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Research Paper

On a class of Leibniz algebras

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Abstract

We pointed out the class of Leibniz algebras such that the Killing form is non degenerate implies algebras are semisimple.

Keywords: *Killing form; Leibniz algebras; Leibniz modules; Representations; Semisimplicity.*

1. Introduction

Throughout this paper, F will be an algebraically closed field of characteristic zero. All vector spaces and algebras will be finite dimensional over F. Note the sum of two vector subspaces V_1, V_2 by $V_1 \dot{+} V_2$ and direct sum by $V_1 \oplus V_2$. It is well-known that a Lie algebra is semisimple if and only if its Killing form is non degenerate. An equivalent criterion is found for Leibniz algebra L which satisfies, for all x, y in L, the trace of the endomorphism $(ad_x \circ ad_y)_{|Ess(L)}$ equals zero. Call such algebras "Killing- Leibniz-Algebra".

Section 2 is devoted to basic facts. In Section 3, the links between radical and nilradical are set. Section 4 is devoted to the nilpotency of the ideal $\{Rad(L), L\}$. In Section 5, the main theorem is settled. For conclusion, we give an hierarchy of Leibniz algebras and two questions are done about Killing Leibniz Algebras.

2. Preliminary notes

Let us note that Leibniz algebras are defined in two classes:

- Right Leibniz algebras, with the rule

  $$[x,[y,z]] = [[x,y],z] - [[x,z],y] \text{ for any } x,y,z \in L. \tag{1}$$

- Left Leibniz algebras, with the rule

  $$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \text{ for any } x,y,z \in L. \tag{2}$$

For an algebra $(A, [\,,\,])$ with vectors multiplication $[a,b]$, for all a, b in A, define the algebra $(A, [\,,\,]^{op})$ as the underlying vector space A where the vectors multiplication is defined by $[a,b]^{op} = [b,a]$. We have that:

Proposition 2.1. *The algebra* $(A, [\,,\,])$ *is left Leibniz algebra if and only if the algebra* $(A, [\,,\,]^{op})$ *is right Leibniz algebra.*

So results on Left Leibniz algebras are available on Right Leibniz algebras, (with minors variations).
Here we write "Leibniz algebras" for "Right Leibniz algebras".

It follows from the equation (1) called Leibniz identity that in any Leibniz algebra one has

$$[y,[x,x]] = 0,\ [z,[x,y]] + [z,[y,x]] = 0, \text{ for all } x,y,z \in L.$$

Definition 2.2. *(Ideal) A subspace* H *of a Leibniz algebra* L *is called left (respectively right) ideal if for* $a \in H$ *and* $x \in L$ *one has* $[x,a] \in H$ *(respectively* $[a,x] \in H$*). If* H *is both left and right ideal, then* H *is called (two-sided) ideal.*

If V *is a vector space, let* $End_F(V)$ *denotes the set of all endomorphisms of* V*. An action of* L *on* $End_F(V)$ *is a linear map of* L *on* $End_F(V)$*.*

Definition 2.3. *(Representation) Let* L *be a Leibniz algebra and* V *a vector space.* V *is an* L*-module if there are:*

- *a left action,* $l : L \longrightarrow End_F(V)$, $x \mapsto l_x$,
- *a right action,* $r : L \longrightarrow End_F(V)$, $x \mapsto r_x$,
  such that:

$$\begin{aligned} r_{[x,y]} &= r_y r_x - r_x r_y, \\ l_{[x,y]} &= r_y l_x - l_x r_y, \\ l_{[x,y]} &= r_y l_x + l_x l_y. \end{aligned}$$

For x in L, $r_x(v)$ will be denoted by vx and $l_x(v)$ will be denoted by xv. The triplet (l,r,V) is called a representation of L on V. Now if L is a Leibniz algebra, we have the adjoint representation "(Ad, ad, L)" defined as follows: for all x and y in L, $ad_x : L \longrightarrow L$, $y \longmapsto [y,x]$ and $Ad_x : L \longrightarrow L$, $y \longmapsto [x,y]$.

