}(1-z_i)(w_i)^{\hat{\alpha}_2}} - \frac{1}{\hat{\alpha}_2} = \frac{1}{m_r(2)}\sum_{i=1}^{r}(1-z_i)\ln(w_i),$$

$$\hat{\theta}_1 = \left[\frac{(n_1 - m_r(1))(w_r)^{\hat{\alpha}_1} + \sum_{i=1}^{r} z_i\,(w_i)^{\hat{\alpha}_1}}{m_r(1)} \right]^{\frac{1}{\hat{\alpha}_1}},$$

and

$$\hat{\theta}_2 = \left[\frac{(n_2 - m_r(2))(w_r)^{\hat{\alpha}_2} + \sum_{i=1}^{r}(1-z_i)(w_i)^{\hat{\alpha}_2}}{m_r(2)} \right]^{\frac{1}{\hat{\alpha}_2}}.$$

2) For $1 \le h \le k$ and $\alpha_h = 1$, we obtain multiple exponential populations under joint type-II censoring introduced by Balakrishnan and Feng [4].

3) For $h = 2$ and $\alpha_h = 1$, we obtain two exponential populations under joint type-II censoring introduced by Balakrishnan and Rasouli [3].

Remark: From the MLEs in (4), it is evident that when $M_r(h) = \sum_{i=1}^{r} z_i(h) = 0$ or r, $\hat{\alpha}_h$ or $\hat{\theta}_h$ do not exist, respectively. Hence, the MLEs in (4) are only conditional MLEs, conditioned on $1 \le M_r(h) \le r-1$.

3. Approximate inference

The approximate asymptotic variance-covariance matrix for α_h and θ_h can be obtained by inverting the information matrix with the elements that are negative of the expected values of the second order derivatives of logarithms of the likelihood functions. Cohen [5] concluded that the approximate variance covariance matrix may be obtained by replacing expected values by their MLE's. To obtain elements for information matrix, let $I(\alpha_k,\theta_k) = \left(I_{i,j}(\alpha_k,\theta_k)\right), i,j = 1,2,\dots,2k$, denote the Fisher information matrix of the parameters $(\alpha_1,\alpha_2,\dots,\alpha_k)$ and $(\theta_1,\theta_2,\dots,\theta_k)$, where $I_{i,j}(\alpha_k,\theta_k) = -E\left(\frac{\partial^2 \ln L}{\partial \alpha_{k\,i}\,\partial \theta_{k\,j}}\right)$. We have $I_{i,j}\left[(\alpha_1,\alpha_2,\dots,\alpha_k,\theta_1,\theta_2,\dots,\theta_k)\right] = 0$ if $ik \ne jk$. Consequently, the observed Fisher information matrix is given by

$$I(\hat{\alpha}_1,\hat{\alpha}_2,\dots,\hat{\alpha}_k,\hat{\theta}_1,\hat{\theta}_2,\dots,\hat{\theta}_k)$$
$$= -Diag\left(\left.\frac{\partial^2 \ln L}{\partial \alpha_1}\right|_{\alpha_1=\hat{\alpha}_1}, \left.\frac{\partial^2 \ln L}{\partial \alpha_2}\right|_{\alpha_2=\hat{\alpha}_2}, \dots, \left.\frac{\partial^2 \ln L}{\partial \alpha_k}\right|_{\alpha_k=\hat{\alpha}_k}; \left.\frac{\partial^2 \ln L}{\partial \theta_1}\right|_{\theta_1=\hat{\theta}_1}, \left.\frac{\partial^2 \ln L}{\partial \theta_2}\right|_{\theta_2=\hat{\theta}_2}, \dots, \left.\frac{\partial^2 \ln L}{\partial \theta_k}\right|_{\theta_k=\hat{\theta}_k} \right)$$

