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On the Conharmonic and Concircular curvature tensors of almost C(λ) Manifolds

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Abstract

The object of the present paper is to characterize certain curvature conditions on conharmonic and concircular curvature tensors on almost C(λ) manifolds. In this paper we study conharmonically flat, ξ-conharmonically flat, concircularly flat and ξ-concircularly flat almost C(λ) manifolds.

Keywords: *Almost $C(\lambda)$ manifolds, Conharmonic and concircular curvature tensor, ξ-conharmonically flat and ξ-concircularly flat.*

1 Introduction

The notion of almost C(λ) manifolds was introduced by D. Janssen and L. Vanhecke [3]. Further Z. Olszak and R. Rosca [8] investigated such manifolds. Again S. V. Kharitonova [5] studied conformally flat almost C(λ) manifolds. In the paper [1] the author studied Ricci tensor and quasi-conformal curvature tensor of almost $C(\lambda)$ manifolds. In paper [4] the authors have studied ξ-conharmonic flat Generalized Sasakian-Space-Forms. Also in paper [7] the authors have studied ξ-concircularly flat 3-dimensional quasi-Sasakian manifold. Our present work is motivated by these works. The present paper is organized as follows:

After introduction we give some preliminaries in Section 2. In Section 3 and Section 4 we study conharmonically and concircularly flat almost $C(\lambda)$ manifolds. In Sections 5 and Section 6 we investigate respectively ξ-conharmonically and ξ-concircularly flat almost C(λ) manifolds.

2 Preliminary notes

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M such that [2]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1. \tag{1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad X, Y \in T(M). \tag{2}$$

Then also

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi). \tag{3}$$

$$g(\phi X, X) = 0. \tag{4}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X. \tag{5}$$

If an almost contact Riemannian manifold M satisfies the condition

$$S = ag + b\eta \otimes \eta \tag{6}$$

for some functions a and b in $C^\infty(M)$ and S is the Ricci tensor, then M is said to be an η-Einstein manifold.
If, in particular, a=0 then this manifold will be called a special type of η-Einstein manifold.
An almost contact manifold is called an almost $C(\lambda)$ manifold if the Riemannian curvature R satisfies the following relation [5]

$$R(X,Y)Z = R(\phi X,\phi Y)Z - \lambda[Xg(Y,Z) - g(X,Z)Y - \phi Xg(\phi Y,Z) + g(\phi X,Z)\phi Y], \tag{7}$$

where, $X, Y, Z \in$TM and λ is a real number. From (7) we have,

$$R(X,Y)\xi = R(\phi X,\phi Y)\xi - \lambda[X\eta(Y) - \eta(X)Y]. \tag{8}$$

On an almost C(λ) manifold, we also have [1]

$$QX = AX + B\eta(X)\xi. \tag{9}$$

where, $A = -\lambda(2n-1)$, $B = -\lambda$ and Q is the Ricci-operator.

$$\eta(QX) = (A+B)\eta(X). \tag{10}$$

$$S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y). \tag{11}$$

$$r = -4n^2\lambda. \tag{12}$$

$$S(X,\xi) = (A+B)\eta(X). \tag{13}$$

$$S(\xi,\xi) = (A+B). \tag{14}$$

$$g(QX,Y) = S(X,Y). \tag{15}$$

3 Conharmonically flat almost $C(\lambda)$ manifolds

Definition 3.1. The conharmonic curvature tensor C of type (1,3) on a Riemannian manifold (M, g) of dimension $(2n+1)$ is defined by [6]

$$C(X,Y)Z = R(X,Y)Z - \tfrac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{16}$$

for all $X, Y, Z \in \chi(M)$, where Q is the Ricci-operator. If C vanishes identically then we say that the manifold is conharmonically flat.

Thus for a conharmonic flat almost $C(\lambda)$ manifold, we get from (16)

$$R(X,Y)Z = \frac{1}{2n-1}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\}. \tag{17}$$

By virtue of (9) and (11), (17) takes the form

$$\begin{aligned} R(X,Y)Z &= \tfrac{1}{2n-1}\{Ag(Y,Z)X + B\eta(Y)\eta(Z)X - Ag(X,Z)Y - B\eta(X)\eta(Z)Y + Ag(Y,Z)X \\ &+ Bg(Y,Z)\eta(X)\xi - Ag(X,Z)Y - Bg(X,Z)\eta(Y)\xi\}. \end{aligned} \tag{18}$$