Remark 2.4.
For $x \in L$, $ad_x : L \longrightarrow L$ *is a derivation of* L *i.e. for all* x,y,z *in* L, $ad_x([y,z]) = [ad_x(y), z] + [y, ad_x(z)]$.
For $x \in L$, $Ad_x : L \longrightarrow L$ *is an anti-derivation of* L *i.e. for all* x,y,z *in* L, $Ad_x([y,z]) = [Ad_x(y), z] - [Ad_x(z), y]$.

For an arbitrary algebra and for all non negative integer n let us define the sequences:

(i) $D^1(L) = L^{[1]} = L^2$, $D^{n+1}(L) = L^{[n+1]} = [L^{[n]}, L^{[n]}]$;

(ii) $L^1 = L$, $L^{n+1} = [L^1, L^n] + [L^2, L^{n-1}] + \cdots + [L^{n-1}, L^2] + [L^n, L^1]$.

Definition 2.5. *([1]) An algebra* L *is called solvable if there exists* $m \in \mathbb{N}^*$ *such that* $D^m(L) = L^{[m]} = \{0\}$.
An algebra L *is called nilpotent if there exists* $m \in \mathbb{N}^*$ *such that* $L^m = \{0\}$.

Definition 2.6. *Let* A *be a subspace of a Leibniz algebra* L*. The normalizer of* A *is denoted by:*

$$n_L(A) = \{y \in L |\, [y,a] \in A \text{ and } [a,y] \in A \text{ for all } a \in A\}.$$

Definition 2.7. *([4]) A Leibniz algebra* L *is said to be semisimple if* $Rad(L) = \mathrm{Ess}(L)$.

Equivalently, we can say that:
Leibniz algebra L simple if $\{0\} \neq [L,L] \neq Ess(L)$ and every ideal of L belongs to the set $\{L, Ess(L), (0)\}$.
Since $D\imath = \imath^2$ is an ideal whenever $\imath$ is (by Equation 1), if $rad(L) \neq Ess(L)$ then L contains an ideal $\jmath$ which satisfies $\jmath^2 \subseteq Ess(L) \subsetneq \jmath$.
So an other equivalent definition is:

Remark 2.8. L *is semisimple if it has no ideal* $\jmath$ *which satisfies* $\jmath^2 \subseteq \mathrm{Ess}(L) \subsetneq \jmath$.

Lemma 2.9. *[3] Let* L *be a Leibniz algebra and* (l,r,V) *a representation of* L*. Let* A *be a subspace of* L*, then* $r_A = \{r_x, \text{ for all } x \in A\}$ *is a subspace of the vector space* $End_F(V)$*. In particular,* r_L *is a Lie subalgebra of* $gl(V)$ *and* L *is solvable (respectively nilpotent) if and only if* r_L *is solvable (respectively nilpotent).*

Proof. The results are clear since for all x, y in L and for all λ in F, we have that
$r_{x+\lambda y} = r_x + \lambda r_y$ and $[r_x, r_y] = r_{[y,x]}$. □

Remark 2.10. *Let L be a Leibniz algebra and (l, r, V) a representation of L. If for all x in L, r_x is nilpotent then l_x is also nilpotent for all x. Since we have $l_x^k = (-1)^{k+1} l_x (r_x)^{k-1}$. Thus when r_x is nilpotent for all x in L, we can say that the representation (l, r, V) of L is nilpotent.*

Lemma 2.11. *Let L be a Leibniz algebra and (l, r, V) a representation of L. Let A be a subspace of the vector space L and let x in the normalizer $n_L(A)$ of A. Then we have for all integer k in $\mathbb{N}$ and for all a in A:*

i) $\delta_{k+1} = r_a^{k+1} r_x - r_x r_a^{k+1} \in r_A^{k+1}$.

ii) $\beta_{k+1} = r_x^{k+1} r_a - r_a r_x^{k+1} \in r_A r_x^k \dot{+} \cdots \dot{+} r_A r_x \dot{+} r_A$.