Where

$$\left.\frac{\partial^2 \ln L}{\partial \alpha_h}\right|_{\alpha_h=\hat{\alpha}_h} = \frac{m_r(h)}{\alpha_h^2} + \sum_{i=1}^{r} z_i(h)\left(\frac{w_i}{\theta_h}\right)^{\alpha_h}\left[\ln\left(\frac{w_i}{\theta_h}\right)\right]^2 + (n_h - m_r(h))\left(\frac{w_r}{\theta_h}\right)^{\alpha_h}\left[\ln\left(\frac{w_r}{\theta_h}\right)\right]^2,$$

$$\left.\frac{\partial^2 \ln L}{\partial \theta_h}\right|_{\theta_h=\hat{\theta}_h} = -m_r(h)\left(\frac{\alpha_h}{\theta_h}\right) + \left(\frac{\alpha_h}{\theta_h}\right)\left(\frac{\alpha_h+1}{\theta_h}\right)\left[\sum_{i=1}^{r} z_i(h)\left(\frac{w_i}{\theta_h}\right)^{\alpha_h} + (n_h - m_r(h))\left(\frac{w_r}{\theta_h}\right)^{\alpha_h}\right],$$

and if $ik = jk$, we obtain

$$\frac{-\partial^2 \ln L}{\partial\alpha_h \partial\theta_h} = \frac{m_r(h)}{\theta_h} - \sum_{i=1}^{r} z_i(h)\left(\frac{1}{\theta_h}\right)\left(\frac{w_i}{\theta_h}\right)^{\alpha_h}\left[1+\alpha_h \ln\left(\frac{w_i}{\theta_h}\right)\right] - (n_h - m_r(h))\left(\frac{1}{\theta_h}\right)\left(\frac{w_r}{\theta_h}\right)^{\alpha_h}\left[1+\alpha_h \ln\left(\frac{w_r}{\theta_h}\right)\right]$$

Using the asymptotic normality of the MLEs, we can express the approximate $100(1-\alpha)\%$ confidence intervals for $(\alpha_1,\alpha_2,...,\alpha_k)$ and $(\theta_1,\theta_2,...,\theta_k)$.

Suppose that $\hat{\delta}$ is the MLE of the parameter vector $\delta=(\alpha_1,\alpha_2,...,\alpha_k;\theta_1,\theta_2,...,\theta_k)$. Denote the Fisher information matrix corresponding to δ by I_δ and $\phi=\lim_{n\to\infty} n I_\delta^{-1}$. Then, $\hat{\delta}$ is asymptotically normal distributed (see Serfling [9]), i.e., $\sqrt{n}(\hat{\delta}-\delta) \sim N(0,\phi)$. In particular, let $\left(\hat{S}_{\hat{\alpha}_h}\right)^2 = \hat{\phi}_{(i,i)}/n, i=1,2,...,k$ are the (i,i) elements in the matrix $\hat{\phi}=n\hat{I}_\delta^{-1}$ and $\hat{I}_\delta$ is the estimator of I_δ. Therefore, asymptotic normality confidence intervals of δ_h, $h=1,2,...,k$ with confidence level $100(1-\alpha)\%$ are given by

$\hat{\alpha}_h \pm Z_{\alpha/2}\hat{S}_{\hat{\alpha}_h}$ And $\hat{\theta}_h \pm Z_{\alpha/2}\hat{S}_{\hat{\theta}_h}$,

where $Z_{(1-\alpha)/2}$ denotes the upper $(1-\alpha)/2$ percentage point of the standard normal distribution.

4. Simulation results and discussion

A simulation study was carried out to evaluate the performance of the MLEs and also the 95% approximate confidence intervals discussed in the preceding sections. We considered different sample sizes for three populations $(i.e., h=3)$ as $n_1=20,30,60,120$, $n_2=20,35,75$ and $n_3=20,35,75$, and different choices of $r=30,40,50,60,80,100,120,160,200$. We also chose the parameters $(\alpha_1,\alpha_2,\alpha_3,\theta_1,\theta_2,\theta_3)$ to be $(1,1.5,2,0.5,0.7,0.9)$. For these cases, we computed the MLEs for the parameters $(\alpha_1,\alpha_2,\alpha_3,\theta_1,\theta_2,\theta_3)$, root mean squared errors $\sqrt{MSE}$, the 95% approximate confidence intervals, the average widths and the corresponding coverage probabilities. We repeated this process 5000 times and computed the average values of all the estimates. The average value of the MLEs and $\sqrt{MSE}$ summarized in tables 1. From these values, it is clear that the MLEs have a moderate bias when the essential sample size r is small even when the sample sizes (n_1,n_2,n_3) are not small. This bias also seems to affect the approximate confidence intervals based on normality as they are not centered properly in this case. However, the biases of the MLEs become negligible when r increases, and MSE of all the estimates decrease with increasing r even when the sample sizes (n_1,n_2,n_3) are small, as is evident from table 1.

