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A weighted inverted exponential distribution

M. A. Hussian

Department of Mathematical Statistics, Institute of Statistical Studies and Research (ISSR),Cairo University, Giza, Egypt
**Current address: Department of statistics and operations research, faculty of science, King Saud University, B.O. 2455, Riyadh 11451, KSA*
E-mail: msabry@ksu.edu.sa

Abstract

In this paper, a generalized version of the inverted exponential distribution called the weighted inverted exponential distribution is introduced. The proposed distribution is used to analyze lifetime data. Several statistical properties of the weighted inverted exponential distribution are studied and derived. Least squares estimation, maximum likelihood estimation and Bayesian estimation methods are used to evaluate the distribution parameters. Numerical simulation and a real-life data analysis are carried out to validate the robustness of the proposed distribution.

***Keywords**: Inverted exponential distribution, azzalini approach, weighted, maximum likelihood, Bayesian estimation.*

1 Introduction

Different methods may be used to introduce a shape parameter to probability distribution model and they may results in a variety of weighted distributions. For example, the gamma distribution and the generalized exponential distribution are different weighted versions of the exponential distribution. Fisher [1] first introduced the concept of weighted distribution, while Cox [2] and Zelen [3] introduced weighted distribution to present length biased sampling. Recently, Gupta and Kundu [4] have followed a similar approach as of Azzalini [5], for introducing a shape parameter to the exponential distribution. They showed that by applying Azzalini's method to the exponential distribution, a new class of weighted exponential distribution denoted by *WE* distribution could be obtained. Furthermore, they showed that the WE distribution possess some good properties and can be used as a good fit to survival time data compared to other popular lifetime distributions. Shakhatreh [6], followed a similar way to introduce the two parameter weighted exponential distribution by simply modifying Azzalini's approach to introduce two shape parameters instead of one as the case in the weighted one parameter exponential and the weighted two parameter Weibull distributions [4], [7], [8].
Most of distributional developments are made using the exponential distribution because of its simplicity and mathematical feasibility that made it the most widely used lifetime model in reliability theory. If a random variable X has an exponential distribution, $Y=1/X$ will have an inverted exponential (*IE*) distribution. The *IE* distribution has been introduced by Killer and Kamath [9] and has been discussed as a lifetime model by [10] in detail. They have obtained maximum likelihood (*ML*) estimates, confidence limits and uniformly minimum variance unbiased estimators for the parameters and reliability function of the *IE* distribution with complete samples. Abouammoh and Alshingiti [11] introduced a shape parameter in the *IE* distribution to obtain the generalized inverted exponential (*GIE*) distribution. They derived many distributional properties and reliability characteristics of the *GIE* distribution. Assuming it a good lifetime model, they have obtained maximum likelihood estimators, least square estimators and confidence intervals of the two parameters involved.
The *IE* distribution has the following cumulative distribution function (*cdf*) and probability density function (*pdf*) for $X > 0$:

$$F(x) = \exp\left(-\frac{\lambda}{x}\right), \tag{1}$$

with

$$f(x) = \frac{\lambda}{x^2}\exp\left(-\frac{\lambda}{x}\right), \tag{2}$$

where $\lambda > 0$ and $\alpha > 0$ are the scale and shape parameters respectively.

The main aim of this paper is to introduce a new generalization of the inverted exponential distribution based on a modified weighted version of Azallini's approach [5], and discuss some of its distributional developments. Azallini's approach is to introduce a shape parameter to a probability distribution function in the following way. Let $g_0(x)$ be a one-dimensional probability density function (*pdf*), and $G_0(x)$ is the corresponding one-dimensional distribution function (*cdf*) such that $G_0(x)$ exists, then

$$f(x;\lambda,\alpha) = K\, g_0(x) G_0(\alpha x), \tag{3}$$

Is a density function where *K* is the normalizing constant?

The rest of the paper is organized as follows, in section 2; the *pdf* and *cdf* of the proposed model are presented. Sections 3, 4 and 5 are devoted to discuss some basic properties of this distribution function. Estimation of the proposed distribution parameters using least squares (*LS*) estimation method, maximum likelihood (*ML*) estimation, and Bayesian (*BS*) estimation method are presents in section 6. In sections, 7 and 8 simulation study and real data analysis are carried out to test the performance of the proposed model and finally the paper is concluded.

