Algorithms of Common Solutions for a Fixed Point of Hemicontractive-type Mapping and a Generalized Equilibrium Problem

Habtu Zegeye[1*], Tesfalem Hadush Meche[2] and Mengistu Goa Sangago[3]

[1]*Department of Mathematics, College of Sciences, Botswana International University of Science and Technology, Private Mail Bag 16, Palapye, Botswana*
[2]*Department of Mathematics, College of Natural and Computational Sciences, Addis Ababa University, P.O.Box 1176, Addis Ababa, Ethiopia*
[3]*Department of Mathematics, College of Natural and Computational Sciences, Addis Ababa University, P.O.Box 31167, Addis Ababa, Ethiopia*
[*]*Corresponding author E-mail: habtuzh@yahoo.com*

Abstract

In this paper, we introduce and study an iterative algorithm for finding a common element of the set of fixed points of a Lipschitz hemicontractive-type multi-valued mapping and the set of solutions of a generalized equilibrium problem in the framework of Hilbert spaces. Our results improve and extend most of the results that have been proved previously by many authors in this research area.

Keywords: Fixed Points of Mappings; Generalized Equilibrium Problem; Hemicontractive-type Multi-valued Mapping; Iterative Algorithm; Strong Convergence.

1. Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let C be a nonempty subset of H.
A mapping $T : C \longrightarrow H$ is called $k-strictly\ pseudocontractive$ if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2, \qquad (1.1)$$

for all $x, y \in C$. If $k = 1$ in (1.1), then T is called *pseudocontractive* mapping.
A mapping $T : C \to H$ is called *Lipschitzian* if there exists $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x - y\|, \ \forall x, y \in C$. If $L = 1$, then T is called *nonexpansive* and if $L \in [0,1)$ then T is called a *contraction*.
Observe that the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings and hence the class of nonexpansive and contraction mappings.

Let $CB(C)$ denote the family of nonempty, closed and bounded subsets of C. The Pompeiu-Hausdorff metric ([2]) on $CB(C)$ is defined by

$$D(A,B) = \max\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\},$$

for all $A, B \in CB(C)$, where $d(x,B) = \inf\{\|x - b\| : b \in B\}$.
A multi-valued mapping $T : C \longrightarrow CB(C)$ is said to be $k-strictly$ $pseudocontractive$ if there exists a constant $k \in [0,1)$ such that

$$D^2(Tx,Ty) \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2, \qquad (1.2)$$

for all $x, y \in C$ and $u \in Tx, v \in Ty$. If $k = 1$ in (1.2), then T is called *pseudocontractive* mapping.
A multi-valued mapping T is called *Lipschitzian* if there exists $L \geq 0$ such that

$$D(Tx,Ty) \leq L\|x - y\|, \ \forall x, y \in C.$$

If $L = 1$, then T is called *nonexpansive* and if $L \in [0,1)$ then T is called a *contraction* mapping.
An element $x \in C$ is called a *fixed point* of $T : C \longrightarrow C$ (resp., $T : C \longrightarrow CB(C)$) if $x = Tx$ (resp., $x \in Tx$). The set of fixed points of T is denoted by $F(T)$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x.
A mapping $T : C \longrightarrow H$ is called *quasi-nonexpansive* if $F(T)$ is nonempty and the inequality

$$\|Tx - p\| \leq \|x - p\|,$$

holds for all $p \in F(T), x \in C$.
The mapping T is called *hemicontractive* if $F(T)$ is nonempty and it satisfies:

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2,$$

for all $p \in F(T), x \in C$.
A multi-valued mapping $T : C \longrightarrow CB(C)$ is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and for all $p \in F(T), x \in C$, we have

$$D(Tx,Tp) \leq \|x - p\|.$$

A multivalued mapping $T : C \longrightarrow CB(C)$ is said to be *hemicontractive-type* if $F(T) \neq \emptyset$ and for all $p \in F(T)$, $x \in C$,

$$D^2(Tx, Tp) \leq \|x - p\|^2 + \|x - u\|^2, \ \forall u \in Tx. \quad (1.3)$$

We observe that every nonexpansive mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping, and every pseudocontractive mapping T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall p \in F(T)$ is hemicontractive-type mapping. It is also easy to see that every quasi-nonexpansive mapping is hemicontractive-type mapping. Next, we give an example of a hemicontractive-type mapping which is not quasi-nonexpansive and pseudocontractive mappings.

Example 1.1. *Let $T : \mathbb{R} \longrightarrow CB(\mathbb{R})$ be given by*

$$Tx = [-3x, -2x], \ if \ x \geq 0 \ and \ Tx = [-2x, -3x], \ if \ x < 0.$$

Then, zero is the only fixed point of T, that is, $F(T) = \{0\}$ and for all $x \in \mathbb{R} \setminus \{0\}$, we have

$$
\begin{aligned}
D(Tx, T0) &= \max\{ \sup_{a \in Tx} d(a, T0), \ \sup_{b \in T0} d(b, Tx) \} \\
&= \max\{ \sup_{a \in Tx} |a|, d(0, Tx) \} \\
&= \max\{3|x|, 2|x|)\} = 3|x| > |x - 0|, \quad (1.4)
\end{aligned}
$$

which shows that T is not quasi-nonexpansive. And for every $x \in \mathbb{R}$,

$$d(x, Tx) = \inf_{a \in Tx} |a - x| = |x + 2x| = 3|x|.$$

Thus, from (1.4), we have

$$
\begin{aligned}
D^2(Tx, T0) = 9|x|^2 &= |x - 0|^2 + 8|x|^2 \\
&\leq |x - 0|^2 + 9|x|^2 = |x - 0|^2 + (d(x, Tx))^2 \\
&\leq |x - 0|^2 + |x - u|^2, \ for \ all \ u \in Tx.
\end{aligned}
$$

This shows that T is hemicontractive-type mapping. However, it is not a pseudocontractive. Indeed, for $x = 1$, $y = 2$, $u = -3 \in Tx$ and $v = -4 \in Ty$, we have

$$D^2(Tx, Ty) = 9 > 1 + 4 = |x - y|^2 + |x - u - (y - v)|^2.$$

Therefore, the class of multi-valued hemicontractive-type mappings is more general in the sense that it includes properly the class of multi-valued quasi-nonexpansive mappings and the class of pseudocontractive mappings T with $F(T) \neq \emptyset$ and $T(p) = \{p\}, \forall p \in F(T)$.

