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Right circulant matrices with Jacobsthal sequence

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Abstract

In this paper, the eigenvalues, Euclidean norm and inverse of right circulant matrices with Jacobsthal sequence were obtained.

Keywords: *Jacobsthal sequence, right circulant matrix*

1 Introduction

The Jacobsthal sequence $\{j_k\}_{k=0}^{+\infty}$ satisfies the recurrence relation

$$j_k = j_{k-1} + 2j_{k-2} \tag{1}$$

with initial values $j_0 = 0$ and $j_1 = 1$.

The right circulant matrix as defined in [1] is given by

$$RCIRC_n(\vec{j}) = \begin{pmatrix} j_0 & j_1 & j_2 & \cdots & j_{n-2} & j_{n-1} \\ j_{n-1} & j_0 & j_1 & \cdots & j_{n-3} & j_{n-2} \\ j_{n-2} & j_{n-1} & j_0 & \cdots & j_{n-4} & j_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ j_2 & j_3 & j_4 & \cdots & j_0 & j_1 \\ j_1 & j_2 & j_3 & \cdots & j_{n-1} & j_0 \end{pmatrix}$$

where j_k are the first n terms of the Jacobsthal sequence.

In [1], the determinant and the inverse of $RCIRC_n(\vec{j})$ were obtained. The aim of this paper is to find explicit forms for the eigenvalues, Euclidean norm and inverse of $RCIRC_n(\vec{j})$. In the inverse, we will be using Inverse Discrete Fourier Transform.

2 Preliminary notes

Lemma 2.1 $$\sum_{k=0}^{n-1}\left[\frac{2^k-(-1)^k}{3}\right]\omega^{-mk} = \frac{1}{3}\left[\frac{1-2^n}{1-2\omega^{-m}} - \frac{1-(-1)^n}{1+\omega^{-m}}\right]$$
where $\omega = e^{2\pi i/n}$

Proof:

$$\sum_{k=0}^{n-1}\left[\frac{2^k-(-1)^k}{3}\right]\omega^{-mk} = \frac{1}{3}\left[\sum_{k=0}^{n-1}\left(2\omega^{-m}\right)^k - \sum_{k=0}^{n-1}\left(-\omega^{-m}\right)^k\right]$$

$$\begin{aligned} &= \frac{1}{3}\left[\frac{1-(2\omega^{-m})^n}{1-2\omega^{-m}} - \frac{1-(-\omega^{-m})^n}{1-(\omega^{-m})}\right] \\ &= \frac{1}{3}\left[\frac{1-2^n}{1-2\omega^{-m}} - \frac{1-(-1)^n}{1+\omega^{-m}}\right] \end{aligned}$$

Lemma 2.2 *Given the equation*

$$s_k = \sum_{k=0}^{n-1}\left[r\omega^{-m}-1\right]\omega^{mk} \tag{3}$$

where r is non-zero,

$$\begin{aligned} &s_0 = -n \\ &s_1 = rn \\ &s_k = 0;\ for\ k \geq 2 \end{aligned}$$

Proof:

For k=0

$$\begin{aligned} s_0 &= \sum_{k=0}^{n-1}\left[r\omega^{-m}-1\right] \\ &= \frac{r(1-\omega^n)}{1-\omega} - n \\ &= -n \end{aligned}$$

For k=1

$$\begin{aligned} s_1 &= \sum_{k=0}^{n-1}\left[r\omega^{-m}-1\right]\omega^{m} \\ &= rn - \frac{1-\omega^n}{1-\omega} \\ &= rn \end{aligned}$$

For $k \geq 2$

$$\begin{aligned} s_k &= \sum_{k=0}^{n-1}\left[r\omega^{-m}-1\right]\omega^{mk} \\ &= \sum_{k=0}^{n-1}\left[r\omega^{m(k-1)} - \omega^{mk}\right] \\ &= r\frac{1-\omega^{n(k-1)}}{1-\omega^{k-1}} - \frac{1-\omega^n}{1-\omega} \\ &= 0 \end{aligned}$$

3 Main results

Theorem 3.1 *If n is even, the eigenvalues of $RCIRC_n(\vec{j})$ are given by*

$$\lambda_m = \frac{1-2^n}{3-6\omega^{-m}} \tag{4}$$

where m=0,1 ..., n-1.