Proof. For i), since $[r_a, r_x] = r_{[x,a]}$, we have $\delta_1 = r_a r_x - r_x r_a = r_{[x,a]}$. Thus $\delta_1 \in r_A$ since $x \in n_L(A)$. And we have:

$$\begin{aligned} \delta_2 &= r_a^2 r_x - r_x r_a^2 = r_a (r_a r_x) - r_x r_a^2 = r_a (r_x r_a + \delta_1) - r_x r_a^2 = (r_a r_x) r_a + r_a \delta_1 - r_x r_a^2 \\ &= (r_x r_a + \delta_1) r_a + r_a \delta_1 - r_x r_a^2 = \delta_1 r_a + r_a \delta_1 \in r_A^2. \end{aligned}$$

With the hypothesis of recurrence: $\delta_k = r_a^k r_x - r_x r_a^k \in r_A^k$, we get:

$$\begin{aligned} \delta_{k+1} &= r_a^{k+1} r_x - r_x r_a^{k+1} = r_a \left(r_a^k r_x\right) - r_x r_a^{k+1} = r_a \left(r_x r_a^k + \delta_k\right) - r_x r_a^{k+1} = (r_a r_x) r_a^k + r_a \delta_k - r_x r_a^{k+1} \\ &= (r_x r_a + \delta_1) r_a^k + r_a \delta_k - r_x r_a^{k+1} = \delta_1 r_a^k + r_a \delta_k \in (r_A)^{k+1}. \end{aligned}$$

And for ii), we have $[r_x, r_a] = r_{[a,x]}$, so $\beta_1 = -\delta_1 \in r_A = r_A r_x^0$ since $x \in n_L(A)$ (where $r_x^0 = 1_V$). Note that we have:

$$\begin{aligned} \beta_2 &= r_x^2 r_a - r_a r_x^2 = r_x (r_x r_a) - r_a r_x^2 = r_x \left(r_a r_x + r_{[a,x]}\right) - r_a r_x^2 = (r_x r_a) r_x + r_x r_{[a,x]} - r_a r_x^2 \\ &= \left(r_a r_x + r_{[a,x]}\right) r_x + \left(r_{[a,x]} r_x + r_{[[a,x],x]}\right) - r_a r_x^2 = 2 r_{[a,x]} r_x + r_{[[a,x],x]} \in r_A r_x \dot{+} r_A. \end{aligned}$$

Set the hypothesis that $\beta_k = r_x^k r_a - r_a r_x^k \in r_A r_x^{k-1} \dot{+} \cdots \dot{+} r_A r_x \dot{+} r_A$, and then it will follow that:

$$\begin{aligned} \beta_{k+1} &= r_x^{k+1} r_a - r_a r_x^{k+1} = r_x^k (r_x r_a) - r_a r_x^{k+1} = r_x^k \left(r_a r_x + r_{[a,x]}\right) - r_a r_x^{k+1} = \left(r_x^k r_a\right) r_x + r_x^k r_{[a,x]} - r_a r_x^{k+1} \\ &= \left(r_a r_x^k + \beta_k\right) r_x + r_{[a,x]} r_x^k + \beta_1' - r_a r_x^{k+1} (\text{ where } \beta_1' = r_x^k r_{[a,x]} - r_{[a,x]} r_x^k = r^k_{[[a,x],x]} \in r_A^k) \\ &= \beta_k r_x + r_{[a,x]} r_x^k + \beta_1' \in \left(r_A r_x^{k-1} \dot{+} \cdots \dot{+} r_A\right) r_x + r_A r_x^k + \dot{+} r_A \in r_A r_x^k \dot{+} r_A r_x^{k-1} \dot{+} \cdots \dot{+} r_A r_x \dot{+} r_A. \end{aligned}$$

Proofs are done. □

Lemma 2.12. *Let L be a Leibniz algebra and (l, r, V) a representation of L. Let A be a subspace of the vector space L and x in the normalizer $n_L(A)$ of A. Then we have for all integer k and p in $\mathbb{N}$:*

$$\left[r_A^p r_x^k\right] \circ r_A \subseteq r_A^{p+1} r_x^k \dot{+} \cdots \dot{+} r_A^{p+1} r_x \dot{+} r_A^{p+1}.$$

Proof. We shall note that:

$$\left[r_A^p r_x^k\right] \circ r_A = r_A^p \circ \left[r_x^k \circ r_A\right] \subseteq r_A^p \left(r_A r_x^k \dot{+} \cdots \dot{+} r_A r_x \dot{+} r_A\right) \subseteq r_A^{p+1} r_x^k \dot{+} \cdots \dot{+} r_A^{p+1} r_x \dot{+} (r_A)^{p+1}.$$