Let $T : C \longrightarrow CB(C)$ be a multi-valued mapping, $I - T$ (where I is the identity mapping on C) is said to be *demiclosed at zero* if $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $\lim_{n \to \infty} d(x_n, Tx_n) = 0$ implies $x \in Tx$.

Several authors have studied iterative algorithms for approximating fixed points of various classes of nonlinear mappings (including hemicontractive mapping); see, *for example*, [5, 8, 14, 15, 21] and the references cited therein. In 2007, Rafiq [13] introduced the following Mann-type implicit iteration process and showed that the process converges strongly to a fixed point of a continuous hemicontractive mapping T from a compact and convex subset C of a real Hilbert space into itself:

$$
\begin{cases}
x_0 \in C, \\
x_n = \alpha_n x_{n-1} + (1 - \alpha_n) Tx_n, \quad \forall n \geq 1,
\end{cases}
$$

where $\{\alpha_n\} \subset [0, 1]$ satisfying some appropriate conditions. Recently, Woldeamanuel et al.[19] extended the results of Rafiq [13] from single-valued hemicontractive to multi-valued hemicontractive-type mapping. More precisely, they studied the following three-step iterative algorithm for finding a common point of fixed points of

a finite family of Lipschitz and hemicontractive-type multi-valued mappings under mild assumption on the parameters:

$$
\begin{cases}
z_n = (1 - \gamma_n)x_n + \gamma_n w_n, & w_n \in T_n y_n, \\
y_n = (1 - \beta_n)x_n + \beta_n u_n, & u_n \in T_n x_n, \\
x_{n+1} = \alpha_n w + (1 - \alpha_n)z_n, & \forall n \geq 1.
\end{cases} \quad (1.5)
$$

They proved that the sequence $\{x_n\}$, generated by (1.5), converges strongly to some point p in $\cap_{i=1}^N F(T_i)$ nearest to w.

A mapping $A : C \longrightarrow H$ is called α-*inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in C.$$

Note that every α-inverse strongly monotone mapping is $\frac{1}{\alpha}$-Lipschitz mapping. However, the converse may not hold. For example, the mapping $A : (0, 1) \longrightarrow \mathbb{R}$ defined by $Ax = -x$ is Lipschitz but not α-inverse strongly monotone.

A mapping $A : C \longrightarrow H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \ \forall x, y \in C.$$

Clearly, the class of monotone mappings includes the class of α-inverse strongly monotone and the inclusion is proper. To see this, consider the mapping $A : (0, 1) \longrightarrow \mathbb{R}$ given by $Ax = -\frac{1}{x}$. Then, for every $x, y \in C$, we have

$$\langle x - y, Ax - Ay \rangle = \langle x - y, -\frac{1}{x} + \frac{1}{y} \rangle = \frac{1}{xy}\|x - y\|^2 \geq 0,$$

which shows that A is monotone mapping. However, since $|Ax - Ay| = |\frac{-1}{x} + \frac{1}{y}| = \frac{1}{xy}|x - y|$ and $\frac{1}{xy} \to \infty$ as $xy \to 0$, then we can not find a constant $L \geq 0$ such that $|Ax - Ay| \leq L|x - y|$, for all $x, y \in (0, 1)$. Hence, A is not Lipschitz and so it is not α-inverse strongly monotone mapping.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction and $A : C \longrightarrow H$ be a nonlinear mapping. The *generalized equilibrium problem* is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z \rangle \geq 0, \quad \forall y \in C. \quad (1.6)$$

In this paper, the set of solutions of problem (1.6) is denoted by $EP(F, A)$, i.e.,

$$EP(F, A) = \{z \in C : F(z, y) + \langle Az, y - z \rangle \geq 0, \ \forall y \in C\}.$$

In the case of $A \equiv 0$, then the problem (1.6) reduces to the equilibrium problem:

$$\text{Finding } z \in C \text{ such that } F(z, y) \geq 0, \ \forall y \in C. \quad (1.7)$$

The set of solutions of problem (1.7) is denoted by $EP(F)$. In the case of $F \equiv 0$, then the problem (1.6) reduced to finding $z \in C$ such that

$$\langle Az, y - z \rangle \geq 0, \ \forall y \in C, \quad (1.8)$$

which is called *the classical variational inequality problem*. The set of solutions of problem (1.8) is denoted by $VI(C, A)$.

Clearly, if a point $z \in VI(C, A) \cap EP(F)$, then $z \in EP(F, A)$, however, the converse may not hold as we can see from the following example.

Example 1.2. *Let $H = \mathbb{R}$ and $C = [0, 1]$. Define $F : C \times C \to \mathbb{R}$ by $F(x, y) = x + y$ and $A : C \to H$ by $Ax = \frac{x}{2}$. Then, for $x = 1$, we have*

$$F(x, y) + \langle Ax, y - x \rangle = (1 + y) + \frac{1}{2}(y - 1) = \frac{1}{2}(3y + 1) \geq 0,$$

for all $y \in C$, which implies that $1 \in EP(F, A)$. But, since $\langle A1, y - 1 \rangle = \frac{1}{2}(y - 1) \leq 0$ for all $y \in C$, then $1 \notin VI(C, A)$ and hence $1 \notin EP(F) \cap VI(C, A)$.

