International Journal of Advanced Statistics and Probability, 12 (1) (2025) 33-39

International Journal of Advanced Statistics and Probability

Website: www.sciencepubco.com/index.php/IJASP
https://doi.org/10.14419/n5f0mc55
Research paper

Finite mixture of gamma distributions and applications

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Received: April 6, 2025, Accepted: May 1, 2025, Published: May 5, 2025

Abstract

In this paper a new one parameter probability distribution "Finite Mixture of Gamma Distributions" is suggested from three component mixture of Exponential (θ), Gamma (4, θ) and Gamma (3,θ) with mixing proportions θ^2/(〖(θ〗 ^2+6θ+2)),6θ/(〖(θ〗 ^2+6θ+2)) and 2/(〖(θ〗 ^2+6θ+2)) respectively. Its first four moments and expressions for skewness, kurtosis, and coefficient of variation have been given. Various mathematical and statistical properties of the proposed distribution, including hazard rate function, mean residual life function, and stress strength reliability, have been discussed. Estimation of the parameter has been discussed. The goodness of fit of the distribution has been discussed with two real-life data set and secondary data. The fit has been compared with one-parameter life time distributions, including Exponential, Lindley, and Chris Jerry distributions.

Keywords: *Finite Mixture Distributions; Estimation of Parameters; Hazard Rate Function; Kurtosis; Moments; Mean Residual Life Function; Skewness.*

1. Introduction

Recently, a large number of one-parameter lifetime distributions have been introduced in the statistical literature. Their ability to provide an adequate fit is often limited by the intrinsic characteristics of the data. The analysis of the complex data requires a statistical model that can capture diverse patterns and heterogeneity in the underlying population. Finite mixture models [1981] serve as a powerful tool for this purpose as they allow the representation of data as a convex combination of multiple probability distributions. These models have found extensive applications in the field, such as reliability analysis, survival studies, and risk modelling. Recently introduced one-parameter distributions are the mixtures of Exponential and Gamma distributions. These distributions give a better fit than the classical exponential and Gamma distributions. Lindley distribution [1958] is a two-component mixture of exponential (θ) and gamma (2, θ) With mixing proportions p $=\frac{\theta}{\theta+1}$. Shanker distribution [2015 b] is a mixture of exponential (θ) and gamma (2, θ) with mixing proportions p $=\frac{\theta^2}{\theta^2+1}$. Akash distribution [2015] is a mixture of exponential (θ) and gamma (3, θ) with mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$. Rama distribution [2017] is a mixture of exponential (θ) and gamma (4, θ) with mixing proportions $\frac{\theta^3}{\theta^3+6}$ and $\frac{6}{\theta^3+6}$. Suja distribution [2017] is a mixture of exponential (θ) and gamma (5, θ) With mixing proportions $\frac{\theta^4}{\theta^4+24}$ and $\frac{24}{\theta^4+24}$. Sujatha distribution [2016] is a mixture of exponential (θ), gamma (2, θ) and gamma (3, θ) With mixing proportions $\frac{\theta^2}{\theta^2+\theta+2}$, $\frac{\theta}{\theta^2+\theta+2}$ and $\frac{2}{\theta^2+\theta+2}$. Amarendra distribution [8] is a mixture of exponential (θ), gamma (2, θ), gamma (3, θ)and gamma (4, θ) with mixing proportions $\frac{\theta^3}{\theta^3+\theta^2+2\theta+6}$, $\frac{\theta^2}{\theta^3+\theta^2+2\theta+6}$, $\frac{2\theta}{\theta^3+\theta^2+2\theta+6}$ and $\frac{6}{\theta^3+\theta^2+2\theta+6}$. Aradhana distribution [2016] is a mixture of exponential (θ), gamma (2, θ) and gamma (3, θ) With mixing proportions $\frac{\theta^2}{\theta^2+2\theta+2}$. Pranav distribution [2018] proposed by Shukla KK is a mixture of exponential (θ) and Gamma (4, θ) with mixing proportion $\frac{\theta^4}{\theta^4+6}$and $\frac{6}{\theta^4+6}$. Chris Jerry distribution [2022] is a one-parameter lifetime distribution by Oneykwere and Obulezi is a mixture of exponential (θ) and Gamma (3, θ) with mixing proportion $\frac{\theta}{\theta+2}$. Komal distribution [2023] proposed by Shanker is a convex combination of exponential (θ) and Gamma (2, θ) with mixing proportion $\frac{\theta(\theta+1)}{\theta^2+\theta+1}$ and $\frac{1}{\theta^2+\theta+1}$. Pratibha distribution [2023] by Rama Shanker is a mixture of exponential (θ), Gamma (2, θ), and Gamma (4, θ) with mixing proportions $\frac{\theta^3}{\theta^3+\theta+2}$, $\frac{\theta}{\theta^3+\theta+2}$ and $\frac{2}{\theta^3+\theta+2}$. In this paper we introduce a new one-parameter lifetime distribution is a mixture of exponential (θ), gamma (4, θ) and gamma (3, θ) with their mixing proportions $\frac{\theta^2}{\theta^2+6\theta+2}$, $\frac{6\theta}{\theta^2+6\theta+2}$ and $\frac{2}{\theta^2+6\theta+2}$.The new one-parameter probability density function (PDF) is given by