□

Thanks to the preceding lemma whe have for all integer k, l, p and q in $\mathbb{N}$:

$$r_A^p r_x^k \circ r_A^q r_x^l \subseteq r_A^{p+q} r_x^{k+l} \dot{+} \cdots \dot{+} r_A^{p+q} r_x^l.$$

Lemma 2.13. *Let L be a Leibniz algebra and (l, r, V) a representation of L. Let A be a subspace of the vector space L and x in the normalizer $n_L(A)$ of A and for a non negative integer k let E_k be the subspace $E_k = r_A r_x^k \dot{+} \cdots \dot{+} r_A$. Then we have for all integer p in $\mathbb{N}^*$:*

$$E_k^p \subseteq r_A^p r_x^{pk} \dot{+} \cdots \dot{+} r_A^p r_x^{2k} \dot{+} \cdots \dot{+} r_A^p r_x \dot{+} r_A^p.$$

Proof. Let us compute E_k^p for $p = 2, 3$; we have $[r_x, r_a] = r_{[a,x]}$, so

$$\begin{aligned} E_k^2 &= \left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right)^2 = \left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right)\left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right) \\ &\subseteq \left(r_A r_x^k\right)\left(r_A r_x^k\right) \dot{+} \cdots \dot{+} r_A (r_A r_x) \dot{+} (r_A r_x) r_A \dot{+} r_A r_A \subseteq r_A^2 r_x^{2k} \dot{+} \cdots \dot{+} r_A^2 r_x^k \dot{+} \cdots \dot{+} r_A^2 r_x \dot{+} r_A^2, \\ E_k^3 &= \left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right)^3 = \left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right)^2 \left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right) \\ &\subseteq \left(r_A^2 r_x^{2k} \dot{+} \cdots \dot{+} r_A^2 r_x^k \dot{+} \cdots \dot{+} r_A^2\right)\left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right) \subseteq \left(r_A^2 r_x^{2k}\right)\left(r_A r_x^k\right) \dot{+} \cdots \dot{+} r_A^2 (r_A r_x) \dot{+} \left(r_A^2 r_x\right) r_A \dot{+} r_A^2 r_A \\ &\subseteq r_A^3 r_x^{3k} \dot{+} \cdots \dot{+} r_A^3 r_x^{2k} \dot{+} \cdots \dot{+} r_A^3 r_x \dot{+} r_A^3 \end{aligned}$$

and set by hypothesis that we have $E_k^{p-1} \subseteq r_A^{p-1} r_x^{(p-1)k} \dot{+} \cdots \dot{+} r_A^{p-1} r_x \dot{+} r_A^{p-1}$.

$$\begin{aligned}\text{So we will get } E_k^p &= \left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right)^p = \left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right)^{p-1}\left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right)\\ &\subseteq \left(r_A^{p-1} r_x^{(p-1)k} \dot{+} \cdots \dot{+} r_A^{p-1} r_x \dot{+} r_A^{p-1}\right)\left(r_A r_x^k \dot{+} \cdots \dot{+} r_A\right)\\ &\subseteq \left(r_A^{p-1} r_x^{(p-1)k}\right)\left(r_A r_x^k\right) \dot{+} \cdots \dot{+} r_A^{p-1}\left(r_A r_x\right) \dot{+} \left(r_A^{p-1} r_x\right) r_A \dot{+} r_A^{p-1} r_A\\ &\subseteq r_A^p r_x^{pk} \dot{+} \cdots \dot{+} r_A^p r_x^{2k} \dot{+} \cdots \dot{+} r_A^p r_x \dot{+} r_A^p.\end{aligned}$$

The proof is then done. □

Lemma 2.14. *Let L be a Leibniz algebra and (l, r, V) a representation of L. Let A be a subspace of the vector space L and x in the normalizer $n_L(A)$ of A. Let m be a non negative integer. Then for all $(\lambda, a) \in F \times A$,*

$$f_m = \left(r_{a+\lambda x}\right)^m - \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k} r_x^k \in r_A r_x^m \dot{+} \cdots \dot{+} r_A.$$