Assumption 1.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. In the sequel, let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the following conditions:*

(A1) $F(x,x) = 0, \forall x \in C$;
(A2) *F is monotone, i.e.,* $F(x,y) + F(y,x) \leq 0, \forall x,y \in C$;
(A3) $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y), \forall x,y,z \in C$;
(A4) *For each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.*

Remark 1.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying Assumption 1.3 and let A be a continuous monotone mapping. Define $G : C \times C \longrightarrow \mathbb{R}$ by*

$$G(x,y) = F(x,y) + \langle Ax, y - x \rangle,$$

then it is easy to see that the bifunction G satisfies Assumption 1.3. Thus, the generalized equilibrium problem (1.6) is equivalent to the following equilibrium problem:

Find $z \in C$ such that $G(z,w) \geq 0$, for all $w \in C$.

Generalized equilibrium problem is more general in the sense that it includes equilibrium problems and hence variational inequalitiy, optimization problem, Nash equilibrium problem etc. Consequently, many authors have shown their interest in constructing an iterative algorithm for approximating common solution of generalized equilibrium and fixed point problems. We describe some of them as follows:

In 2008, Moudafi [11] introduced an iterative scheme for finding a common element of the set of solutions of generalized equilibrium problems and the set of fixed points of nonexpansive single-valued mapping in a Hilbert space setting and then obtained the following weak convergence theorem.

Theorem 1.5. *[11] Let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.3. Let A be an α-inverse strongly monotone mapping of C into H, and let T be a nonexpansive mapping of C in to itself such that $F(T) \cap EP(F,A)$ is nonempty. Let $x_0 \in C$ and $\{y_n\}$, $\{x_n\}$ be sequences generated by*

$$\begin{cases} F(y_n, y) + \langle Ax_n, y - y_n \rangle \\ + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle \geq 0, \forall y \in C \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \forall n \geq 0, \end{cases}$$

where $\{r_n\} \subset [a,b]$ for some $a,b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c,d]$ for some $c,d \in (0,1)$. Then, $\{x_n\}$ converges weakly to $z \in F(T) \cap EP(F,A)$, where $z = \lim_{n \to \infty} P_{F(T) \cap EP(F,A)} x_n$.

In [17], motivated by the result given by Moudafi [11], Takahashi and Takahashi proposed an iterative scheme and proved the following convergence theorem.

Theorem 1.6. *[17] Let C be a nonempty, closed and convex subset of a real Hilbert space H and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.3. Let A be an α-inverse strongly monotone mapping of C into H and $T : C \longrightarrow C$ be a nonexpansive mapping such that $F(T) \cap EP(F,A)$ is nonempty. Let $u, x_1 \in C$ and $\{z_n\}$, $\{x_n\}$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n \rangle \\ + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \forall y \in C \\ y_n = \alpha_n u + (1 - \alpha_n) z_n \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T y_n, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1], \{\beta_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfying:

i. $0 < a \leq \lambda_n \leq b < 2\alpha, 0 < c \leq \beta_n \leq d < 1$;
ii. $\lim_{n \to \infty} |\lambda_n - \lambda_{n+1}| = 0, \lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(T) \cap EP(F,A)}(u)$.

Recently, Wong et al.[4] introduced the following implicit viscosity approximation method, starting from $x_0 \in C$, for finding common point of the set of solutions of generalized equilibrium problem (1.6) and the set of fixed points of a continuous pseudocontractive mapping in Hilbert spaces:

$$\begin{cases} F(z_{n-1}, y) + \langle Ax_{n-1}, y - z_{n-1} \rangle \\ + \frac{1}{\lambda_n} \langle y - z_{n-1}, z_{n-1} - x_{n-1} \rangle \geq 0, \forall y \in C \\ y_n = \beta_n f(x_{n-1}) + \gamma_n S z_{n-1} + (1 - \beta_n - \gamma_n) x_{n-1} \\ x_n = \alpha_n y_n + (1 - \alpha_n) T x_n, \forall n \in \mathbb{N}, \end{cases} \quad (1.9)$$

where $F : C \times C \longrightarrow \mathbb{R}$ is a bifunction, $A : C \longrightarrow H$ is an α-inverse strongly monotone mapping, $f : C \longrightarrow C$ is a fixed contraction mapping, $S : C \longrightarrow C$ is a nonexpansive mapping, and $T : C \longrightarrow C$ is a continuous pseudocontractive mapping such that $F(T) \cap EP(F,A)$ is nonempty. They proved that the sequence $\{x_n\}$, generated by (1.9), converges strongly to a point in $F(T) \cap EP(F,A)$ provided that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1]$ and $\{\lambda_n\} \subset (0, 2\alpha]$ satisfy some appropriate control conditions.

In this paper, motivated and inspired by the idea of Iiduka and Takahashi [7], Takahashi and Takahashi [16, 17], Wong et al.[4], we introduce an iterative algorithm for finding a common element of the solution set of a generalized equilibrium problem (1.6) and the set of fixed points of a multi-valved Lipschitz hemicontractive-type mapping. We also establish a strong convergence theorem in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by Meche et al.[10], Moudafi [11], Takahashi and Takahashi [16, 17], Wong et.al.[4] and some other results that have been obtained previously in this research area.

2. Preliminaries

Throughout this section unless otherwise stated, let C be a nonempty, closed and convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

P_C is called the metric projection of H onto C. We note that the metric projection P_C is a nonexpansive mapping from H onto C. It also satisfies the following property: for any $x \in H$ and $z \in C$,

$$z = P_C x \text{ if and only if } \langle x - z, z - y \rangle \geq 0, \text{ for all } y \in C. \quad (2.1)$$

Furthermore, for any $x, y \in H$, we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle. \quad (2.2)$$

We note that $(I - T)$ is demiclosed at zero whenever $T : C \longrightarrow C$ is a nonexpansive mapping, (see [1]).
We need the following lemmas for the proof of our main results.