2 PDF and CDF

A random variable T is said to have a weighted inverted exponential (*WIE*) distribution, if the *cdf* and *pdf* of $T > 0$ are given by

$$F(t;\lambda,\alpha) = (1+\alpha)\left(\exp\left(-\frac{\lambda}{t}\right) - \frac{\alpha}{(1+\alpha)} \exp\left(-\frac{\lambda(1+\alpha)}{\alpha t}\right)\right), \tag{4}$$

And

$$f(t;\lambda,\alpha) = (1+\alpha)\frac{\lambda}{t^2} \exp\left(-\frac{\lambda}{t}\right)\left(1-\exp\left(-\frac{\lambda}{\alpha t}\right)\right), \tag{5}$$

Where. λ And α are scale and shape parameters α respectively.

This model can be obtained in many ways. One of these ways is by following the approach proposed by Azallini [5], and replacing the cumulative distribution function $G_0(\alpha x)$ by its corresponding reliability function $R(x) = \bar{G}_0(\alpha x)$. The weighted inverted exponential distribution is then given by

$$f(x;\lambda,\alpha) = K\, g_0(x) \bar{G}_0(\alpha x), \tag{6}$$

Where x is an inverted exponential random variable with parameter λ, and K is the normalizing coefficient.

Another way is to consider U as a random variable distributed as Beta (α, 2), then the random variable $T = -\lambda/(\alpha \ln U)$, [4, 12, 13], will be distributed as a weighted inverted exponential distribution with *cdf* and *pdf* as in Eqs. (4) and (5) respectively. Examples of the *WIE* distribution *pdf* and *cdf* for different couples of scale and shape parameters are shown in figures 1 and 2.

3 Relationship to other distributions

The *WIE* distribution is reduced to the *IE* distribution as $\alpha \to 0$. Furthermore, the distribution of $X = 1/T$ is the weighted exponential distribution and as $\alpha \to 0$, it reduces to the one-parameter exponential distribution.

4 Reliability measures

The reliability function (*RF*), the failure rate function (*FR*), the reversed hazard rate (*RHR*) and the mean residual life (*MRL*) of the *WIE* distribution are given by.

$$R(t) = 1 - F(t;\lambda,\alpha,)$$

$$R(t) = 1-(1+\alpha)\exp\left(-\frac{\lambda}{t}\right) + \alpha \exp\left(-\frac{\lambda(1+\alpha)}{\alpha t}\right), \tag{7}$$

Where, $x > 0$ and $\lambda,\ \alpha > 0$, and the probability that a system having age x unites of time will survive to $x+t$ unites of time for, $x >$, $t > 0$ and $\lambda,\ \alpha > 0$, is

$$R(t \mid x) = \frac{R(t+x)}{R(t)} = \frac{1-(1+\alpha)\exp\left(-\frac{\lambda}{t+x}\right)+\alpha\exp\left(-\frac{\lambda(1+\alpha)}{\alpha(t+x)}\right)}{1-(1+\alpha)\exp\left(-\frac{\lambda}{t}\right)+\alpha\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t}\right)} \tag{8}$$

The *FR* (hazard rate) function $x > 0$ and $\lambda,\ \alpha > 0$, is also given by:

$$r(t) = \frac{f(t)}{R(t)} = \frac{\lambda(1+\alpha)\exp\left(-\frac{\lambda}{t}\right)\left(1-\exp\left(-\frac{\lambda}{\alpha t}\right)\right)}{t^2\left(1-(1+\alpha)\exp\left(-\frac{\lambda}{t}\right)+\alpha\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t}\right)\right)}, \tag{9}$$

And the *RHR* (also known as Mills' ratio) is as follows

$$h(x) = \frac{f(t)}{1-R(t)} = \frac{\lambda\exp\left(-\frac{\lambda}{t}\right)\left(1-\exp\left(-\frac{\lambda}{\alpha t}\right)\right)}{\left(\exp\left(-\frac{\lambda}{t}\right)-\frac{\alpha}{(1+\alpha)}\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t}\right)\right)} \tag{10}$$