$$f(x, \theta) = \frac{\theta^3}{\theta^2+6\theta+2}(1+\theta^2 x^3 + x^2)e^{-\theta x},\ x>0,\ \theta > 0 \tag{1}$$

We would name the one-parameter lifetime distribution as a Finite Mixture of Gamma distributions.
The cumulative distribution function (CDF) F (x, θ) is given as

$$F(x, \theta) = 1 - \left[1+\frac{x\theta(x^2\theta^3+3x\theta^2+x\theta+6\theta+2)}{\theta^2+6\theta+2}\right] e^{-\theta x},\ x>0,\ \theta > 0 \tag{2}$$

The survival function is given by $\left[1+\frac{x\theta(x^2\theta^3+3x\theta^2+x\theta+6\theta+2)}{\theta^2+6\theta+2}\right] e^{-\theta x}$
The variations in the PDF and CDF for different values of the parameter θ are illustrated in Figures 1 and 2.

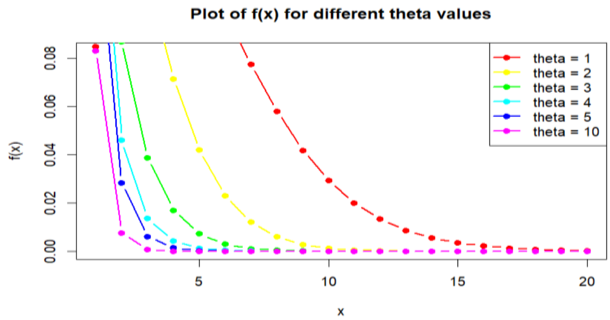

Fig. 1: The pdf of a Finite Mixture of Gamma Distributions.

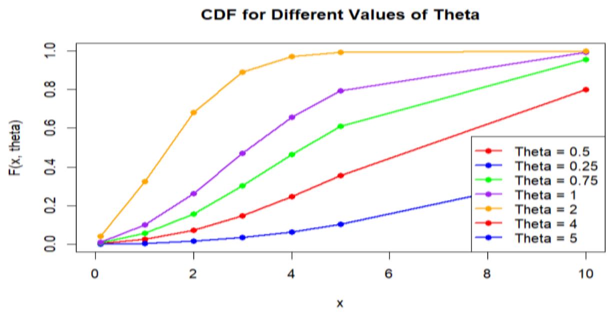

Fig. 2: The CDF of a Finite Mixture of Gamma Distributions.

2. Reliability properties

A continuous random variable having the PDF f(x, θ) and cdf F (x, θThen), the hazard rate function of the random variable h(x) is given as

$$h(x) = \lim_{\Delta x \to 0} \frac{P(X<x+\Delta x/X>x)}{\Delta x}$$

$$= \frac{f(x,\theta)}{1-F(x,\theta)}$$

Thus, the hazard rate function of a Finite mixture of Gamma distributions is

$$h(x) = \frac{\theta^3(\theta^2x^3+x^2+1)}{\theta^3(3x^2+x^3\theta)+\theta^2(6x+x^2+1)(2x+6)\theta+2} \tag{3}$$

For x=0 $h(0) = \frac{\theta^3}{\theta^2+6\theta+2} = f(0)$

The behaviour of the hazard rate function of the distribution for various values of the parameter is shown in Figure 3.