Lemma 2.1. *[22] Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 1, 2, \cdots, n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ the following equality holds:*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\|^2 = \sum_{i=1}^{n} \alpha_i \|x_i\|^2 - \sum_{1 \leq i,j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.2. *Let H be a real Hilbert space. Then, for any given $x, y \in H$, we have the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

Lemma 2.3. *[3, 6] Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying Assumption 1.3. For $r > 0$ and for all $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r x = \{ z \in C : F(z,y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \}.$$

Then, the following hold:

(1) *T_r is nonempty and single valued;*
(2) *T_r is firmly nonexpansive, i.e., $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$, $\forall x, y \in H$;*
(3) *$F(T_r) = EP(F)$;*
(4) *$EP(F)$ is closed and convex.*

From Lemma 2.3 we observe that for each given $x \in H$, there exists a unique element $z \in C$ such that $z = T_r x$.

Lemma 2.4. *[12] Let H be a Hilbert space. Let $A, B \in CB(H)$ and $a \in A$. Then, for $\varepsilon > 0$, there exists a point $b \in B$ such that $\|a - b\| \leq D(A, B) + \varepsilon$. As a consequence of this, taking $\varepsilon = D(A, B)$, we obtain that $\|a - b\| \leq 2D(A, B)$.*

Lemma 2.5. *[20] Let $\{b_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$b_{n+1} \leq (1 - \alpha_n) b_n + \alpha_n \delta_n, \text{ for } n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\delta_n \subset \mathbb{R}$ satisfying the following conditions:

$$\lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } \limsup_{n \to \infty} \delta_n \leq 0.$$

Then, $\lim_{n \to \infty} b_n = 0$.

Lemma 2.6. *[9] Let $\{a_n\}$ be a sequence of real numbers such that there exist a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = max\{j \leq k : a_j \leq a_{j+1}\}$.

3. Main Result

In this section, we prove strong convergence theorems for a generalized equilibrium problem and a fixed point problem for a multi-valued Lipschitz hemicontractive-type mapping.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.3. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L such that $\Omega = F(T) \cap EP(F,A)$ is nonempty and $Tp = \{p\}$ for all $p \in \Omega$. Let $x_1, u \in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F(z_n,y) + \langle Az_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall y \in C, \\ y_n = (1 - \lambda_n) z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \gamma_n z_n, \end{cases} \quad (3.1)$$

for all $n \geq 1$, where $u_n \in Tz_n, v_n \in Ty_n$ such that $\|u_n - v_n\| \leq 2D(Tz_n, Ty_n)$ and $\{r_n\} \subset [r, \infty)$ for some $r > 0$, $\{\beta_n\}, \{\gamma_n\} \subset [a,b]$, and $\{\alpha_n\} \subset (0,c)$ for some $a, b, c \in (0,1)$ satisfying the following conditions: (i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n \to \infty} \alpha_n = 0, \sum \alpha_n = \infty$; (iii) $\alpha_n + \beta_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1 + 4L^2} + 1}$. Then, the sequence $\{x_n\}$ is bounded.

Proof. Let $p \in \Omega$. Then, $Tp = \{p\}$ and $F(p,y) + \langle Ap, y - p \rangle \geq 0$. Define $G(x,y) = F(x,y) + \langle Ax, y - x \rangle$ for all $x, y \in C$. Then, by remark 1.4, G satisfies Assumption 1.3 and $p \in EP(F,A)$ is equivalent to $G(p,y) \geq 0$ for all $y \in C$. Thus, using the definition, z_n can be

rewritten as $z_n = T_{r_n} x_n := \{ F(z_n,y) + \langle Az_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall y \in C, \}$ and hence we have $p = T_{r_n} p$. From the fact that T_{r_n} is nonexpansive, we obtain that

$$\|z_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|. \quad (3.2)$$

Since T is hemicontractive-type mapping and $u_n \in Tz_n$, from (3.1), (3.2) and Lemma 2.1, we get

$$\begin{aligned} \|y_n - p\|^2 &= \|(1 - \lambda_n) z_n + \lambda_n u_n - p\|^2 \\ &= (1 - \lambda_n)\|z_n - p\|^2 + \lambda_n \|u_n - p\|^2 \\ &\quad - \lambda_n(1 - \lambda_n)\|z_n - u_n\|^2 \\ &\leq (1 - \lambda_n)\|z_n - p\|^2 + \lambda_n D^2(Tz_n, Tp) \\ &\quad - \lambda_n(1 - \lambda_n)\|z_n - u_n\|^2 \\ &\leq (1 - \lambda_n)\|z_n - p\|^2 + \lambda_n \left(\|z_n - p\|^2 + \|z_n - u_n\|^2 \right) \\ &\quad - \lambda_n(1 - \lambda_n)\|z_n - u_n\|^2 \\ &= \|z_n - p\|^2 + \lambda_n \|z_n - u_n\|^2 - \lambda_n(1 - \lambda_n)\|z_n - u_n\|^2 \\ &\leq \|x_n - p\|^2 + \lambda_n^2 \|z_n - u_n\|^2. \quad (3.3) \end{aligned}$$

On the other hand, since T is hemicontractive-type mapping and $v_n \in Ty_n$, from (3.1) and (3.3), we have that

$$\begin{aligned} \|v_n - p\|^2 &= (d(v_n, Tp))^2 \leq D^2(Ty_n, Tp) \\ &\leq \|y_n - p\|^2 + \|y_n - v_n\|^2 \\ &\leq \|x_n - p\|^2 + \lambda_n^2 \|z_n - u_n\|^2 \\ &\quad + \|y_n - v_n\|^2. \quad (3.4) \end{aligned}$$