In table 2, the coverage probabilities of 95% approximate confidence intervals and the average widths of $(\alpha_1,\alpha_2,\alpha_3,\theta_1,\theta_2,\theta_3)$ for different sample sizes for three populations and different choices of r. From these values, it is clear that the approximate confidence intervals have its coverage probability to be very nearly 95%.

Table 1: The Average Value of the MLEs $(\alpha_1,\alpha_2,\alpha_3,\theta_1,\theta_2,\theta_3)$ and ($\sqrt{MSE}$) For Small, Moderate and Large Values of (n_1,n_2,n_3) and Different Choices of r

(n_1,n_2,n_3)	r	$\alpha_1=1$		$\alpha_2=1.5$		$\alpha_3=2$		$\theta_1=0.5$		$\theta_2=0.7$		$\theta_3=0.9$	
		$\hat{\alpha}_1$	$\sqrt{MSE}$	$\hat{\alpha}_2$	$\sqrt{MSE}$	$\hat{\alpha}_3$	$\sqrt{MSE}$	$\hat{\theta}_1$	$\sqrt{MSE}$	$\hat{\theta}_2$	$\sqrt{MSE}$	$\hat{\theta}_3$	$\sqrt{MSE}$
(20,20,20)	30	1.097	0.312	1.671	0.529	2.501	0.6	0.518	0.166	0.708	0.18	1.115	0.453
	40	1.078	0.262	1.577	0.378	2.176	0.562	0.513	0.137	0.709	0.135	0.995	0.224
	50	1.071	0.237	1.532	0.305	2.083	0.557	0.51	0.125	0.702	0.115	0.91	0.132
(30,35,35)	40	1.07	0.253	1.419	0.328	2.385	0.662	0.515	0.145	0.75	0.173	1.403	0.876
	60	1.052	0.208	1.369	0.256	2.136	0.668	0.51	0.112	0.721	0.113	1.095	0.292
	80	1.045	0.183	1.425	0.221	1.994	0.612	0.508	0.102	0.697	0.09	0.932	0.113
(60,35,35)	60	1.038	0.168	1.794	0.501	2.603	0.456	0.506	0.089	0.667	0.133	1.074	0.409
	80	1.029	0.144	1.64	0.294	2.273	0.443	0.506	0.078	0.695	0.103	1.008	0.215
	100	1.026	0.129	1.611	0.248	2.043	0.448	0.505	0.072	0.701	0.089	0.937	0.121
(60,75,75)	80	1.034	0.166	1.355	0.247	2.123	0.675	0.507	0.095	0.759	0.127	1.392	0.629
	120	1.026	0.141	1.31	0.238	2.071	0.705	0.505	0.077	0.729	0.082	1.132	0.283
	160	1.022	0.126	1.375	0.185	1.953	0.64	0.504	0.071	0.701	0.063	0.956	0.099
(120,75,75)	100	1.02	1.087	1.786	0.429	2.543	0.385	0.504	0.072	0.647	0.12	1.087	0.372
	160	1.014	0.696	1.58	0.189	2.2	0.454	0.502	0.055	0.696	0.07	1.022	0.184
	200	1.012	0.088	1.559	0.160	2.012	0.465	0.502	0.051	0.701	0.062	0.95	0.098
(75,75,120)	100	1.027	0.146	1.42	0.223	1.916	0.617	0.505	0.083	0.738	0.113	1.281	0.477
	160	1.02	0.123	1.375	0.191	1.817	0.641	0.504	0.068	0.716	0.073	1.058	0.191
	200	1.018	0.113	1.419	0.16	1.79	0.597	0.503	0.064	0.702	0.062	0.952	0.083