In view of (7) we get from (18)

$$\begin{aligned} & R(\phi X,\phi Y)Z \\ = \;& \lambda\{Xg(Y,Z) - g(X,Z)Y - \phi Xg(\phi Y,Z) + g(\phi X,Z)\phi Y\} + \tfrac{1}{2n-1}\{AXg(Y,Z)) + BX\eta(Y)\eta(Z) \\ - \;& AYg(X,Z) - B\eta(X)\eta(Z)Y + Ag(Y,Z)X + Bg(Y,Z)\eta(X)\xi - Ag(X,Z)Y - Bg(X,Z)\eta(Y)\xi\}. \end{aligned} \tag{19}$$

Putting $Y=\xi$ and using the value of A and B in (19) we get

$$2n\lambda\{X\eta(Z) - g(X,Z)\xi\} = 0. \tag{20}$$

Taking inner product of (20) with a vector field ξ, we obtain

$$2n\lambda\{\eta(X)\eta(Z) - g(X,Z)\} = 0. \tag{21}$$

Putting $X = QX$ in (21) we get

$$2n\lambda\{\eta(QX)\eta(Z) - g(QX, Z)\} = 0. \tag{22}$$

Using (10) and (15) in (22) we obtained

$$2n\lambda\{(A + B)\eta(X)\eta(Z) - S(X, Z)\} = 0. \tag{23}$$

Therefore, either $\lambda = 0$ or, $S(X, Z) = (A + B)\eta(X)\eta(Z)$.

Thus we are in a position to state the following result:

Theorem 3.1. For a conharmonically flat almost $C(\lambda)$ manifold, either λ=0 or the manifold is special type of η-Einstein.

Again we know from [3] that a manifold is cosymplectic if λ vanishes. Thus we have the following corollary.

Corollary 3.1. Every conharmonic flat almost $C(\lambda)$ manifold is, either cosymplectic or the manifold is special type of η-Einstein.

4 Concircularly flat almost $C(\lambda)$ manifold

Definition 4.1. The concircular curvature tensor C of type (1,3) on a Riemannian manifold (M, g) of dimension $(2n + 1)$ is defined by [6]

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)}\{g(Y, Z)X - g(X, Z)Y\}, \tag{24}$$

for any vector fields $X, Y, Z \in \chi(M)$, r is the scalar curvature of the manifold. The concircular curvature tensor C of M represents the deviation of the manifold from constant curvature . If C vanishes identically, we say that the manifold is concircularly flat.

Thus for a concircularly flat almost $C(\lambda)$ manifold, we have from (24)

$$R(X, Y)Z = \frac{r}{2n(2n+1)}\{g(Y, Z)X - g(X, Z)Y\}, \tag{25}$$

In view of (7) we obtain from (25)

$$\begin{aligned} & R(\phi X, \phi Y)Z \\ = \; & \lambda\{g(Y, Z)X - g(X, Z)Y - g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y\} + \tfrac{r}{2n(2n+1)}\{g(Y, Z)X - g(X, Z)Y\}. \end{aligned} \tag{26}$$

Putting $Y = \xi$ and using (12) we get from (26)

$$\lambda\{\eta(Z)X - g(X, Z)\xi\} = 0. \tag{27}$$

Taking inner product of (27) with a vector field ξ, we obtain

$$\lambda\{\eta(X)\eta(Z) - g(X, Z)\} = 0. \tag{28}$$

Putting $X = QX$ in (28) we get

$$\lambda\{\eta(QX)\eta(Z) - g(QX, Z)\} = 0. \tag{29}$$

Using (10) and (15) in (29) we obtain

$$\lambda\{(A + B)\eta(X)\eta(Z) - S(X, Z)\} = 0. \tag{30}$$

Therefore, either $\lambda = 0$ or, $S(X, Z) = (A + B)\eta(X)\eta(Z)$.

Thus we are in a position to state the following :

Theorem 4.1. For a concircularly flat almost $C(\lambda)$ manifold, either λ=0 or the manifold is special type of η-Einstein.

Again we know from [3] that a manifold is cosymplectic if λ vanishes. Thus we have the following corollary.

Corollary 4.1. Every concircularly flat almost $C(\lambda)$ manifold is, either cosymplectic or the manifold is special type of η-Einstein.

5 ξ-conharmonically flat almost $C(\lambda)$ manifold

Definition 5.1. The conharmonic curvature tensor C of type (1,3) on a Riemannian manifold (M, g) of dimension $(2n+1)$ will be called ξ-conharmonic flat [4] if C(X,Y)$\xi = 0$ for all $X, Y \in$ TM.