It follows from (3.1) that

$$\begin{aligned} \|z_n - y_n\|^2 &= \|z_n - ((1 - \lambda_n) z_n + \lambda_n u_n)\|^2 \\ &= \lambda_n^2 \|z_n - u_n\|^2. \quad (3.5) \end{aligned}$$

Since T is $L-$Lipschitzian mapping and $\|u_n - v_n\| \leq 2D(Tz_n, Ty_n)$, using (3.5) and Lemma 2.1, we get that

$$\begin{aligned} \|y_n - v_n\|^2 &= \|(1 - \lambda_n)(z_n - v_n) + \lambda_n(u_n - v_n)\|^2 \\ &= (1 - \lambda_n)\|z_n - v_n\|^2 + \lambda_n \|u_n - v_n\|^2 \\ &\quad - \lambda_n(1 - \lambda_n)\|z_n - u_n\|^2 \\ &\leq (1 - \lambda_n)\|z_n - v_n\|^2 + 4\lambda_n D^2(Tz_n, Ty_n) \\ &\quad - \lambda_n(1 - \lambda_n)\|z_n - u_n\|^2 \\ &\leq (1 - \lambda_n)\|z_n - v_n\|^2 + 4\lambda_n L^2 \|z_n - y_n\|^2 \\ &\quad - \lambda_n(1 - \lambda_n)\|z_n - u_n\|^2 \\ &= (1 - \lambda_n)\|z_n - v_n\|^2 + 4\lambda_n^3 L^2 \|z_n - u_n\|^2 \\ &\quad - \lambda_n(1 - \lambda_n)\|z_n - u_n\|^2 \\ &= (1 - \lambda_n)\|z_n - v_n\|^2 \\ &\quad + \lambda_n(4L^2 \lambda_n^2 + \lambda_n - 1)\|z_n - u_n\|^2. \quad (3.6) \end{aligned}$$

Hence, substituting (3.6) into (3.4), we obtain that

$$\begin{aligned} \|v_n - p\|^2 &\leq \|x_n - p\|^2 + \lambda_n^2 \|z_n - u_n\|^2 + (1 - \lambda_n)\|z_n - v_n\|^2 \\ &\quad + \lambda_n(4L^2 \lambda_n^2 + \lambda_n - 1)\|z_n - u_n\|^2 \\ &= \|x_n - p\|^2 + (1 - \lambda_n)\|z_n - v_n\|^2 \\ &\quad + \lambda_n(4L^2 \lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2. \quad (3.7) \end{aligned}$$

Thus, from (3.2), (3.7) and Lemma 2.1, we have that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n v_n + \gamma_n z_n - p\|^2 \\
&\leq \alpha_n \|u - p\|^2 + \beta_n \|v_n - p\|^2 + \gamma_n \|z_n - p\|^2 \\
&\quad - \beta_n \gamma_n \|z_n - v_n\|^2 \\
&\leq \alpha_n \|u - p\|^2 \\
&\quad + \beta_n \Big(\|x_n - p\|^2 + (1 - \lambda_n)\|z_n - v_n\|^2 \\
&\quad + \lambda_n (4L^2 \lambda_n^2 + 2\lambda_n - 1)\|z_n - u_n\|^2 \Big) \\
&\quad + \gamma_n \|z_n - p\|^2 - \beta_n \gamma_n \|z_n - v_n\|^2 \\
&= \alpha_n \|u - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\
&\quad + \beta_n (1 - \gamma_n - \lambda_n)\|z_n - v_n\|^2 \\
&\quad - \beta_n \lambda_n (1 - 4L^2 \lambda_n^2 - 2\lambda_n)\|z_n - u_n\|^2.
\end{aligned}
$$

Then, by assumption (i), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\
&\quad - \beta_n \lambda_n (1 - 4L^2 \lambda_n^2 - 2\lambda_n)\|z_n - u_n\|^2 \\
&\quad + \beta_n (\alpha_n + \beta_n - \lambda_n)\|z_n - v_n\|^2, \qquad (3.8)
\end{aligned}
$$

and from assumption (iii), we get

$$
\begin{aligned}
&1 - 4L^2 \lambda_n^2 - 2\lambda_n \geq 1 - 4L^2 \lambda^2 - 2\lambda > 0 \\
&\text{and } (\alpha_n + \beta_n) - \lambda_n \leq 0, \qquad (3.9)
\end{aligned}
$$

for all $n \geq 1$. Therefore, from (3.8) and (3.9), we infer that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\
&\leq \max\{\|u - p\|^2, \|x_n - p\|^2\}.
\end{aligned}
$$

Thus, by induction, we have that

$$
\|x_n - p\|^2 \leq \max\{\|u - p\|^2, \|x_1 - p\|^2\},
$$

which implies that $\{x_n\}$ is bounded. This completes the proof. □

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.3. Let $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L such that $\Omega = F(T) \cap EP(F,A)$ is nonempty, closed and convex. Assume that $(I - T)$ is demiclosed at zero and $Tp = \{p\}$ for all $p \in \Omega$. Let $x_1, u \in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
F(z_n, y) + \langle A z_n, y - z_n \rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall y \in C, \\
y_n = (1 - \lambda_n) z_n + \lambda_n u_n, \\
x_{n+1} = \alpha_n u + \beta_n v_n + \gamma_n z_n,
\end{cases} \qquad (3.10)
$$

for all $n \geq 1$, where $u_n \in T z_n, v_n \in T y_n$ such that $\|u_n - v_n\| \leq 2D(T z_n, T y_n)$ and $\{r_n\} \subset [r, \infty)$ for some $r > 0$, $\{\beta_n\}, \{\gamma_n\} \subset [a, b]$, and $\{\alpha_n\} \subset (0, c)$ for some $a, b, c \in (0, 1)$ satisfying the following conditions: (i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $\alpha_n + \beta_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1 + 4L^2 + 1}}$. Then, the sequence $\{x_n\}$ converges strongly to the point $z = P_\Omega(u)$.

