Operations on covering numbers of certain graph classes

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Abstract

The bounds on the sum and product of chromatic numbers of a graph and its complement are known as Nordhaus-Gaddum inequalities. In a similar way, the operations on the covering numbers of graphs with their complement are studied and with respect to this, new characterizations of certain graph classes have also been given in this paper.

Keywords: Covering Number; Graph Complement; Independence Number; Line Graph; Matching Number.

1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [11]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

Many problems in extremal graph theory seek the extreme values of graph parameters on families of graphs. The classic paper of Nordhaus and Gaddum [7] study the extreme values of the sum (or product) of a parameter on a graph and its complement, following solving these problems for the chromatic number on n-vertex graphs. In this paper, we study such problems for some graphs and their associated graphs.

Definition 1.1. [6] A *Walk*, $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$, in a graph G is a finite sequence whose terms are alternately vertices and edges such that, for $1 \le i \le k$, the edge e_i has ends v_{i-1} and v_i.

Definition 1.2. [6] If the vertices $v_0, v_1, \ldots, v_k$ of a walk W are distinct then W is called a *Path*. A path with n vertices will be denoted by P_n. P_n has length $n-1$.

Definition 1.3. [6] Let G be a simple graph with n vertices. The *complement* $\bar{G}$ of G is defined to be the simple graph with the same vertex set as G and where two vertices u and v are adjacent precisely when they are not adjacent in G. Roughly speaking then, the complement of G can be obtained from the complete graph K_n by rubbing out all the edges of G.

Definition 1.4. [5] Two vertices that are not adjacent in a graph G are said to be *independent*. A set S of vertices is independent if any two vertices of S are independent. The *vertex independence number* or simply the *independence number*, of a graph G, denoted by $\alpha(G)$ is the maximum cardinality among the independent sets of vertices of G.

Definition 1.5. [2] A subset M of the edge set of G, is called a *matching* in G if no two of the edges in M are adjacent. In other words, if for any two edges e and f in M, both the end vertices of e are different from the end vertices of f.

Definition 1.6. [2] A *perfect matching* of a graph G is a matching of G containing $n/2$ edges, the largest possible, meaning perfect matchings are only possible on graphs with an even number of vertices. A perfect matching sometimes called a *complete matching* or *1-factor*.

Definition 1.7. [2] The matching number of a graph G, denoted by $\nu(G)$, is the size of a maximal independent edge set. It is also known as *edge independence number*. The matching number $\nu(G)$ satisfies the inequality $\nu(G) \le \lfloor \frac{n}{2} \rfloor$.

Equality occurs only for a perfect matching and graph G has a perfect matching if and only if $|G| = 2\,\nu(G)$, where $|G| = n$ is the vertex count of G.

Definition 1.8. [2] A *maximum independent set* in a line graph corresponds to maximum matching in the original graph.

In this paper, we discussed the sum and product of the covering numbers of certain class of graphs and their complements.

2. New Results

Theorem 2.1. [15] The independence number $\alpha(G)$ of a graph G and vertex cover number $\beta(G)$ are related by $\alpha(G) + \beta(G) = |G|$, where $|G| = n$, the vertex count of G.

Proposition 2.2. For a complete graph K_n, $n \ge 3$, $\beta(K_n) + \beta(\bar{K}_n) = n - 1$ and $\beta(K_n).\beta(\bar{K}_n) = 0$.