Figure 3, shows the FR for the WIE distribution for different couples of shape and scale parameters. The *MRL* is very important in reliability and survival analysis [14-16]. The *MRL* of the random variable *X* is given by

$$m(t) = E(T-t \mid T>t) = \frac{\int_t^\infty \overline{F}(y)dy}{\overline{F}(t)} = \frac{\int_t^\infty 1-(1+\alpha)\exp\left(-\frac{\lambda}{y}\right)+\alpha\exp\left(-\frac{\lambda(1+\alpha)}{\alpha y}\right)dy}{1-(1+\alpha)\exp\left(-\frac{\lambda}{t}\right)+\alpha\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t}\right)},$$

Which at the end will lead to?

$$m(t) = \frac{\left(-t+\lambda(1+\alpha)\left[\Gamma\left(-1,\frac{1}{t}\right)-\Gamma\left(-1,\frac{\lambda(1+\alpha)}{\alpha t}\right)-\ln\lambda-\ln\frac{\alpha}{\lambda(1+\alpha)}\right]\right)}{1-(1+\alpha)\exp\left(-\frac{\lambda}{t}\right)+\alpha\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t}\right)} \tag{11}$$

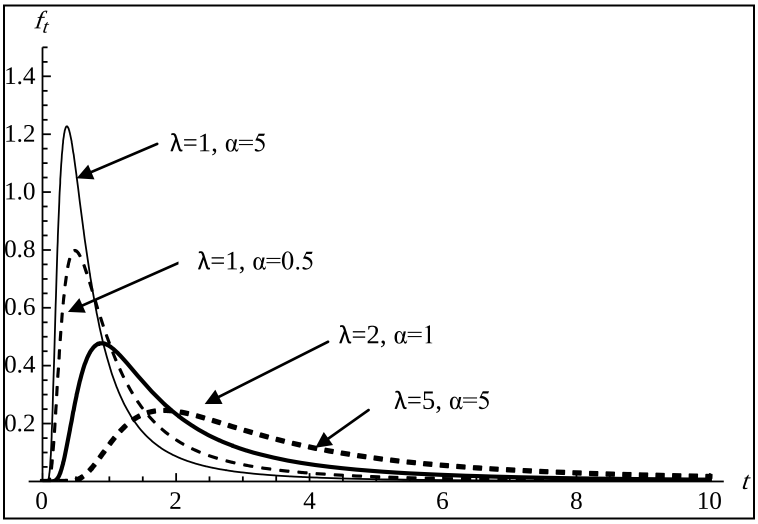

Fig. 1: Examples of the *WIE* distribution probability density function for different couples of shape and scale parameters.

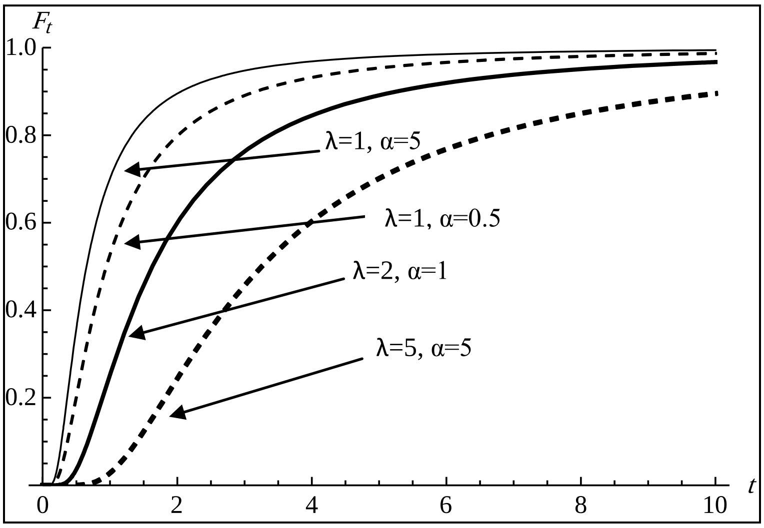

Fig. 2: Examples of the *WIE* cumulative distribution function for different couples of shape and scale parameters.