Proof. Since Ω is nonempty, closed and convex subset of C, we get that P_Ω is well defined. Clearly, from Theorem 3.1 the sequence $\{x_n\}$ and hence $\{y_n\}, \{z_n\}$ are bounded. As in Theorem 3.1, let $p \in \Omega$. Then, using (3.10), Lemma 2.1 and Lemma 2.2, we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n u + \beta_n v_n + \gamma_n z_n - p\|^2 \\
&\leq \|\beta_n (v_n - p) + \gamma_n (z_n - p)\|^2 \\
&\quad + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\
&\leq \beta_n \|v_n - p\|^2 + \gamma_n \|z_n - p\|^2 - \beta_n \gamma_n \|z_n - v_n\|^2 \\
&\quad + 2\alpha_n \langle u - p, x_{n+1} - p \rangle. \qquad (3.11)
\end{aligned}
$$

Since T_{r_n} is firmly nonexpansive, from (2.2) we have that

$$
\begin{aligned}
\|z_n - p\|^2 &= \|T_{r_n} x_n - T_{r_n} p\|^2 \\
&\leq \langle z_n - p, x_n - p \rangle \\
&= \frac{1}{2}\Big(\|z_n - p\|^2 + \|x_n - p\|^2 - \|x_n - z_n\|^2 \Big),
\end{aligned}
$$

which gives that

$$
\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - z_n\|^2. \qquad (3.12)
$$

Thus, substituting (3.7) and (3.12) into (3.11), we obtain that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n \Big(\|x_n - p\|^2 + \lambda_n (4L^2 \lambda_n^2 + 2\lambda_n - 1) \\
&\quad \times \|z_n - u_n\|^2 + (1 - \lambda_n)\|z_n - v_n\|^2 \Big) \\
&\quad + \gamma_n (\|x_n - p\|^2 - \|x_n - z_n\|^2) \\
&\quad - \beta_n \gamma_n \|z_n - v_n\|^2 + 2\alpha_n \langle u - p, x_{n+1} - p \rangle \\
&= (1 - \alpha_n)\|x_n - p\|^2 - \beta_n \lambda_n (1 - 4L^2 \lambda_n^2 - 2\lambda_n) \\
&\quad \times \|z_n - u_n\|^2 + \beta_n (\alpha_n + \beta_n - \lambda_n)\|z_n - v_n\|^2 \\
&\quad - \gamma_n \|z_n - x_n\|^2 + 2\alpha_n \langle u - p, x_{n+1} - p \rangle. \quad (3.13)
\end{aligned}
$$

It follows from (3.9) that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1 - \alpha_n)\|x_n - p\|^2 \\
&\quad + 2\alpha_n \langle u - p, x_{n+1} - p \rangle. \qquad (3.14)
\end{aligned}
$$

Next, we consider two cases:

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - p\|\}$ is decreasing for all $n \geq n_0$. Then, we get that, $\{\|x_n - p\|\}$ is convergent. Thus, from (3.13) and (3.9), we have that

$$
\begin{aligned}
\beta_n \lambda_n \Big(1 - 4L^2 \lambda_n^2 - 2\lambda_n \Big)\|z_n - u_n\|^2 &\leq (1 - \alpha_n)\|x_n - p\|^2 \\
&\quad - \|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n \langle u - p, x_{n+1} - p \rangle.
\end{aligned}
$$

Hence, from (3.9) and the fact that $\alpha_n \to 0$ as $n \to \infty$, we have that

$$
\|z_n - u_n\| \to 0 \text{ as } n \to \infty, \qquad (3.15)
$$

which implies that

$$
d(z_n, T z_n) \leq \|z_n - u_n\| \to 0 \text{ as } n \to \infty. \qquad (3.16)
$$

In addition, from (3.9) and (3.13), we find that

$$
\begin{aligned}
\gamma_n \|z_n - x_n\|^2 &\leq (1 - \alpha_n)\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n \langle u - p, x_{n+1} - p \rangle.
\end{aligned}
$$

Thus, since $\alpha_n \to 0$ as $n \to \infty$, it follows from the assumption of $\{\gamma_n\}$ that

$$
\lim_{n \to \infty} \|z_n - x_n\| = 0. \qquad (3.17)
$$

And using the Lipschitz continuity of T, (3.5) and (3.15), we get that

$$
\begin{aligned}
\|z_n - v_n\| &\leq \|z_n - u_n\| + \|u_n - v_n\| \\
&\leq \|z_n - u_n\| + 2L\|z_n - y_n\| \\
&= \|z_n - u_n\| + 2L\lambda_n \|z_n - u_n\| \to 0 \text{ as } n \to \infty. \quad (3.18)
\end{aligned}
$$

Therefore, from (3.17), (3.18), definition of $\{x_{n+1}\}$ and the fact that $\alpha_n \to 0$ as $n \to \infty$, we get

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \\
&= \|\alpha_n (u - z_n) + \beta_n (v_n - z_n)\| + \|z_n - x_n\| \\
&\leq \alpha_n \|u - z_n\| + \beta_n \|v_n - z_n\| + \|z_n - x_n\| \to 0 \text{ as } n \to \infty. \\
&\qquad (3.19)
\end{aligned}
$$