Proof. The independence number of a complete graph K_n on n vertices is 1, since each vertex is joined with every other vertex of the graph G. By theorem 2.1, the covering number, $\beta(K_n)$ of K_n = $n - 1$. By [13], Since, the complement of K_n is an empty graph with n vertices, it will have an independence number n. Therefore by theorem 2.1, $\beta(\bar{K}_n) = n - n = 0$. That is for a complete graph K_n, $n \ge 3$, $\beta(K_n) + \beta(\bar{K}_n) = n - 1$ and $\beta(K_n).\beta(\bar{K}_n) = 0$. □

Proposition 2.3. *For a complete bipartite graph* $K_{m,n}$,

$$\beta(K_{m,n}) + \beta(\bar{K}_{m,n}) = 2(m-1) + n \ and$$

$$\beta(K_{m,n}).\beta(\bar{K}_{m,n}) = m(m+n-2)$$

Proof. Without the loss of generality, let $m < n$. The independence number of a complete bipartite graph, $\alpha(K_{m,n}) = max(m,n) = n$. Since a complete bipartite graph consists of $m+n$ number of vertices, by theorem 2.1, $\beta(K_{m,n}) = m+n-n = m$. The complement of $K_{m,n}$ is $K_m \cup K_n$, which is a disjoint union of 2 complete graphs. The independence number of $K_m \cup K_n$ is 1+1=2. By 2.1, $\beta(\bar{K}_{m,n}) = m+n-2$.
Therefore, $\beta(K_{m,n}) + \beta(\bar{K}_{m,n}) = m + (m+n-2) = 2(m-1) + n$ and $\beta(K_{m,n}).\beta(\bar{K}_{m,n}) = m(m+n-2)$ $\square$

Definition 2.4. [11] For $n \geq 3$, a *wheel graph* W_{n+1} is the graph $K_1 + C_n$. A wheel graph W_{n+1} has $n+1$ vertices and $2n$ edges.

Theorem 2.5. For a wheel graph W_{n+1}, $n \geq 3$,

$$\beta(W_{n+1}) + \beta(W_{n+1}^-) = \begin{cases} \frac{(3n-2)}{2} & \text{; if } n \text{ is even} \\ \frac{(3n-1)}{2} & \text{; if } n \text{ is odd} \end{cases}$$

and

$$\beta(W_{n+1}).\beta(W_{n+1}) = \begin{cases} \frac{(n^2-4)}{2} & \text{; if } n \text{ is even} \\ \\ \frac{(n-2)(n+3)}{2} & \text{; if } n \text{ is odd} \end{cases}$$

Proof. By [12], the independence number of a wheel graph W_{n+1} is $\lfloor \frac{n}{2} \rfloor$. The number of vertices in W_{n+1} is $n+1$. Then by theorem 2.1, $\beta(W_{n+1}) = (n+1) - \lfloor \frac{n}{2} \rfloor = (n+1) - \frac{n}{2} = \frac{(n+2)}{2}$, if n is even and $(n+1) - \frac{(n-1)}{2} = \frac{(n+3)}{2}$, if n is odd. Since the central vertex is adjacent to all other vertices of W_{n+1}, in $\bar{W}_{n+1}$, that vertex contributes 1 to the maximal independence set. All the vertices other than the two which are adjacent in W_{n+1} will be adjacent to any vertex v in $\bar{W}_{n+1}$. There fore any maximal independence set in $\bar{W}_{n+1}$ can have atmost 2 elements. That is independence number of $\bar{W}_{n+1}$ is 3. There fore by 2.1, $\beta(\bar{W}_{n+1}) = n+1-3 = n-2$.
That is For $n \geq 3$,

$$\beta(W_{n+1}) + \beta(\bar{W}_{n+1}) = \begin{cases} \frac{(3n-2)}{2} & \text{; if } n \text{ is even} \\ \frac{(3n-1)}{2} & \text{; if } n \text{ is odd} \end{cases}$$

and

$$\beta(W_{n+1}).\beta(\bar{W}_{n+1})) = \begin{cases} \frac{(n^2-4)}{2} & \text{; if } n \text{ is even} \\ \\ \frac{(n-2)(n+3)}{2} & \text{; if } n \text{ is odd} \end{cases}$$

$\square$

Definition 2.6. [10] *Helm graphs* are graphs obtained from a wheel by attaching one pendant edge to each vertex of the cycle.