Thus for a ξ-conharmonic flat almost $C(\lambda)$ manifold we get from (16)

$$R(X,Y)\xi = \frac{1}{2n-1}\{S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY\}. \tag{31}$$

In view of (8) we get from (31)

$$R(\phi X, \phi Y)\xi = \lambda\{X\eta(Y) - \eta(X)Y\} + \tfrac{1}{2n-1}\{S(Y,\xi)X - S(X,\xi)Y + \eta(Y)QX - \eta(X)QY\}. \tag{32}$$

Putting $Y = \xi$ in (32) we obtain

$$0 = \lambda\{X - \eta(X)\xi\} + \tfrac{1}{2n-1}\{S(\xi,\xi)X - S(X,\xi)\xi + QX - \eta(X)Q\xi\}. \tag{33}$$

Using (9), (13) and (14) in (33) we get

$$\{\lambda + \frac{2A+B}{2n-1}\}\{X - \eta(X)\xi\} = 0. \tag{34}$$

Putting the value of $A = -\lambda(2n-1)$ and $B = -\lambda$ in (34) we obtain

$$-\frac{2n\lambda}{2n-1}\{X - \eta(X)\xi\} = 0. \tag{35}$$

Taking inner product of (35) with a vector field U, we obtain

$$2n\lambda[g(X,U) - \eta(X)\eta(U)] = 0. \tag{36}$$

Putting $X = QX$ in (36) we get

$$2n\lambda\{g(QX,U) - \eta(QX)\eta(U)\} = 0. \tag{37}$$

Using (10) and (15) in (37)

$$2n\lambda\{S(X,U) - (A+B)\eta(X)\eta(U)\} = 0. \tag{38}$$

Therefore, either λ =0 or, S(X,U)=(A+B)η (X)η (U)

Thus we are in a position to state the following result:

Theorem 5.1. For ξ-conharmonically flat almost $C(\lambda)$ manifold, either λ=0 or the manifold is special type of η-Einstein.

Again we know from [3] that a manifold is cosymplectic if λ vanishes.Thus we have the following corollary:

Corollary 5.1. Every ξ-conharmonically flat almost $C(\lambda)$ manifold is either cosymplectic or the manifold is special type of η-Einstein.

6 ξ-concircularly flat almost $C(\lambda)$ manifold

Definition 6.1. The concircular curvature tensor C of type (1,3) on a Riemannian manifold (M, g) of dimension n will be called ξ-concircularly flat [7] if C(X,Y)$\xi = 0$ for all $X, Y \in$ TM.

Thus for a ξ-concircularly flat almost $C(\lambda)$ manifold we get from (24)

$$R(X,Y)\xi = \frac{r}{2n(2n+1)}[\eta(Y)X - \eta(X)Y], \tag{39}$$

In view of (8) we obtain from (39)

$$R(\phi X, \phi Y)\xi = \{\lambda + \frac{r}{2n(2n+1)}\}\{\eta(Y)X - \eta(X)Y\}. \tag{40}$$

Setting $Y = \xi$ in (40) we get

$$0 = \{\lambda + \frac{r}{2n(n+1)}\}\{X - \eta(X)\xi\}. \tag{41}$$

By virtue of (12) we get from (41)

$$\{\lambda - \frac{2n\lambda}{(2n+1)}\}\{X - \eta(X)\xi\} = 0. \tag{42}$$

Taking inner product of (42) with a vector field U, we obtain

$$\frac{\lambda}{(2n+1)}\{g(X,U) - \eta(X)\eta(U)\} = 0. \tag{43}$$

Putting $X = QX$ in (43) we get

$$\lambda\{g(QX,U) - \eta(QX)\eta(U)\} = 0. \tag{44}$$

Using (10) and (15) in (44) we get

$$\lambda\{S(X,U) - (A+B)\eta(X)\eta(U)\} = 0. \tag{45}$$

Therefore, either $\lambda = 0$ or, S(X,U)=(A+B)η (X)η (U) .

Thus we are in a position to state the following result:
Theorem 6.1. For ξ-concircularly flat almost $C(\lambda)$ manifold, either λ=0 or the manifold is special type of η-Einstein.
Again we know from [3] that a manifold is cosymplectic if λ vanishes.Thus we have the following corollary:
Corollary 6.1. Every ξ-concircularly flat almost $C(\lambda)$ manifold is either cosymplectic or the manifold is special type of η-Einstein.

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