Proof. By induction:

$$f_1 = \left(r_{a+\lambda x}\right)^1 - \sum_{k=0}^{1}\binom{1}{k}\lambda^k r_a^{1-k} r_x^k = r_{a+\lambda x} - \left(r_a + \lambda r_x\right) = 0 \in r_A r_x \dot{+} r_A.$$

And if by hypoyhesis we have: $f_m = \left(r_{a+\lambda x}\right)^m - \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k} r_x^k \in r_A r_x^m \dot{+} \cdots \dot{+} r_A$,we can write:

$$\begin{aligned} f_{m+1} &= \left(r_a + \lambda r_x\right)^{m+1} - \sum_{k=0}^{m+1}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k = \left(r_a + \lambda r_x\right)^m\left(r_a + \lambda r_x\right) - \sum_{k=0}^{m+1}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k\\ &= \left(\sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k} r_x^k + f_m\right)\left(r_a + \lambda r_x\right) - \sum_{k=0}^{m+1}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k\\ &= \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k} r_x^k r_a + f_m r_a + \sum_{k=0}^{m}\binom{m}{k}\lambda^{k+1} r_a^{m-k} r_x^{k+1} + \lambda f_m r_x - \sum_{k=0}^{m+1}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k. \end{aligned}$$

Then we have

$$f_{m+1} = \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k}\left(r_x^k r_a\right) + f_m r_a + \sum_{k=0}^{m}\binom{m}{k}\lambda^{k+1} r_a^{m-k} r_x^{k+1} + \lambda f_m r_x - \sum_{k=0}^{m+1}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k.$$

Since $r_x^k r_a = r_a r_x^k + \beta_k$ we have

$$\begin{aligned} f_{m+1} &= \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k}\left(r_a r_x^k + \beta_k\right) + f_m r_a + \sum_{k=0}^{m}\binom{m}{k}\lambda^{k+1} r_a^{m-k} r_x^{k+1} + \lambda f_m r_x - \sum_{k=0}^{m+1}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k\\ &= \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k+1} r_x^k + \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k}\beta_k + f_m r_a\\ &\qquad + \sum_{k=0}^{m}\binom{m}{k}\lambda^{k+1} r_a^{m-k} r_x^{k+1} + \lambda f_m r_x) - \sum_{k=0}^{m+1}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k\\ &= r_a^{m+1} + \sum_{k=1}^{m}\binom{m}{k}\lambda^k r_a^{m-k+1} r_x^k + \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k}\beta_k + f_m r_a\\ &\qquad + \lambda^{m+1} r_x^{m+1} + \sum_{k=0}^{m-1}\binom{m}{k}\lambda^{k+1} r_a^{m-k} r_x^{k+1} + \lambda f_m r_x - \sum_{k=0}^{m+1}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k\\ &= r_a^{m+1} + \sum_{j=1}^{m}\binom{m}{j}\lambda^k r_a^{m-j+1} r_x^j + \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k}\beta_k + f_m r_a\\ &\qquad + \lambda^{m+1} r_x^{m+1} + \sum_{j=1}^{m}\binom{m}{j-1}\lambda^j r_a^{m-j+1} r_x^j + \lambda f_m r_x - \sum_{k=0}^{m+1}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k\\ &= r_a^{m+1} + \sum_{j=1}^{m}\binom{m}{j}\lambda^k r_a^{m-j+1} r_x^j + \sum_{j=1}^{m}\binom{m}{j-1}\lambda^j r_a^{m-j+1} r_x^j + \lambda^{m+1} r_x^{m+1}\\ &\qquad - \left(r_a^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k}\lambda^k r_a^{m-k+1} r_x^k + \lambda^{m+1} r_x^{m+1}\right) + \sum_{k=0}^{m}\binom{m}{k}\lambda^k r_a^{m-k}\beta_k + f_m r_a + \lambda f_m r_x. \end{aligned}$$

Finally we have

$$f_{m+1} = \sum_{k=0}^{m} \binom{m}{k} \lambda^k r_a^{m-k} \beta_k + f_m r_a + \lambda f_m r_x$$
$$\in \sum_{k=0}^{m} \binom{m}{k} \lambda^k (r_A)^{m-k} \left(r_A r_x^m \dot{+} \cdots \dot{+} r_A\right) \dot{+} \left(r_A r_x^m \dot{+} \cdots \dot{+} r_A\right) r_A \dot{+} \lambda \left(r_A r_x^m \dot{+} \cdots \dot{+} r_A\right) r_x$$
$$\in r_A r_x^{m+1} \dot{+} \cdots \dot{+} \cdots \dot{+} r_A r_x \dot{+} r_A.$$