Table 2: Simulated Coverage Probabilities (CP) and the Average Widths of the 95% Confidence Intervals of for Some Small, Moderate and Large Values of (n_1, n_2, n_3) and R

(n_1,n_2,n_3)	r	$\alpha_1=1$		$\alpha_2=1.5$		$\alpha_3=2$		$\theta_1=0.5$		$\theta_2=0.7$		$\theta_3=0.9$	
		CP (%)	Length	CP (%)	Length	CP (%)	Length	CP (%)	Length	CP (%)	Length	CP (%)	Length
(20,20,20)	30	93.9	1.046	95.8	1.929	98.06	2.58	95.02	0.591	96.24	0.695	97.32	2.007
	40	94.1	0.917	95.84	1.494	97.9	1.811	94.56	0.511	95.46	0.527	97.7	0.967
	50	94.06	0.83	96.18	1.239	97.38	1.389	94.3	0.472	95.28	0.457	98.32	0.631
(30,35,35)	40	94.42	0.904	97.1	1.416	99.14	1.986	95.3	0.529	97.12	0.771	97.94	3.282
	60	94.9	0.76	97.96	1.049	99.36	1.301	95	0.427	96.6	0.481	98.02	1.061
	80	94.72	0.674	96.82	0.898	98.26	1.019	94.82	0.391	96	0.367	98.48	0.538
(60,35,35)	60	94.46	0.609	96.38	1.723	98.9	2.41	94.82	0.344	95.74	0.511	97.62	1.745
	80	94.66	0.533	98.08	1.269	99.46	1.602	94.6	0.299	95.08	0.397	97.48	0.859
	100	94.36	0.482	98	1.048	99.18	1.222	95.16	0.278	94.3	0.337	97.66	0.52
(60,75,75)	80	94.88	0.624	98.16	0.938	99.66	1.329	95	0.363	97.78	0.549	98.18	2.078
	120	94.66	0.533	98.42	0.704	99.54	0.885	95.08	0.301	97.42	0.351	98.12	0.806
	160	94.8	0.476	97.54	0.613	98.82	0.704	94.98	0.278	96.52	0.263	98.6	0.406
(120,75,75)	100	95.8	0.463	96.48	1.361	99.28	2.019	94.74	0.274	96.08	0.434	97.74	1.656
	160	95.5	0.376	98.46	0.854	99.62	1.087	94.86	0.213	95.76	0.282	97.48	0.628
	200	95.46	0.341	98.24	0.714	99.42	0.84	94.92	0.198	94.54	0.239	97.78	0.384
(75,75,120)	100	94.84	0.551	97.9	0.973	98.58	1.072	95.18	0.319	97.54	0.5	97.3	1.335
	160	94.72	0.463	98.4	0.709	98.04	0.689	95.16	0.263	96.86	0.311	97.58	0.499
	200	94.44	0.426	97.98	0.635	96.6	0.579	95.04	0.249	96.48	0.257	98.32	0.313

5. Illustrative example

Nelson [7], (Ch. 10, Table 4.1) has given times to breakdown in minutes of an insulating fluid subjected to high voltage stress. The failure times were observed in the form of groups with each group reporting data on 10 insulating fluids. For the purpose of illustrating the methods of inference detailed in the preceding sections, let us consider the following three groups of samples of failure time data presented in table 3.

Table 3: Failure Time Data as Three Groups of Insulating Fluids

Group	Data									
1	0.31	0.66	1.54	1.70	1.82	1.89	2.17	2.24	4.03	9.99
2	0.00	0.18	0.55	0.66	0.71	1.30	1.63	2.17	2.75	10.60
3	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	4.75

Suppose the samples of sizes $n_1 = 10, n_2 = 10$ and $n_3 = 10$ in table 3 are from three Weibull populations with $(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)$. Suppose joint type-II censoring with r as 12, 13 and 15 had been enforced on these data. For example, table 4 presents the jointly type-II censored data that would have been obtained from the data in table 3 with r = 15.