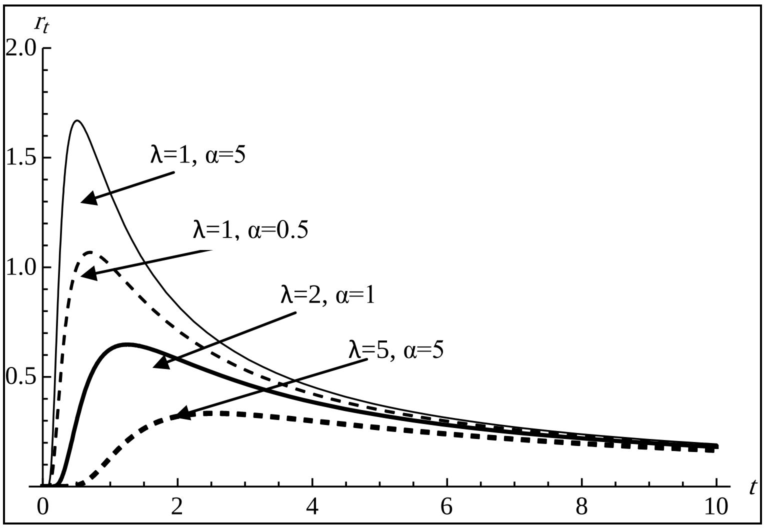

Fig. 3: Examples of the *WIE* failure rate function for different couples of shape and scale parameters

5 Moments

Let T denote a random variable with *cdf* and *pdf* as in Eqs (4) and (5). Some important characteristics of a distribution can be studied through moments; therefore, the first, second and the variance of the random variable T are given by

$$\mu_1' = -\lambda(1+\alpha)\ln\frac{\alpha}{(1+\alpha)} \tag{12}$$

$$\mu_2' = \frac{\lambda^2(1+\alpha)}{\alpha}\left(1+\alpha\ln\frac{\alpha}{(1+\alpha)}\right) \tag{13}$$

And the variance of T is given by:

$$\sigma^2 = \frac{\lambda^2(1+\alpha)}{\alpha}\left(1+\alpha\ln\frac{\alpha}{(1+\alpha)} - (1+\alpha)\left(\ln\frac{\alpha}{(1+\alpha)}\right)^2\right). \tag{14}$$

The mode of the random variable T denoted by t_0, can be obtained directly from the derivatives of the *pdf*, which at the end is the solution of

$$\lambda + \alpha(\lambda - 2t_0)\left[1-\exp\left(-\frac{\lambda}{\alpha t_0}\right)\right] = 0 \tag{15}$$

One can note that $d^2 f(t;\lambda,\alpha)/dx^2 < 0$, for all $t > 0$, which suggests that this distribution is strongly unimodal and $f(t)$ is log-concave and has an increasing likelihood ratio property [17].
The median of the random variable *T* denoted by *m*, is the solution of $F(m;\lambda,\alpha) = 0.5$ where F is the *cdf* of the random variable *T* which at the end is the solution of the nonlinear equation

$$\exp\left(-\frac{\lambda}{\alpha m}\right)\left[1+\alpha-\alpha\exp\left(-\frac{\lambda}{\alpha m}\right)\right] = 0.5 \tag{16}$$

6 Statistical inferences

6.1 The method of least squares

In this subsection, the least squares (*LS*) estimators for the unknown parameters vector parameters $\Omega = (\lambda,\alpha)$ are discussed. Let $T_1,\ldots,T_n$ be a random sample drawn from a probability distribution function with cdf $F(\cdot)$. Assume that $T_{(1)},\ldots,T_{(n)}$ are the ordered sample based on the random sample drawn. It is known that $E\left(F(T_{(i)}\right) = i/(n+1)$, $i = 1,2,\ldots,n$. The *LS* estimators are obtained by minimizing the loss function

$$Q(\Omega) = \sum_{i=1}^{n}\left(F(T_{(i)}) - E\left(F(T_{(i)})\right)\right)^2 = \sum_{i=1}^{n}\left(F(T_{(i)}) - \frac{i}{(n+1)}\right)^2. \tag{17}$$