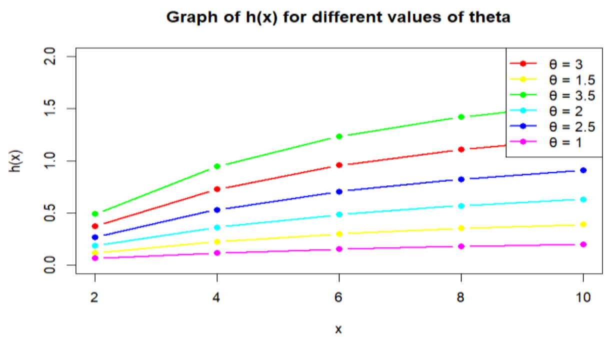

Fig. 3: The Hazard Function of a Finite Mixture of Gamma Distributions.

For a continuous random variable over the interval $(0,\infty)$ represents the life of a system. Mean Residual function (MR) measures the expected value of the remaining lifetime for the system if it has survived up to X. Consider the conditional random variable X=(X-x/X>x), x>0. Then the mean residual life function is denoted by M(x), is defined as

$$M(x) = \frac{1}{S(x)} \int_x^{\infty} 1 - F(t)dt,$$

$$S(x)=1-F(x)$$

$$=\frac{\theta^4 x^2(x+3)+\theta^2(x^2+12x+1)+\theta(4x+24)+6}{\theta[(\theta^2+6\theta+2)+x\theta(\theta^3x^2+3x\theta^2+\theta x+6\theta+2)} \quad (4)$$

For x=0, $M(0) = \frac{(\theta^2+24\theta+6)}{\theta(\theta^2+6\theta+2)}$,

What is the meaning of Finite Mixture of Gamma Distributions
The graphical representation of the mean residual function is given in Figure 4.

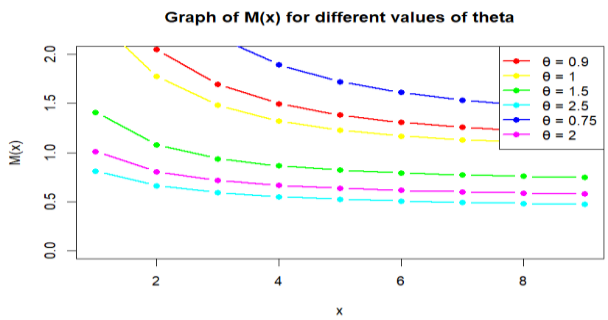

Fig. 4: The Graph of Mean Residual Function of Finite Mixture of Gamma Distributions.

3. Statistical properties

3.1. Moments and related measures

The r^{th} *The* moment of the new distribution is given by

$$\mu'_r = E[X^r]$$

$$= \frac{r!}{\theta^r}\left[\frac{\theta^2+(r+3)(r+2)(r+1)\theta+(r+2)(r+1)}{(\theta^2+6\theta+2)}\right], \ r = 1,2,3, \ldots \quad (5)$$

And then the first four moments about origin as

$$\mu'_1 = \frac{(\theta^2+24\theta+6)}{\theta(\theta^2+6\theta+2)} \quad \mu'_2 = \frac{2(\theta^2+60\theta+12)}{\theta^2(\theta^2+6\theta+2)}$$

$$\mu'_3 = \frac{6(\theta^2+120\theta+20)}{\theta^3(\theta^2+6\theta+2)} \quad \mu'_4 = \frac{24(\theta^2+210\theta+30)}{\theta^4(\theta^2+6\theta+2)}$$

Thus, the central moments are obtained as

$$\mu_2 = \frac{(\theta^4+84\theta^3+160\theta^2+96\theta+12)}{\theta^2(\theta^2+6\theta+2)^2}$$

$$\mu_3 = \frac{(2\theta^6+396\theta^5+708\theta^4+2376\theta^3+1827\theta^2+576\theta+48)}{\theta^3(\theta^2+6\theta+2)^3}$$

$$\mu_4 = \frac{\begin{pmatrix} 9\theta^8+2808\theta^7+21120\theta^6+105696\theta^5+535752\theta^4 \\ +449568\theta^3+60384\theta^2+115200\theta+720 \end{pmatrix}}{\theta^4(\theta^2+6\theta+2)^4}$$

The graph of Mean and Variance of Finite Mixture of Gamma distributions is given in figure 5.

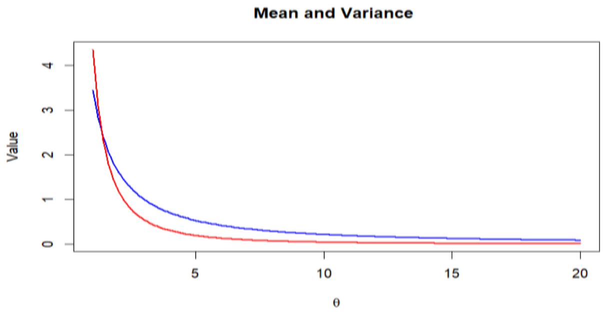

Fig. 5: The Graph of Mean and Variance of Finite Mixture of Gamma Distributions.