□

Definition 2.15. *Call $x \in End(V)$ semisimple if the roots of its minimum polynomial over F are all distinct, or equivalently, if x is diagonalizable.*

Remark 2.16. 1. *Two commuting semisimple endomorphisms are simultaneously diagonalizable, so their sum and difference are both semisimple.*

2. *If x is semisimple and x leaves a subspace W invariant, then the restriction of x to W denoted by $x_{|W}$ is semisimple.*

Definition 2.17. *Call $x \in L$ ad-semisimple (respectively Ad-semisimple) if the endomorphisms ad_x is semisimple (respectively Ad_x is semisimple).*

Call $x \in L$ ad-nilpotent (respectively Ad-nilpotent) if the endomorphisms ad_x is nilpotent (respectively Ad_x is nilpotent).

Lemma 2.18. *Let $V = V_1 \oplus V_2$ be a direct sum of two vector spaces V_1, V_2, an non negative integer p and σ an endomorphism of V shuch that $\sigma^p(V) \subseteq V_1$, then the trace of σ denoted by $tr(\sigma) = tr(\sigma_{|V_1})$, where $\sigma_{|V_1}$ is the restriction of σ to V_1.*

Proof. Since we have an algebraically closed field, we can find a basis $\{v_1, \cdots, v_m, \cdots, v_n\}$ of V whith $\{v_1, \cdots, v_m,\}$ is a basis of V_1 and scalars $\lambda_1, \cdots, \lambda_n$ shuch that the matrix of σ in this basis is

$$N_{0k} = \begin{pmatrix} \lambda_1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & \lambda_2 & a_{2,3} & \cdots & a_{2,n} \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \lambda_{n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

For $m+1 \le i \le n$, we have a vector $0 \ne v_i \in V_2$ shuch that $\sigma(v_i) = \lambda_i v_i$.
Then $\sigma^p(v_i) = \lambda_i^p v_i \in V_2 \cap V_1 = \{0\}$. So $\lambda_i = 0$ for $m+1 \le i \le n$, and

$$tr(\sigma) = \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{m} \lambda_j = tr(\sigma_{|V_1}).$$

□

3. Radical and Nilradical

The proof of following proposition can be found in [5].

Proposition 3.1. *Let $\mathfrak{W}$ be a Lie subalgebra of $End_F(V)$ where V is an F-vector space. Then $\mathfrak{W}$ is solvable if and only if $tr(x \circ y) = 0$ for all $x \in \mathfrak{W}$ and $y \in [\mathfrak{W}, \mathfrak{W}]$.*

Theorem 3.2. *[1, Theorem 3.7] Let L be a Leibniz algebra. Then L is solvable if and only if for all x in L and all y in $[L, L]$, $tr(ad_x \circ ad_y) = 0$.*

If $\imath$ is an ideal of L and $L/\imath$ is solvable (respectively nilpotent), then $D^{(n)}(L/\imath) = 0$ (respectively $(L/\imath)^n = 0$) implies that $D^{(n)}(L) \subset \imath$ (respectively $L^n \subset \imath$ nilpotent). If $\imath$ itself is solvable with $D^{(m)}(\imath) = 0$ (respectively nilpotent with $\imath^m = 0$), then $D^{(m+n)}(L) = 0$ (respectively $L^{m+n} = 0$).
So we have proved:

Proposition 3.3. *If $\imath \subset L$ is an ideal, and both $\imath$ and $L/\imath$ are solvable (respectively nilpotent), so is L solvable (respectively nilpotent).*

If $\imath$ and $\jmath$ are solvable ideals, then $(\imath + \jmath)/\jmath \equiv \imath/(\imath \cap \jmath)$ is solvable, being the homomorphic image of a solvable algebra. So, by the previous propositio, we have the

Proposition 3.4. *If $\imath$ and $\jmath$ are solvable ideals (respectively nilpotent ideals) in L so $\imath + \jmath$ is solvable (respectively nilpotent). In particular, every Leibniz algebra L has a largest solvable ideal which contains all other solvable ideals and a largest nilpotent ideal which contains all other nilpotent ideals.*
The largest solvable one is denoted by $Rad\,(L)$.
The largest nilpotent one is denoted by $Nil\,(L)$.