Theorem 2.7. *For a helm graph H_n, $n \geq 3$, $\beta(H_n) + \beta(\bar{H}_n) = 3n-2$ and $\beta(H_n).\beta(\bar{H}_n) = 2n(n-1)$.*

Proof. A helm graph H_n consists of $2n+1$ vertices and $3n$ edges. By [12], the independence number of a helm graph, $\alpha(H_n) = n+1$. Since, the number of vertices of H_n is $2n+1$, by theorem 2.1, $\beta(H_n) = (2n+1) - (n+1) = n$. By [13], $\alpha(\bar{H}_n) = 3$ and by theorem 2.1, $\beta(\bar{H}_n) = (2n+1) - 3 = 2n-2$.
Therefore, For a helm graph H_n, $n \geq 3$, $\beta(H_n) + \beta(\bar{H}_n) = 3n-2$ and $\beta(H_n).\beta(\bar{H}_n) = 2n(n-1)$. $\square$

Definition 2.8. [15] Given a vertex x and a set U of vertices, an x, $U-$fan is a set of paths from x to U such that any two of them share only the vertex x. A $U-$fan is denoted by $F_{1,n}$.

Theorem 2.9. *For a fan graph $F_{1,n}$,*

$$\beta(F_{1,n}) + \beta(\bar{F}_{1,n}) = \begin{cases} \frac{3n-2}{2} & \text{; if } n \text{ is even} \\ \frac{3n-3}{2} & \text{; if } n \text{ is odd} \end{cases}$$

and

$$\beta(F_{1,n}).\beta(\bar{F}_{1,n}) = \begin{cases} \frac{n^2-4}{2} & \text{; if } n \text{ is even} \\ \frac{(n+1)(n-2)}{2} & \text{; if } n \text{ is odd} \end{cases}$$

Proof. A fan graph $F_{1,n}$ is defined to be a graph $K_1 + P_n$. By [12], the independence number of a fan graph $F_{1,n}$ is either $\frac{n}{2}$ or $\frac{n+1}{2}$, depending on n is even or odd. Since the number of vertices of $F_{1,n}$ is $n+1$, by theorem 2.1, $\beta(F_{1,n}) = (n+1) - \frac{n}{2} = \frac{n+2}{2}$, if n is even and $(n+1) - \frac{(n+1)}{2} = \frac{n+1}{2}$, if n is odd. By [13], the independence number of $\bar{F}_{1,n}$ is 3. By theorem 2.1, $\beta(\bar{F}_{1,n}) = n+1-3 = n-2$.
Therefore, For a fan graph $F_{1,n}$,

$$\beta(F_{1,n}) + \beta(\bar{F}_{1,n}) = \begin{cases} \frac{3n-2}{2} & \text{; if } n \text{ is even} \\ \frac{3n-3}{2} & \text{; if } n \text{ is odd} \end{cases}$$

and

$$\beta(F_{1,n}).\beta(\bar{F}_{1,n}) = \begin{cases} \frac{n^2-4}{2} & \text{; if } n \text{ is even} \\ \frac{(n+1)(n-2)}{2} & \text{; if } n \text{ is odd} \end{cases}$$

$\square$

Definition 2.10. [3, 17] An $n-$sun or a *trampoline*, denoted by S_n, is a chordal graph on $2n$ vertices, where $n \geq 3$, whose vertex set can be partitioned into two sets $U = \{u_1, u_2, u_3, ..., u_n\}$ and $W = \{w_1, w_2, w_3, ..., w_n\}$ such that U is an independent set of G and u_i is adjacent to w_j if and only if $j = i$ or $j = i+1 (\mod n)$. A *complete sun* is a sun G where the induced subgraph $\langle U \rangle$ is complete.