The coefficient of variation (CV), Coefficient of Skewness (√β1) and Coefficient of Kurtosis (β2) and index of dispersion (γ) of the above distribution is thus obtained as

$$CV = \frac{SD}{mean}$$

$$= \frac{\sqrt{(\theta^4+84\theta^3+160\theta^2+96\theta+12)}}{(\theta^2+24\theta+6)}$$

$$\beta 1 = \frac{{\mu_3}^2}{{\mu_2}^3}$$

$$\sqrt{\beta 1} = \frac{\mu_3}{{\mu_2}^{3/2}}$$

$$= \frac{(2\theta^6+396\theta^5+708\theta^4+2376\theta^3+1827\theta^2+576\theta+48)(\theta^2+6\theta+2)^{1/2}}{(\theta^4+84\theta^3+160\theta^2+96\theta+12)^{3/2}} \tag{6}$$

$$\beta 2 = \frac{\mu_4}{{\mu_2}^2}$$

$$= \frac{\begin{pmatrix}9\theta^8+2808\theta^7+21120\theta^6+105696\theta^5+535752\theta^4+ \\ 449568\theta^3+603840\theta^2+11520\theta+720\end{pmatrix}}{(\theta^4+84\theta^3+160\theta^2+96\theta+12)^2} \tag{7}$$

$$\text{Index of Dispersion} = \frac{\mu_2}{\mu_1'}$$

$$= \frac{(\theta^4+84\theta^3+160\theta^2+96\theta+12)}{\theta(\theta^2+6\theta+2)(\theta^2+24\theta+6)} \tag{8}$$

The nature of Skewness, Kurtosis ,CV and Index of dispersion is given in Figure 6,Figure 7, Figure 8 and Figure 9 respectively.

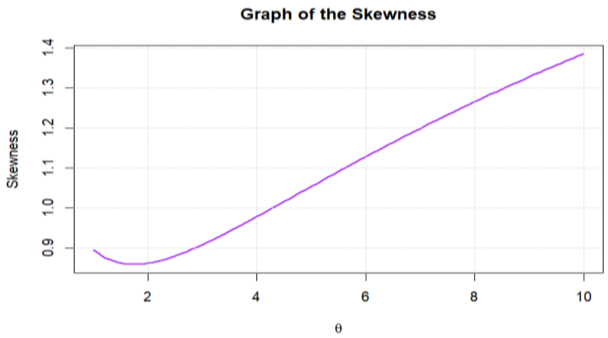

Fig. 6: The Graph of Skewness of Finite Mixture of Gamma Distributions.

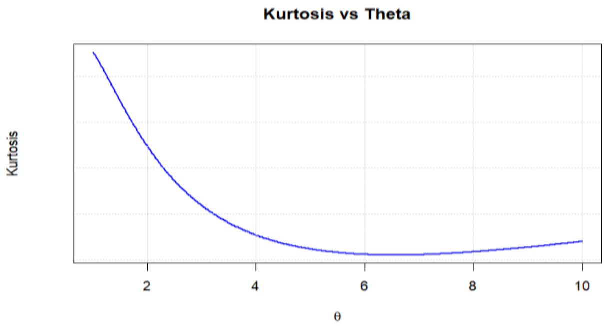

Fig. 7: The Graph of Kurtosis of Finite Mixture of Gamma Distributions.

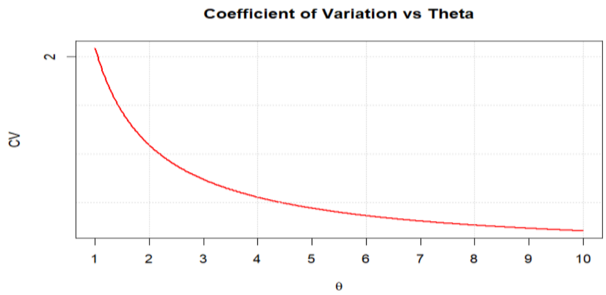

Fig. 8: The Graph of Coefficient of Variation of Finite Mixture of Gamma Distributions.