Now, let $z = P_\Omega(u)$. We claim that $\limsup_{n\to\infty}\langle u-z, x_{n+1}-z\rangle \le 0$. Since $\{x_{n+1}\}$ is a bounded sequence in a real Hilbert space H, which is reflexive Banach space, there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ and an element in H, say q, such that

$$x_{n_i+1} \rightharpoonup q \text{ and } \limsup_{n\to\infty}\langle u-z, x_{n+1}-z\rangle = \lim_{i\to\infty}\langle u-z, x_{n_i+1}-z\rangle.$$

Since, C is weakly closed, we have $q \in C$ and from (3.19) it follows that $x_{n_i} \rightharpoonup q$ as $i\to\infty$. From (3.17), we obtain that $z_{n_i} \rightharpoonup q$ as $i\to\infty$ and hence using the assumption that $(I-T)$ is demiclosed at zero and (3.16), we conclude that

$$q \in F(T).$$

In addition, from (3.1) and replacing n by n_i in (3.17), we have that

$$\|x_{n_i} - T_{r_{n_i}}x_{n_i}\| = \|x_{n_i} - z_{n_i}\| \to 0 \text{ as } i\to\infty.$$

Thus, since $T_{r_{n_i}}$ is nonexpansive, then $(I-T_{r_{n_i}})$ is demiclosed at zero and hence since $x_{n_i} \rightharpoonup q$ as $i\to\infty$, we obtain that $q = T_{r_{n_i}}q$, that is, $F(q,y) + \langle Aq, y-q\rangle \ge 0$ for all $y\in C$. Hence, we have

$$q \in EP(F,A),$$

which implies that $q \in \Omega$. Hence, since $z = P_\Omega(u)$ and $x_{n_i} \rightharpoonup q$, from (2.1) we get that

$$\begin{aligned}\limsup_{n\to\infty}\langle u-z, x_{n+1}-z\rangle &= \lim_{i\to\infty}\langle u-z, x_{n_i+1}-z\rangle \\ &= \langle u-z, q-z\rangle \le 0 \quad (3.20)\end{aligned}$$

Thus, since $z\in\Omega$, from (3.14), (3.20), assumptions of $\{\alpha_n\}$ and Lemma 2.5, we get that

$$\|x_n - z\| \to 0 \text{ as } n\to\infty.$$

Hence, $x_n \to z = P_\Omega(u)$.

Case 2. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\|x_{n_j} - p\| < \|x_{n_j+1} - p\|,$$

for all $j\in\mathbb{N}$. Then, by Lemma 2.6, there exist a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to\infty$, and

$$\|x_{m_k} - p\| \le \|x_{m_k+1} - p\| \text{ and } \|x_k - p\| \le \|x_{m_k+1} - p\|, \quad (3.21)$$

for all $k\in\mathbb{N}$. Thus, from (3.13), (3.9) and the fact that $\alpha_n \to 0$, we get that

$$\|z_{m_k} - u_{m_k}\| \to 0 \text{ and } \|z_{m_k} - x_{m_k}\| \to 0 \text{ as } k\to\infty.$$

Hence, since $z = P_\Omega(u)$, following the method in Case 1, we obtain that

$$\limsup_{k\to\infty}\langle u-z, x_{m_k+1}-z\rangle \le 0. \quad (3.22)$$

Now, Because $z\in\Omega$, from (3.14), we have that

$$\begin{aligned}\|x_{m_k+1} - z\|^2 &\le (1-\alpha_{m_k})\|x_{m_k} - z\|^2 \\ &\quad + 2\alpha_{m_k}\langle u-z, x_{m_k+1}-z\rangle, \quad (3.23)\end{aligned}$$

and hence, since $z\in\Omega$, (3.21) and (3.23) imply that

$$\begin{aligned}\alpha_{m_k}\|x_{m_k} - z\|^2 &\le \|x_{m_k} - z\|^2 - \|x_{m_k+1} - z\|^2 \\ &\quad + 2\alpha_{m_k}\langle u-z, x_{m_k+1}-z\rangle \\ &\le 2\alpha_{m_k}\langle u-z, x_{m_k+1}-z\rangle.\end{aligned}$$

Hence, the fact that $\alpha_{m_k} > 0$ imply that

$$\|x_{m_k} - z\|^2 \le 2\langle u-z, x_{m_k+1}-z\rangle.$$

Thus, using (3.22) we get that $\|x_{m_k} - z\| \to 0$ as $k\to\infty$. This together with (3.23) and (3.22) implies that $\|x_{m_k+1} - z\| \to 0$ as $k\to\infty$. Since $z\in\Omega$, we obtain $\|x_k - z\| \le \|x_{m_k+1} - z\|$, for all $k\in\mathbb{N}$. Thus, we get that $x_k \to z$ as $k\to\infty$. Consequently, from both the above cases, we deduce that $\{x_n\}$ converges strongly to $z = P_\Omega(u)$. This completes the proof. $\qquad\square$

As a direct consequence of our main result, we obtain the following results:

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $F : C\times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.3. Let $T : C \longrightarrow CB(C)$ be a Lipschitz pseudocontractive mapping with Lipschitz constant L such that $\Omega = F(T)\cap EP(F,A)$ is nonempty, closed and convex. Assume that $(I-T)$ is demiclosed at zero and $Tp = \{p\}$ for all $p\in F(T)$. Let $x_1, u\in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F(z_n, y) + \langle Az_n, y-z_n\rangle + \frac{1}{r_n}\langle y-z_n, z_n-x_n\rangle \ge 0, \ \forall y\in C, \\ y_n = (1-\lambda_n)z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \gamma_n z_n, \end{cases}$$

for all $n\ge 1$, where $u_n \in Tz_n, v_n \in Ty_n$ such that $\|u_n - v_n\| \le 2D(Tz_n, Ty_n)$ and $\{r_n\} \subset [r,\infty)$ for some $r > 0$, $\{\beta_n\}, \{\gamma_n\} \subset [a,b]$, and $\{\alpha_n\} \subset (0,c)$ for some $a,b,c\in(0,1)$ satisfying the following conditions: (i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum\alpha_n = \infty$; (iii) $\alpha_n + \beta_n \le \lambda_n \le \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $z = P_\Omega(u)$.