Theorem 2.11. *For a complete sun graph S_n, $n \geq 3$, $\beta(S_n) + \beta(\bar{S}_n) = 2n$ and $\beta(S_n).\beta(\bar{S}_n) = n^2$.*

Proof. Let S_n be a complete sun graph on $2n$ vertices. By [12], the independence number of S_n, $\alpha(S_n) = n$. Since S_n consists of $2n$ vertices, by theorem 2.1, $\beta(S_n) = 2n - n = n$. Now consider the complement of S_n. Since $V = \{v_1, v_2, v_3, ..., v_n\}$ be the vertex set of the complete graph K_n in S_n, it contributes n to the maximal independence set of $\alpha(\bar{S}_n)$ and $U = \{u_1, u_2, u_3, ..., u_n\}$ can't do anything regarding the same. That is $\alpha(\bar{S}_n) = n$. By theorem 2.1, $\beta(\bar{S}_n) = 2n - n = n$.
Therefore, For a complete sun graph S_n, $n \geq 3$, $\beta(S_n) + \beta(\bar{S}_n) = 2n$ and $\beta(S_n).\beta(\bar{S}_n) = n^2$. $\square$

Definition 2.12. [17] The $n-$sunlet graph is the graph on $2n$ vertices obtained by attaching n pendant edges to a cycle graph C_n and is denoted by L_n.

Theorem 2.13. *For a sunlet graph L_n on $2n$ vertices, $n \geq 3$, $\beta(L_n) + \beta(\bar{L}_n) = 3n-2$ and $\beta(L_n).\beta(\bar{L}_n) = 2n(n-1)$.*

Proof. Let L_n be a sunlet graph on $2n$ vertices. By [12], the independence number of L_n, $\alpha(L_n) = n$. Since L_n consists of $2n$ vertices, by theorem 2.1, $\beta(L_n) = 2n - n = n$. By [13], $\alpha(\bar{L}_n) = 2$ and by theorem 2.1, $\beta(\bar{L}_n) = 2n - 2 = 2(n-1)$.
Therefore, For a sunlet graph L_n on $2n$ vertices, $n \geq 3$, $\beta(L_n) + \beta(\bar{L}_n) = 3n-2$ and $\beta(L_n).\beta(\bar{L}_n) = 2n(n-1)$. $\square$

Definition 2.14. [11] The *armed crown* is a graph G obtained by adjoining a path P_m to every vertex of a cycle C_n.

Theorem 2.15. For an armed crown graph G with a path P_m and a cycle C_n, $\beta(G) + \beta(\bar{G}) = \frac{3mn-4}{2}$ and $\beta(G).\beta(\bar{G}) = \frac{mn(mn-2)}{2}$.

Proof. Note that the number of vertices of P_m is m. By [12], $\alpha(G) = \frac{mn}{2}$, except for m and n are odd . The number of vertices of an armed crown graph is mn. By theorem 2.1, $\beta(G) = mn - \frac{mn}{2} = \frac{mn}{2}$. Now consider the complement of G. Let v be a pendent vertex of any one of the paths P_m attached to the cycle C_n. Also let u be the vertex adjacent to v in the path. Then u and v are independent in $\bar{G}$ and all the other vertices in P_m is adjacent either to u or to v, since the armed crown contains no complete graph other than K_2. There will be no independent set with cardinality greater than or equal to 3. There fore the independence number of $\bar{G}$ is 2. By theorem 2.1, $\beta(\bar{G}) = mn - 2$.

Therefore For an armed crown graph G with a path P_m and a cycle C_n, $\beta(G) + \beta(\bar{G}) = \frac{mn}{2} + (mn - 2) = \frac{3mn-4}{2}$ and $\beta(G).\beta(\bar{G}) = \frac{mn}{2}.(mn-2) = \frac{mn(mn-2)}{2}$. $\square$

3. Conclusion

The theoretical results obtained in this research may provide a better insight into the problems involving matching number and independence number by improving the known lower and upper bounds on sums and products of independence numbers of a graph G and an associated graph of G. More properties and characteristics of operations on independence number and also other graph parameters are yet to be investigated. The problems of establishing the inequalities on sums and products of independence numbers for various graphs and graph classes still remain unsettled. All these facts highlight a wide scope for further studies in this area.

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