In the case of our weighted inverted exponential distribution, Equation (17) becomes

$$Q(\Omega) = \sum_{i=1}^{n}\left((1+\alpha)\left(\exp\left(-\frac{\lambda}{t_i}\right) - \frac{\alpha}{(1+\alpha)}\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t_i}\right)\right) - \frac{i}{(n+1)}\right)^2. \tag{18}$$

The normal equations for the *LS* methods are obtained by minimizing equation (18) with respect to λ and α . That is,

$$\sum_{i=1}^{n}\frac{1}{t_i}\left[\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t_i}\right) - \exp\left(-\frac{\lambda}{t_i}\right)\right]\left((1+\alpha)\exp\left(-\frac{\lambda}{t_i}\right) - \alpha\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t_i}\right) - \frac{i}{(n+1)}\right) = 0 \tag{19}$$

And

$$\sum_{i=1}^{n}\left((1+\alpha)\exp\left(-\frac{\lambda}{t_i}\right) - \alpha\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t_i}\right) - \frac{i}{(n+1)}\right)\left\{\exp\left(-\frac{\lambda}{t_i}\right) - \exp\left(-\frac{\lambda(1+\alpha)}{\alpha t_i}\right)\right.$$
$$\left. + \frac{\lambda\alpha}{(1+\alpha)}\exp\left(-\frac{\lambda(1+\alpha)}{\alpha t_i}\right)\left[\frac{(1+\alpha)}{\alpha^2 t_i} - \frac{1}{\alpha t_i}\right]\right\} = 0 \tag{20}$$

By solving Eqs. (19) And (20), the LS estimators $\hat{\lambda}_{LS}$ and $\hat{\alpha}_{LS}$ of λ and α will be obtained.

6.2 The maximum likelihood method

In this subsection, parameter estimation of the *WIE* distribution by the maximum likelihood method is considered. It is shown in this section that the *ML* estimators of the unknown vector parameters $\Omega = (\lambda,\alpha)$ are in the region R = {($\Omega = (\lambda,\alpha)$: where $\lambda \geq 0$, and $\alpha \geq 0$ }. Let $T_1,\ldots,T_n$ be a random sample from *WIE* distribution with scale and shape parameters λ and α respectively. The likelihood function based on $T_1,\ldots,T_n$ is given by

$$L(\lambda,\alpha) = \lambda^n (1+\alpha)^n \exp\left(-\lambda\sum_{i=1}^{n}\frac{1}{t_i}\right)\prod_{i=1}^{n}\left(\frac{1}{t_i^{\,2}}\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]\right) \tag{21}$$

And therefore, the log-likelihood function is as follows:

$$\ln L(\lambda,\alpha) = n\ln\lambda + n\ln(1+\alpha) - \lambda\sum_{i=1}^{n}\frac{1}{t_i} - 2\sum_{i=1}^{n}\ln t_i + \sum_{i=1}^{n}\ln\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right] \tag{22}$$

Our goal is to maximize the log-likelihood function with respect to λ and α . This is done by partially differentiate (22) with respect to λ and α and equating the result to zero, we obtain the following normal equations,

$$\frac{\partial \ln L}{\partial \lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\frac{1}{t_i}+\sum_{1}^{n}\frac{\frac{1}{\alpha t_i}\exp\left(-\frac{\lambda}{\alpha t_i}\right)}{\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]}=0\,, \tag{23}$$

$$\frac{\partial \ln L}{\partial \alpha}=\frac{n}{1+\alpha}-\sum_{i=1}^{n}\frac{\frac{\lambda}{\alpha^2 t_i}\exp\left(-\frac{\lambda}{\alpha t_i}\right)}{\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]}=0 \tag{24}$$

By solving Eqs. (23) And (24), the maximum likelihood estimators of the parameters of the *WIE* distribution are obtained.