Table 4: Jointly Type-II Censored Data Observed from Table 3 with R = 15

w	0.00	0.18	0.31	0.49	0.55	0.64	0.66	0.66	0.71	0.82	0.93	1.08	1.30	1.54	1.63
Z(1)	0	0	1	0	0	0	1	0	0	0	0	0	0	1	0
Z(2)	1	1	0	0	1	0	0	1	1	0	0	0	1	0	1
Z(3)	0	0	0	1	0	1	0	0	0	1	1	1	0	0	0

We then computed the MLEs of $(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)$ and the estimates of their standard deviations for the choices of r = 12; 13; 15 and these are presented in table 5.

Table 5: The Mles and the Estimates of Their Standard Deviations Based on Jointly Type-II Censored Data from Table 4.

r	MLEs	$\widehat{SD}$
	$(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3)$	$(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3)$
12	(1.236,0.552,3.337,3.568,2.328,1.207)	(0.856,0.236,1.372,3.716,2.146,0.18)
13	(1.022,0.624,2.308,5.502,1.752,1.481)	(0.696,0.241,0.935,6.852,1.211,0.319)
15	(1.239,0.678,1.653,3.736,1.521,1.923)	(0.683,0.239,0.663,2.535,0.859,0.576)

We have also computed the estimates of the covariance matrix of $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ and these are presented in table 6. From the results in tables 5 and 6, we find the estimates to be quite stable excepting α_3.

Table 6: Estimates of the Covariance Matrix of the MLEs Based on Jointly Type-II Censored Data from Table 4

r	Covariance matrix $\left(\rho(\alpha_i, \theta_j)\right)_{i,j}$
12	$\text{II} = \begin{pmatrix} 0.733 & 0 & 0 & -2.658 & 0 & 0 \\ 0 & 0.056 & 0 & 0 & -0.241 & 0 \\ 0 & 0 & 1.882 & 0 & 0 & -0.109 \\ -2.658 & 0 & 0 & 13.811 & 0 & 0 \\ 0 & -0.241 & 0 & 0 & 4.606 & 0 \\ 0 & 0 & -0.109 & 0 & 0 & 0.032 \end{pmatrix}$
13	$\text{II} = \begin{pmatrix} 0.485 & 0 & 0 & -3.966 & 0 & 0 \\ 0 & 0.058 & 0 & 0 & -0.094 & 0 \\ 0 & 0 & 0.874 & 0 & 0 & -0.129 \\ -3.966 & 0 & 0 & 46.956 & 0 & 0 \\ 0 & -0.094 & 0 & 0 & 1.466 & 0 \\ 0 & 0 & -0.129 & 0 & 0 & 0.101 \end{pmatrix}$
15	$\text{II} = \begin{pmatrix} 0.466 & 0 & 0 & -1.258 & 0 & 0 \\ 0 & 0.057 & 0 & 0 & -0.034 & 0 \\ 0 & 0 & 0.439 & 0 & 0 & -0.163 \\ -1.258 & 0 & 0 & 6.425 & 0 & 0 \\ 0 & -0.034 & 0 & 0 & 0.738 & 0 \\ 0 & 0 & -0.163 & 0 & 0 & 0.332 \end{pmatrix}$

Table 7 presents the 95% confidence intervals for $(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)$ based on the approximate method corresponding to the cases *r* = 12, 13 and *r* = 15.

Table 7: The 95% Approximate Confidence Intervals for $(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2, \theta_3)$ Based on Jointly Type-II Censored Data from Table 4

r	CI for α_1	CI for α_2	CI for α_3	CI for θ_1	CI for θ_2	CI for θ_3
12	(0, 2.914)	(0.088,1.014)	(0.651, 6.026)	(0,10.852)	(0, 6.534)	(0.854,1.56)
13	(0, 2.386)	(0.077,1.097)	(2.167, 4.141)	(0,18.933)	(0, 4.125)	(0.857, 2.106)
15	(0, 2.577)	(0.081,1.147)	(2.041, 2.952)	(0, 8.704)	(0, 3.204)	(0.795, 3.052)

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