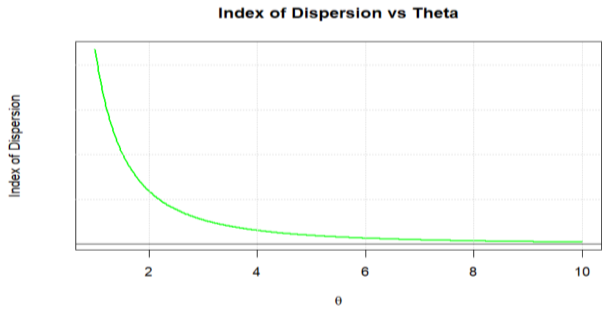

Fig. 9: The Graph of Index of Dispersion of Finite Mixture of Gamma Distributions.

4. Parameter estimation

4.1. Moment estimates of the parameter

By equating the population mean of a finite mixture of gamma distributions, which has a single parameter, to the corresponding sample mean, we obtain the moment estimator. That is, we get a third-degree polynomial equation in θ.
$\vec{x}\theta^3+\theta^2(6\vec{x}-1)+(\vec{x}-12)2\theta-6=0$, where $\vec{x}$ is the sample mean.
By solving the third-degree polynomial using Newton Newton-Raphson method, we can easily get the moment estimate for the parameter θ.

4.2. Maximum likelihood estimate (MLE)

Suppose $(x_1, x_2, x_3 \ldots x_n)$ be a random sample size n from a Finite mixture of gamma distributions. The log likelihood function logL is given by

$$\text{Log } L = n[3\log\theta - \log(\theta^2+6\theta+2)] + \sum_{i=1}^{n}\log\left(\theta^2 x_i^3 + x_i^2 + 1\right) - n\vec{x}\theta$$

The MLE of the parameter is the solution of the log likelihood function $\frac{d\log L}{d\theta} = 0$ which gives

$$\frac{3n}{\theta} - \frac{n(2\theta+6)}{\theta^2+6\theta+2} + 2\,\theta\sum_{i=1}^{n}\frac{1}{(\theta^2 x_i^3 + x_i^2 + 1)} - n\vec{x} = 0 \qquad (9)$$

This result is a non-linear equation θ , which can be solved using the Newton-Raphson method in R software to obtain the maximum likelihood estimate (MLE) of θ.

5. Application

A finite mixture of gamma distributions has been fitted for two real data sets and one secondary data set. a finite mixture of gamma distributions gives a better fit than Exponential, Chris-Jerry, and Lindley distributions. The PDFs of the selected distributions are given in Table 1.

Table 1: The PDF of the Fitted Distributions

Distribution	pdf
Exponential	$f(x, \theta) = \theta e^{-\theta x}$
Lindley	$f(x, \theta) = \frac{\theta^2}{(\theta+1)}(1+x)e^{-\theta x}$
Chris-Jerry	$f(x, \theta) = \frac{\theta^2}{(\theta+2)}(1+\theta x^2)e^{-\theta x}$

The data set given in the Table 2, Table 3 and Table 4 are considered and the values of the MLE of the parameter, -2logL, AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and AICC (Akaike Information Criterion Corrected) for Exponential, Lindley, Chris Jerry and Finite Mixture of Gamma distributions have been computed and given in Table 5. The equations used for calculating AIC, BIC and AICC are as follows

AIC = 2k-2log L, BIC = k log n – 2 log L and AICC = AIC $+\frac{2k(k+1)}{n-k-1}$

Where k is the no. of parameters in the statistical model, n is the sample size and -2logL is the maximized value of log likelihood function under the considered model.

Table 2: The Data 1 Represents the Haemoglobin- Grams Per Decilitre (G/Dl) of Randomly Selected 165 Female

12.4	12.9	11	11.1	11.7	14.2	12.5	10.3	10.4	12.9	9.9
10.7	9.8	13.2	13.6	14	9.1	13.3	11.2	11.4	10.1	12.4
11.3	10.7	9.3	9.3	9.8	9.2	10.4	9.2	14.1	10.4	10.9
11.9	11.7	10.1	10.1	10.7	9.4	11.3	10	9.7	10.6	11.2
13	11.8	10.7	9.4	9.7	10	12.4	10.9	10.6	11.1	11.6
9.4	11.4	10	10.3	10.6	10.1	13.8	12.1	11.5	11.3	
13	14.1	12.2	12.2	12.8	10.9	9.2	13.2	12.7	11.5	
13.7	9.3	13.3	13.3	14.1	11.2	9.9	14.4	13.9	12.2	
14.2	9.6	9.7	9.8	9.7	12.1	10.8	9.4	9.7	12.4	
9.2	10.1	10.6	10.7	10.6	12.3	12	10.2	10.6	12.7	
10	10.5	11.4	11.7	11.6	13.1	13.1	11.2	11.5	13.5	
10.9	10.9	12.6	12.9	12.8	13.5	14.2	12.3	12.7	13.8	
12	11.4	13.9	14.1	9.5	14.2	9.5	13.5	13.9	14	
13.1	12.2	9.4	9.6	14.1	10.4	13.8	12.1	12.3	12.8	
9	10.4	9.2	9.4	9.7	9.9	13.9	12.3	12.5	14.2	
10.7	9.8	13.6	13.8	12.9	11.9	10.7	9.9	9.4	11.8	