Proof:

The eigenvalues of a right circulant matrix are the Discrte Fourier Transform of the entries in the first row, hence

$$\lambda_m = \sum_{k=0}^{n-1} \left[\frac{2^k - (-1)^k}{3}\right] \omega^{-mk}$$

where m=0,1 ..., n-1. Using (2) and n being even, we have

$$\lambda_m = \frac{1 - 2^n}{3 - 6\omega^{-m}}$$

Theorem 3.2 *If n is odd, the eigenvalues of $RCIRC_n(\vec{j})$ are given by*

$$\lambda_m = \frac{1 - 2^n}{3 - 6\omega^{-m}} - \frac{2}{3 + 3\omega^{-m}} \quad (5)$$

where m=0,1 ..., n-1.

Proof:

The same process as the previous theorem but with n odd.

Theorem 3.3 *The Euclidean norm of $RCIRC_n(\vec{j})$ is given by*

$$||RCIRC_n(\vec{j})||_E = \frac{\sqrt{n^2 + 3nj_n^2}}{3} \quad (6)$$

Proof:

$$\begin{aligned}
||RCIRC_n(\vec{j})||_E &= \sqrt{\sum_{i=1,j=1}^{n} a_{ij}^2} \\
&= \sqrt{n \sum_{k=0}^{n-1} j_k^2} \\
&= \frac{1}{3}\sqrt{n \sum_{k=0}^{n-1} [4^k + (-2)^{k+1} + 1]} \\
&= \frac{1}{3}\sqrt{n \left[\frac{4^n - 1}{3} + \frac{2 + (-2)^{n+1}}{3} + n\right]} \\
&= \frac{1}{3}\sqrt{n^2 + n \left[\frac{4^n + (-2)^{n+1} + 1}{3}\right]} \\
&= \frac{1}{3}\sqrt{n^2 + 3n \left[\frac{2^{2n} - 2(-2)^n + 1^{2n}}{9}\right]} \\
&= \frac{1}{3}\sqrt{n^2 + 3n \left[\frac{2^n - (-1)^n}{3}\right]^2} \\
&= \frac{\sqrt{n^2 + 3nj_n^2}}{3}
\end{aligned}$$

Theorem 3.4 *If n is even, $RCIRC_n^{-1}(\vec{j})$ is given by $RCIRC_n(s_0, s_1, ..., s_{n-1})$ where*

$$s_0 = \frac{3}{1 - 2^n}$$

$$s_1 = -\frac{6}{1 - 2^n}$$

$$s_k = 0; \; for \; k \geq 2.$$

Proof:

The inverse of a right circulant matrix is the Inverse Discrete Fourier Transform of the inverse of its eigenvalues, hence

$$\begin{aligned} s_k &= \frac{1}{n}\sum_{m=0}^{n-1} \lambda_m^{-1}\omega^{mk} \\ &= \frac{3}{n}\sum_{m=0}^{n-1}\left[\frac{1-2\omega^{-m}}{1-2^n}\right]\omega^{mk} \\ &= -\frac{3}{(1-2^n)n}\sum_{m=0}^{n-1}\left[2\omega^{-m}-1\right]\omega^{mk} \end{aligned}$$

Using Lemma 2.2, the thereom follows.

Theorem 3.5 *If n is odd, $RCIRC_n^{-1}(\vec{j})$ is given by $RCIRC_n(s_0, s_1, ..., s_{n-1})$ where*

$$s_k = \frac{3}{n}\sum_{k=0}^{n-1}\left[\frac{(5-2^n)\omega^{-m}-1}{(1-2\omega^{-m})(1+\omega^{-m})}\right]\omega^{mk}$$

Proof:

$$\begin{aligned} s_k &= \frac{1}{n}\sum_{m=0}^{n-1}\left[\frac{1-2^n}{3-6\omega^{-m}} - \frac{2}{3+3\omega^{-m}}\right]^{-1}\omega^{mk} \\ &= \frac{3}{n}\sum_{k=0}^{n-1}\left[\frac{1-2^n+\omega^{-m}-2^n\omega^{-m}-2+4\omega^{-m}}{(1-2\omega^{-m})(1+\omega^{-m})}\right]\omega^{mk} \\ &= \frac{3}{n}\sum_{k=0}^{n-1}\left[\frac{(5-2^n)\omega^{-m}-1}{(1-2\omega^{-m})(1+\omega^{-m})}\right]\omega^{mk} \end{aligned}$$

4 Conclusion

The eigenvalues and the inverse of right circulant with Jacobsthal sequence take different forms depending on whether n is even or odd while the Euclidean norm doesn't. Furthermore, we have expressed the Euclidean norm in terms of n and n^{th} Jacobsthal number.

References

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