6.3 Asymptotic distribution and asymptotic confidence intervals

The first derivatives of the log likelihood function of the *WIE* distribution with respect to λ and α are given by Eqs (23) and (24). The observed Fisher information matrix of the parameter vector $\Omega=(\lambda,\alpha)$, of sample size *n* is given by the 2 × 2 symmetric matrix $I(\Omega)$ whose *ij*-th element is given by the second derivatives $I_{ij}=\partial^2 \ln L(\Omega)/\partial\omega_i\partial\omega_j$, $i\ j=1{,}2$ where

$$I_{11}=\frac{-n}{\lambda^2}-\sum_{i=1}^{n}\left(\frac{\frac{1}{\alpha^2 t_i^2}\exp\left(-\frac{2\lambda}{\alpha t_i}\right)}{\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]^2}+\frac{\frac{1}{\alpha^2 t_i^2}\exp\left(-\frac{\lambda}{\alpha t_i}\right)}{\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]}\right),$$

$$I_{22}=\frac{-n}{(1+\alpha)^2}-\sum_{i=1}^{n}\left(\frac{\frac{\lambda^2}{\alpha^4 t_i^2}\exp\left(-\frac{2\lambda}{\alpha t_i}\right)}{\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]^2}+\frac{\frac{\lambda^2}{\alpha^4 t_i^2}\exp\left(-\frac{\lambda}{\alpha t_i}\right)-\frac{2\lambda^2}{\alpha^3 t_i}\exp\left(-\frac{\lambda}{\alpha t_i}\right)}{\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]}\right),$$

$$I_{12}=I_{21}=\sum_{i=1}^{n}\left(\frac{\frac{\lambda}{\alpha^3 t_i^2}\left[\exp\left(-\frac{2\lambda}{\alpha t_i}\right)+\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]}{\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]^2}-\frac{\frac{1}{\alpha^2 t_i}\exp\left(-\frac{\lambda}{\alpha t_i}\right)}{\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]}\right),$$

and

$$I(\Omega)=\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22}\end{pmatrix}_{\lambda=\hat{\lambda},\ \alpha=\hat{\alpha}} \tag{25}$$

While all the parameters are assumed unknown, the asymptotic distribution of the *ML* estimators immediately follows from the Fisher information matrix of λ andα. That is, if $\hat{\lambda}_{ML}$ and $\hat{\alpha}_{ML}$ are the *ML* estimators of λ andα, then,

$$\sqrt{n}\ \left(\left(\hat{\lambda}_{ML}-,\lambda\right),\ \left(\hat{\alpha}_{ML}-\alpha\right)\right)\rightarrow N_2(0,\Sigma_2) \tag{26}$$

Where $\Sigma_2=I^{-1}(\Omega)$. Once we obtain the asymptotic distribution of the *ML* estimators, the approximate (1-ε) 100% confidence intervals for the parameters λ and α are given by $\hat{\lambda}\pm Z_{\varepsilon/2}\sqrt{V(\hat{\lambda})}$, and $\hat{\alpha}\pm Z_{\varepsilon/2}\sqrt{V(\hat{\alpha})}$ respectively where $V(\hat{\lambda})$ and $V(\hat{\alpha})$, are the diagonal elements of the matrix $I^{-1}(\Omega)$, and $Z_{\varepsilon/2}$ is the upper ε/2 percentile of the standard normal distribution, respectively.

6.4 Bayesian estimation

The Bayes estimators of λ and α denoted as $\hat{\lambda}_{BS}$ and $\hat{\alpha}_{BS}$ respectively, are obtained under the assumption that λ and α are independent random variables. In this model, it is assumed that λ follows a non-informative prior distribution with the density function $\pi(\lambda)=1/\lambda$. This is called Jeffrey's prior [18], and is obtained by performing a logarithmic transformation on λ. The prior distribution ofα, denoted as $\pi(\alpha)$, is assumed to be exponential with pdf

$$\pi(\alpha)=b\ \exp(-b\alpha); \qquad \alpha>0, \tag{27}$$

Based on the above assumptions and from the likelihood function in equation (21), the joint density of the data, λ and α can be obtained as

$$L(data,\lambda,\alpha)=L(data;\lambda,\alpha)\ \pi(\lambda)\ \pi(\alpha), \tag{28}$$

