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Wronskian determinant solutions of the (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation

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Abstract

A set of sufficient conditions consisting of systems of linear partial differential equations is obtained which guarantees that the Wronskian determinant solves the (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation in the bilinear form. Our results suggest that more general conditions could be derived by further study.

Keywords: *(2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation, Bilinear form, Wronskian determinant solution.*

1 Introduction

It is well-known that many important phenomena in physics and other fields are described by nonlinear partial differential equations. As mathematical models of these phenomena, investigating exact solutions are important in mathematical physics. The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory and explicit formulas of nonlinear partial differential equations play an essential role in the nonlinear science. Also, the explicit formulas may provide physical information and help us to understand the mechanism of related physical models. In the study, we consider the following (2+1)- dimensional Boiti-Leon-Manna-Pempinelli equation

$$u_{yt} + u_{xxxy} - 3\,u_{xx}\,u_y - 3\,u_x\,u_{xy} = 0, \tag{1}$$

which was derived by Gilson et al [1]. during their researched a (2 + 1)-dimensional generalization of the AKNS shallow-water wave equation through the bilinear approach. Recently, many papers have focused their topics on various exact solutions of eq. (1), which include soliton solutions, Quasi-Periodic Waves and new exact solution by means of the bilinear Backlund transformation [1–3].
In this paper, With the development of the Wronskian technique as a powerful tool to construct exact solutions for (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation. In the following sections we apply Wronskian technique to find explicit formulas of solutions of the (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation in Sections 2. The paper is a conclusion in Section 3.

2 Wronskian technique

In this section, we apply Wronskin form [4–6] to the (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation,

$$u_{yt} + u_{xxxy} - 3\,u_{xx}\,u_y - 3\,u_x\,u_{xy} = 0. \tag{2}$$

To solve eq.(2), we introduce a new dependent variable u by

$$u = -2(\ln f)_x, \tag{3}$$

where $f(x,y,t)$ is an unknown real function which will be determined. Substituting eq. (3) into eq. (2), we have

$$\begin{aligned}(-2\ \ln f)_{xyt} + (-2\ \ln f)_{xxxxy} - 3(-2\ \ln f)_{xxx}\,(-2\ \ln f)_{xy}\\ -3(-2\ \ln f)_{xx}\,(-2\ \ln f)_{xxy} = 0,\end{aligned} \tag{4}$$

which can be integrated once with respect to x to give

$$(\ln f)_{yt} + (\ln f)_{xxxy} + 6(\ln f)_{xx}\,(\ln f)_{xy} = C, \tag{5}$$

Taking $C = 0$, therefore, eq. (5) can be written as

$$(D_tD_y + D_x^3 D_y) f\cdot f = 0, \tag{6}$$

where the D-operator, e.g. for a two-variable function, is defined by

$$\begin{aligned}&D_x^m D_t^n f(x,t)\cdot g(x,t) =\\ &(\tfrac{\partial}{\partial x_1} - \tfrac{\partial}{\partial x_2})^m (\tfrac{\partial}{\partial t_1} - \tfrac{\partial}{\partial t_2})^n [f(x_1,t_1) g(x_2,t_2)]\ |_{x_1=x_2=x,\ t_1=t_2=t}\ .\end{aligned} \tag{7}$$

It is easy to see that eq. (6) can be written as

$$\begin{aligned}f_{N,\,xxxy} f_N - f_{N,\,xxx} f_{N,\,y} - 3\, f_{N,\,xxy} f_{N,\,x} + 3\, f_{N,\,xx} f_{N,\,tx}\\ + f_{N,\,ty} f_N - f_{N,\,t} f_{N,\,y} = 0,\end{aligned} \tag{8}$$

Assume that $\phi_j = \phi_j(t,x,y)$,$(j = 1,2,\dots,N)$ in $t \geq 0$,$-\infty$ <x,y< ∞ has continuous derivative up to any order, and satisfies

$$\begin{aligned}&\phi_{j\,,\,t} = -4\,\phi_{j\,,\,xxx}\,,\\ &\phi_{j\,,\,xx} = \frac{k_j^2}{4}\phi_j\,,\\ &\phi_{j\,,\,y} = \phi_{j\,,\,x}\,.\end{aligned} \tag{9}$$