Table 3: The Data 2 Gives the Height in Cm of 20 Higher Secondary Students from School in Palakkad, Kerala

155	156	156	152	156	148	150	152	155	167
153	170	161	150	150	160	153	153	150	165

Table 4: The Data 3 Gives Life Expectancy at Birth Across 100 Countries in the Year 2021, Data Source Https://Www.Kaggle.Com/Dataset/ I am Sourav Banarjee/Life-Expectancy-At-Birth-Across-The-Globe

61.9824	61.6434	76.4626	80.3684	78.7104	75.3899	72.0431	78.4968	84.5265	81.5797
69.3658	61.6627	81.8787	59.821	59.2696	72.3811	71.798	78.7605	71.5983	75.3003
72.438	70.4697	63.6304	72.7504	77.5714	74.6424	71.815	61.1409	53.8947	82.6565
78.9435	78.2107	58.5983	60.3334	59.193	63.5187	72.8296	63.4174	74.0518	77.0232
77.7283	80.6301	62.3049	72.814	81.3753	72.6146	76.3767	73.67	70.2207	66.5358
67.114	82.4988	70.71	65.821	80.7422	71.694	63.7954	58.8922	62.083	59.6523
69.2368	65.6734	85.4734	70.1229	77.5804	63.1924	74.5301	67.5703	67.2398	81.9976
82.6782	82.255	82.8502	70.5003	74.2563	84.7839	69.3622	61.427	69.9774	69.5835
83.9872	73.6829	81.2033	83.01	77.1436	64.9748	82.0381	60.5943	80.1106	74.9362
73.8749	70.3779	67.4172	71.6822	83.6978	78.6729	68.0608	75.0472	60.7472	71.9112

Table 5: The MLE, -2logl, AIC, BIC and AICC of the Fitted Distributions Using the Data Set Given in Table 2, 3 and 4

Data	Distribution	MLE	-2log L	AIC	BIC	AICC
Data 1	Exponential	0.086	1136.34	1138.34	1141.32	1138.35
	Lindley	0.161	1035.05	1037.05	1040.16	1037.08
	Chris-Jerry	0.248	972.85	974.85	977.96	974.88
	Finite Mixture of Gamma Distributions	0.3050	938.03	940.03	943.74	940.06
Data 2	Distribution	MLE	-2log L	AIC	BIC	AICC
	Exponential	0.0064	241.96	243.96	244.96	244.19
	Lindley	0.012	226.79	228.79	229.79	229.02
	Chris-Jerry	0.019	218.21	220.21	221.20	220.43
	Finite Mixture of Gamma Distributions	0.019	217.9	219.9	220.8	220.1
Data 3	Distribution	MLE	-2log L	AIC	BIC	AICC
	Exponential	0.0140	2053.96	2055.96	2059.23	2055.98
	Lindley	0.027	982.19	984.19	986.80	984.23
	Chris-Jerry	0.0412	940.58	942.58	945.19	942.62
	Finite Mixture of Gamma Distributions	0.0432	937.14	939.14	941.18	939.18

From results given in table 5, it has been seen that the Finite Mixture of Gamma Distributions have the lesser AIC, BIC, AICC and –2logL values as compared to the exponential, Lindley and Chris-Jerry distributions. Hence, it can be concluded that the Finite Mixture of Gamma Distributions leads to a better fit than the exponential, Lindley, and Chris-Jerry distributions.

6. Conclusion

In this paper, a new one-parameter lifetime distribution named Finite Mixture of Gamma Distributions has been proposed. The statistical properties of the distribution, Shape, Moments, Skewness, kurtosis, hazard rate function, and mean residual life function have been discussed. Method of Moments and Maximum Likelihood Estimation have been discussed for estimating the parameter. The distribution is fitted with real data using R Programming.

Acknowledgement

The author thanks the editor and reviewer for their valuable comments to improve the quality of the paper. There are no funding agencies for this research article.

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