Therefore, the joint posterior density of λ and α given the data can be obtained as

$$L(\lambda,\alpha/data)=\frac{L(data;\lambda,\alpha)\ \pi(\lambda)\ \pi(\alpha)}{\int_0^\infty\int_0^\infty L(data;\lambda,\alpha)\ \pi(\lambda)\ \pi(\alpha)\ d\lambda\, d\alpha}, \tag{29}$$

And the joint posterior density of λ and α becomes,

$$\pi(\lambda,\alpha/data)=b\lambda^{n-1}(1+\alpha)^n\ \exp\left(-b\alpha-\lambda\sum_{i=1}^{n}\frac{1}{t_i}-2\sum_{i=1}^{n}\ln t_i+\sum_{i=1}^{n}\ln\left[1-\exp\left(-\frac{\lambda}{\alpha t_i}\right)\right]\right),$$

Therefore, the Bayesian estimator of λ and α are the expected value of the posterior distribution for both λ and α and are given by

$$\hat{\lambda}_{BS}=E(\lambda/x,y)=\int_0^\infty\int_0^\infty \lambda\pi(r/\lambda,\alpha)\ d\lambda\ d\alpha, \tag{30}$$

and

$$\hat{\alpha}_{BS}=E(\alpha/x,y)=\int_0^\infty\int_0^\infty \alpha\pi(r/\lambda,\alpha)\ d\lambda\ d\alpha. \tag{31}$$

Solving the above integrals will lead to the Bayesian estimators of the parameters of the *WIE* distribution.

7 Simulation results

In this section, a study of the behavior of the estimators from the unknown parameter λ and α is considered. The estimation is made when the two parameters λ and α are unknown. Without loss of generality, let us consider that the population value of the scale parameter λ is equal to 1 and the shape parameter α assumes population values 0.2, 0.5, 1 and 3. We compute the estimated values of λ and α , biases and the mean squared error (MSE). This is done by generating samples from the *WIE* distribution and considering samples of sizes of 20, 35, 50 and 100. The simulations are based on 10000 replications. Tables 1 and 2 represents biases and MSEs of the estimators of λ and α respectively. The estimation methods are the least squares method (*LS*), the maximum likelihood method (*ML*) and the Bayesian estimation method (*BS*).

From tables 1 and 2, it can be noted that, the MSE and the bias of both α and λ decreases as the sample size increases. On the other hand, when $0<\alpha\le 1$, the estimator of α is underestimated while for $\alpha>1$, the estimator of α is overestimated.

8 Numerical example

In this section, we illustrate the new distribution given by (3) and (4), to test its performance as another lifetime distribution compared to the exponential, inverted exponential and the weighted exponential distributions. The data used were taken from Gupta and Akham [19], and represent millions of revolutions to failure for 23 ball bearings in fatigue test. The data are:

17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.48	51.96	54.12	55.56	84.12
67.80	68.64	68.64	68.88	93.12	98.64	105.12	105.84	127.92	128.04	173.40	

Table 3 shows the fitting results and goodness of fit summary represented by the Kolmogorov Smirnov, Anderson Darling and the chi-Squared statistics. It can be noted that the performance of the weighted inverted exponential distribution is better than the exponential and the inverted exponential distributions for this set of data, while it is slightly different from the weighted exponential distribution.

Table 1: Parameter estimation results based on bias and MSE for α when the population values of α varies from 0.2 to 3 and the population value of λ is equal to 1.