We now construct an Nth-order Wronskian determinant

$$W(\phi_1\,,\,\phi_2\,,\,\dots,\,\phi_N) = (\widehat{N-1};\Phi) = (\widehat{N-1}) = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \dots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \dots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \dots & \phi_N^{(N-1)} \end{vmatrix}, \tag{10}$$

where

$$\Phi = (\phi_1\,,\,\phi_2\ \dots\ \phi_N)^T\,,\ \phi_i^{(0)} = \phi_i\,,\ \phi_i^{(j)} = \frac{\partial^j}{\partial\,x^j}\,\phi_i\,,\ j \geq 1\,,\ 1\ \leq\ i\ \leq\ N\,. \tag{11}$$

Then we have by (10)

$$f_N =|\ 0\,,1\,,\dots\ N-1\ | = |\ \widehat{N-1}\ |\ . \tag{12}$$

Now we consider each order derivative with respect to x of the Wronskian determinant (10). The derivative with respect to x of the Wronskian f_N is equal to the sum of the determinant for $j = 1\,,2\,,\dots\,,N$ where the jth column of f_N is replaced by its derivative. However, the derivative of the first column is equal to the second, the derivative of the second one equals the third, and so on. A consequence, only the determinant with the last column differentiated remain. that is, the derivative $f_{N,\,x}$ is given by

$$f_{N,\,x} =|\ \widehat{N-2}\,,\,N\ |\ . \tag{13}$$

Similarly, we have

$$\begin{aligned}&f_{N,\,xx} =|\ \widehat{N-3}\,,\,N-1\,,\,N\ | + |\ \widehat{N-2}\,,\,N+1\ |,\\ &f_{N,\,xxx} =|\ \widehat{N-4}\,,\,N-2\,,\,N-1\,,\,N+1\ | +2\ |\ \widehat{N-3}\,,\,N-1\,,\,N+1\ |\\ &\qquad + |\ \widehat{N-2}\,,\,N+2\ |,\\ &f_{N,\,xxxx} =|\ \widehat{N-5}\,,\,N-3\,,\,N-2\,,\,N-1\,,\,N\ \ | +3\ |\ \widehat{N-4}\,,\,N-2\,,\,N-1\,,\,N+1\ |\\ &\qquad +2\ |\ \widehat{N-3}\,,\,N\,,\,N+1\,| +3\ |\ \widehat{N-3}\,,\,N-1\,,\,N+2\ | +2\ |\ \widehat{N-2}\,,\,N+3\,|,\end{aligned} \tag{14}$$

It is easy to verify from (9) that y-derivatives can be written

$$f_{N,y} = f_{N,x} = \mid \widehat{N-2}\,, N \mid, \tag{15}$$

$$f_{N,xy} = \mid \widehat{N-3}\,,\, N-1\,,\, N \mid + \mid \widehat{N-2}\,,\, N+1 \mid . \tag{16}$$

Using (9), the derivatives of f_N with respect to t as

$$\begin{aligned} f_{N,t} = -4(&\mid \widehat{N-4}\,,\, N-2\,,\, N-1\,,\, N \mid - \mid \widehat{N-3}\,,\, N-1\,,\, N+1 \mid \\ &+ \mid \widehat{N-2}\,,\, N+2 \mid), \end{aligned} \tag{17}$$

and

$$\begin{aligned} f_{N,ty} = -4(&\mid \widehat{N-5}\,,\, N-3\,,\, N-2\,,\, N-1\,,\, N \mid - \mid \widehat{N-3}\,,\, N\,,\, N+1 \mid \\ &+ \mid \widehat{N-2}\,,\, N+3 \mid). \end{aligned} \tag{18}$$