Proof. Since a Lipschitz pseudocontractive multi-valued mapping T with $F(T) \ne \emptyset$ and $T(p) = \{p\}, \forall p\in F(T)$ is Lipschitz hemicontractive-type, we obtain that the desired result from Theorem 3.2. This completes the proof. $\qquad\square$

If, in Theorem 3.2, we assume that $A \equiv 0$, then we obtain the following corollary:

Corollary 3.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $F : C\times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.3 and $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L such that $\Omega = F(T)\cap EP(F)$ is nonempty, closed and convex. Assume that $(I-T)$ is demiclosed at zero and $Tp = \{p\}$ for all $p\in\Omega$. Let $x_1, u\in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F(z_n, y) + \frac{1}{r_n}\langle y-z_n, z_n-x_n\rangle \ge 0, \ \forall y\in C, \\ y_n = (1-\lambda_n)z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \gamma_n z_n, \end{cases}$$

for all $n\ge 1$, where $u_n \in Tz_n, v_n \in Ty_n$ such that $\|u_n - v_n\| \le 2D(Tz_n, Ty_n)$ and $\{r_n\} \subset [r,\infty)$ for some $r > 0$, $\{\beta_n\}, \{\gamma_n\} \subset [a,b]$, and $\{\alpha_n\} \subset (0,c)$ for some $a,b,c\in(0,1)$ satisfying the following conditions: (i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum\alpha_n = \infty$; (iii) $\alpha_n + \beta_n \le \lambda_n \le \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $z = P_\Omega(u)$.

If, in Theorem 3.2, we assume that $F(x,y) = 0$ for all $x,y\in C$, then we obtain the following corollary.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $T : C \longrightarrow CB(C)$ be a Lipschitz hemicontractive-type mapping with Lipschitz constant L such that $\Omega = F(T)\cap VI(C,A)$ is nonempty, closed and convex. Assume that $(I-T)$ is demiclosed at zero and $Tp = \{p\}$ for all $p\in\Omega$. Let $x_1, u\in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \langle Az_n, y-z_n\rangle + \frac{1}{r_n}\langle y-z_n, z_n-x_n\rangle \ge 0, \ \forall y\in C, \\ y_n = (1-\lambda_n)z_n + \lambda_n u_n, \\ x_{n+1} = \alpha_n u + \beta_n v_n + \gamma_n z_n, \end{cases}$$

for all $n\ge 1$, where $u_n \in Tz_n, v_n \in Ty_n$ such that $\|u_n - v_n\| \le 2D(Tz_n, Ty_n)$ and $\{r_n\} \subset [r,\infty)$ for some $r > 0$, $\{\beta_n\}, \{\gamma_n\} \subset [a,b]$, and $\{\alpha_n\} \subset (0,c)$ for some $a,b,c\in(0,1)$ satisfying the following conditions: (i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n\to\infty}\alpha_n = 0$, $\sum\alpha_n = \infty$; (iii) $\alpha_n + \beta_n \le \lambda_n \le \lambda < \frac{1}{\sqrt{1+4L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $z = P_\Omega(u)$.

If, in Theorem 3.2, we assume that T is a single-valued hemicontractive mapping from C into itself, then we obtain the following corollary.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.3. Let $T : C \longrightarrow C$ be a Lipschitz hemicontractive mapping with Lipschitz constant L such that $\Omega = F(T) \cap EP(F,A)$ is nonempty, closed and convex. Assume that $(I - T)$ is demiclosed at zero. Let $x_1, u \in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
F(z_n, y) + \langle Az_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall y \in C, \\
y_n = (1 - \lambda_n) z_n + \lambda_n T z_n, \\
x_{n+1} = \alpha_n u + \beta_n T y_n + \gamma_n z_n,
\end{cases}
$$

for all $n \geq 1$, where $\{r_n\} \subset [r, \infty)$ for some $r > 0$, $\{\beta_n\}, \{\gamma_n\} \subset [a, b]$, and $\{\alpha_n\} \subset (0, c)$ for some $a, b, c \in (0, 1)$ satisfying the following conditions: (i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n \to \infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $\alpha_n + \beta_n \leq \lambda_n \leq \lambda < \frac{1}{\sqrt{1+L^2}+1}$. Then, the sequence $\{x_n\}$ converges strongly to the point $z = P_\Omega(u)$.

If, in Theorem 3.2, we assume that $T = I$, where I is the identity mapping on C, then we obtain the following corollary.

Corollary 3.7. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1.3 such that $EP(F,A)$ is nonempty. Let $x_1, u \in C$ be arbitrary and $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
F(z_n, y) + \langle Az_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall y \in C, \\
x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n,
\end{cases}
$$

for all $n \geq 1$, where $\{r_n\} \subset [r, \infty)$ for some $r > 0$ and $\{\alpha_n\} \subset (0, c)$ for some $c \in (0, 1)$ such that $\lim_{n \to \infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to the point $z = P_{EP(F,A)}(u)$.

Remark 3.8. *Theorem 3.2 extends the results of Iiduka and Takahashi [7], Meche et al.[10], Takahashi and Takahashi [16, 17], Wang et al.[18], Wong et al.[4] in the sense that our algorithm scheme provides strong convergence to a common element of the set of solutions of a generalized equilibrium problem for continuous monotone mapping and the set of fixed points of a Lipschitz hemicontractive-type multi-valued mapping. We have used the demiclosedness principle to show that $q \in EP(F,A)$ in Theorem 3.2, which is a little simpler than using Assumption 1.3.*

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