	n	Bias			MSE		
		$\hat{\alpha}_{LS}$	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{LS}$	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{BS}$
α=0.2	20	-0.04621	-0.02275	-0.03429	5.43315	4.67636	4.91612
	35	-0.03025	-0.02046	-0.03173	3.79508	3.26645	3.43393
	50	-0.01352	-0.01724	-0.02747	2.87598	2.47538	2.60230
	100	-0.00724	-0.00986	-0.01306	2.37371	2.04307	2.14782
α=0.5	20	-0.04149	-0.01827	-0.03127	4.30531	3.70561	3.89560
	35	-0.02785	-0.01606	-0.02241	3.10938	2.67623	2.81344
	50	-0.01837	-0.01018	-0.01719	2.26470	1.94924	2.04918
	100	-0.01072	-0.00454	-0.00954	1.53057	1.31737	1.38491
α=1	20	-0.23976	-0.11711	-0.19055	3.76709	3.24236	3.40860
	35	-0.15519	-0.09852	-0.14629	2.45484	2.11290	2.22123
	50	-0.10408	-0.04683	-0.09953	1.41222	1.21551	1.27783
	100	-0.08825	-0.01407	-0.04094	1.02080	0.87861	0.92366
α=3	20	0.60245	0.41728	0.52082	5.52174	4.75265	4.99627
	35	0.44817	0.33593	0.40859	3.75954	3.23586	3.40177
	50	0.31251	0.21387	0.25538	3.01707	2.59681	2.72995
	100	0.26953	0.07165	0.13016	1.56148	1.34397	1.41288

Table 2: Parameter estimation results based on bias and MSE for λ when the population values of α varies from 0.2 to 3 and the population value of λ is equal to 1.

	n	Bias			MSE		
		$\hat{\lambda}_{LS}$	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{BS}$	$\hat{\lambda}_{LS}$	$\hat{\lambda}_{ML}$	$\hat{\lambda}_{BS}$
α=0.2	20	0.85458	0.67809	0.76413	5.9867882	5.41707	5.5265021
	35	0.63708	0.48129	0.55723	4.6352084	4.19411	4.2788366
	50	0.57384	0.42407	0.49708	4.1520166	3.7569	3.8327944
	100	0.28950	0.16679	0.22661	3.9097632	3.5377	3.6091663
α=0.5	20	0.59047	0.43911	0.51290	4.7363315	4.28561	4.3721851
	35	0.33129	0.20460	0.26637	3.91656	3.54385	3.6154405
	50	0.26961	0.14879	0.20769	2.5272496	2.28675	2.3329454
	100	-0.87251	-0.88465	-0.87873	2.3730562	2.14723	2.1906069
α=1	20	0.51173	0.36787	0.43800	3.995867	3.61561	3.6886502
	35	0.31417	0.18911	0.25008	3.3418711	3.02385	3.0849358
	50	0.25412	0.13478	0.19296	2.8313816	2.56194	2.6136946
	100	0.21636	0.10061	0.15704	2.245641	2.03194	2.0729879
α=3	20	0.38882	0.25666	0.32109	3.4764367	3.14561	3.2091555
	35	0.23854	0.12067	0.17813	3.0151826	2.72825	2.7833643
	50	0.20703	0.09217	0.14816	2.3486429	2.12514	2.1680707
	100	0.15282	0.04311	0.09659	2.0330061	1.83954	1.8767012

Table 3: Fits and Goodness of Fit Summary for *WIE* distribution in comparison with some lifetime distributions

Distribution	Kolmogorov-Smirnov	Anderson-Darling	Chi-Squared
inverted exponential (*IE*)	0.30679	2.8114	9.0521
Exponential (*E*)	0.2223	2.4855	2.8529
Weighted exponential (*WE*)	0.219647	0.30214	0.91891
Weighted inverted exponential (*WIE*)	0.220478	0.29245	0.95973

9 Conclusion

In this paper, we have introduced a weighted version of the inverted exponential distribution. Abouammoh and Alshingiti [6] generalized the inverted exponential distribution where, they combined the kumaraswamy and the inverted exponential distributions to introduce the kumaraswamy inverted exponential distribution known as the generalized inverted exponential (GIE) distribution. The proposed distribution can be seen as another generalization to the inverted exponential distribution where, the idea is to use a modified Azallini's approach [2] to introduce the weighted inverted exponential distribution. The weighted exponential distribution is the reciprocal of the proposed model, while the inverted exponential distribution and the exponential distribution are limiting distributions of this model. Moreover, the proposed distribution is flexible and can be used quite effectively to analyze positively skewed data.

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