Substituting (12) and (13)-(18) into (8), we have

$$\begin{aligned} &3 \mid N-1 \mid (\mid \widehat{N-5}, N-3, N-2, N-1, N \mid - \mid \widehat{N-4}, N-2, N-1, N+1 \mid \\ &-2 \mid \widehat{N-3}, N, N+1 \mid - \mid \widehat{N-3}, N-1, N+2 \mid + \mid \widehat{N-2}, N+3 \mid) \\ &-12 \mid \widehat{N-2}, N \mid (\mid \widehat{N-3}, N-1, N+1 \mid) \\ &+3 \mid N-1 \mid (\mid \widehat{N-3}, N-1, N \mid + \mid \widehat{N-2}, N+1 \mid)^2 = 0\,, \end{aligned} \tag{19}$$

On the other hand, by using

$$\mid M,a,b \mid\mid M,c,d \mid - \mid M,a,c \mid\mid M,b,d \mid + \mid M,a,d \mid\mid M,b,c \mid = 0\,, \tag{20}$$

where M is an $N \times (N-2)$ matrix, and a,b,c,d are N-dimensional column vectors.Assume that a_j be an N-dimensional column vectors and γ_j be a real constant not to zero.Then we have,

$$\sum_{i=1}^{N} (\gamma_i \mid a_1, a_2, \ldots, a_N \mid) = \sum_{i=1}^{N} (\mid a_1, a_2, \ldots, \gamma\, a_j, \ldots, a_N \mid) \tag{21}$$

where

$$\gamma\, a_j = (\, \gamma_1\, a_1 j, \gamma_2\, a_2 j, \ldots, \gamma_N\, a_N j\,)^T \quad \textit{and} \quad 1 \le j \le N \tag{22}$$

and

$$\sum_{i=1}^{N} \frac{{k_i}^2}{4} (\sum_{i=1}^{N} \frac{{k_i}^2}{4} \mid \widehat{N-1} \mid) \mid \widehat{N-1} \mid = (\sum_{i=1}^{N} \frac{{k_i}^2}{4} \mid \widehat{N-1} \mid)^2. \tag{23}$$

We have

$$\begin{aligned} &\mid \widehat{N-5}, N-3, N-2, N-1, N \mid - \mid \widehat{N-4}, N-2, N-1, N+1 \mid \\ &- \mid \widehat{N-3}, N-1, N+2 \mid +2 \mid \widehat{N-3}, N, N+1 \mid + \mid \widehat{N-2}, N+3 \mid = \\ &(\mid \widehat{N-3}, N-1, N \mid + \mid \widehat{N-2}, N+1 \mid)^2\,, \end{aligned} \tag{24}$$

Substituting (24) into (19) and using (8) we arrive at

$$\begin{aligned} &\mid \widehat{N-3}, N-2, N-1 \mid\mid \widehat{N-3}, N, N+1 \mid \\ &- \mid \widehat{N-3}, N-2, N \mid\mid \widehat{N-3}, N-1, N+1 \mid \\ &+ \mid \widehat{N-3}, N-1, N \mid\mid \widehat{N-3}, N-2, N+1 \mid = 0\,. \end{aligned} \tag{25}$$

This means that $f = f_N$ satisfies (20). If we choose the special solution of (9) as

$$\phi_j(t,x,y) = \exp^{\frac{\xi_j}{2}} + (-1)^{j+1} \exp^{\frac{-\xi_j}{2}} \tag{26}$$

with

$$\xi_j = k_j\, x + k_j\, y - k_j^3\, t\,, \qquad 1 \le j \le N\,. \tag{27}$$

With the help of eq. (3) and (27), we obtain solution for the (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation terms of the Wronskian determinant

$$u(x,y,t) = -2\, \partial_x^2\, \ln W(\phi_1, \phi_2, \ldots, \phi_N). \tag{28}$$

3 Conclusion

In this paper, we gave the Wronskian determinant solutions of the (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation through the Wronskian technique. Moreover, we construct exact solutions for (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation of this equation by solving the resultant systems of linear partial differential equations which guarantee that the Wronskian determinant solves the equation in the bilinear form.

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