Remark 3.5. *Note that* $\mathrm{Ess}(L) \subseteq Nil\,(L) \subseteq Rad\,(L)$.

4. The ideal $\{Rad(L), L\}$

Let us denote the subspace $[Rad(L), L] \dot{+} [L, Rad(L)]$ by $\{Rad(L), L\}$.

Lemma 4.1. *Let L be a Leibniz algebra and (l, r, V) a representation of L. Let A be a subspace of L for which there exists an integer $n \in \mathbb{N}^*$ with $r_A^n = \{0\}$ and let x be in $n_L(A)$ such that r_x is nilpotent. Then there exists an integer $N \in \mathbb{N}^*$ with $(r_{A+Fx})^N = \{0\}$.*

Proof. Let us notice that for any non negative integer p we have
$(r_{a+\lambda x})^p = \sum_{k=0}^{p} \binom{p}{k} \lambda^k r_x^k (r_a)^{p-k} + f_p$ where $f_p \in E_p = r_A r_x^p \dot{+} \cdots \dot{+} r_A$.

Let m an integer with $(r_x)^m = 0$. Then with $p = 2\sup(m, n) + 1 > m + n$ we have that $(r_{a+\lambda x})^p = f_p \in E_p$. And so

$$[(r_{a+\lambda x})^p]^n = (f_p)^n = \left(r_A r_x^p \dot{+} \cdots \dot{+} r_A\right)^n \subseteq r_A^n r_x^{np} \dot{+} \cdots \dot{+} r_A^n r_x^{2p} \dot{+} \cdots \dot{+} r_A^n r_x \dot{+} r_A^n.$$

Since $r_A^n = \{0\}$, $(r_{a+\lambda x})^{pn} = 0$. So $r_{a+\lambda x}$ is nilpotent for all $a + \lambda x$ in $A \dot{+} Fx$. By [7, Theorem 3.2., page 41] the associative algebra $r_{A\dot{+}Fx}$ is nilpotent algebra. So there is some integer $N \in \mathbb{N}^*$ such that $(r_{A+Fx})^N = \{0\}$. □

Proposition 4.2. *For any representation (l, r, V) of the Leibniz algebra L, the restriction of r to the ideal $\{Rad(L), L\}$ is nilpotent, i.e. there exists an integer $m \in \mathbb{N}^*$ with $\left(r_{\{Rad(L),L\}}\right)^m = \{0\}$.*

Proof. According to [3, Corollary 4.4] the representation of V is nilpotent on the ideal $[L, L]$. Now let $T \subseteq \{Rad(L), L\}$ be a subspace containing $[Rad(L), Rad(L)]$, which is maximal with respect to the property that the representation of V is nilpotent on T. Note that T always is an ideal of $Rad\,(L)$, hence in particular a subalgebra, because it contains $[Rad(L), Rad(L)]$.

Assume that $T \neq \{Rad(L), L\}$. Then there exist at least an x in $Rad(L)$ and y in L with $[x, y] \notin T$ or $[y, x] \notin T$.

If $[x, y] \notin T$, the subspace $B = Rad\,(L) \dot{+} Fx$ is a subalgebra of L, $Rad\,(L)$ is a solvable ideal of B and $B/Rad\,(L) \approx F$ is abelian. Therefore B is a solvable ideal by Proposition 3.3.
Again we use [3, Corollary 4.4] to see that the representation of V is nilpotent on $[B, B]$ and hence that $r_{[x,y]}$ is nilpotent.

Since $T \subseteq Rad(L)$ and $[x, y] \in [Rad\,(L), y] \subseteq Rad\,(L)$, we have
$[[x, y], T] \subseteq [Rad\,(L), T] \subseteq T$ and $[T, [x, y]] \subseteq [T, Rad\,(L)] \subseteq T$.
Finally the preceding lemma show that the representation of V is nilpotent on the subspace $T \oplus F[x, y]$. This contradicts the maximality of T.

If $[y, x] \notin T$, the subspace $B = Rad\,(L) \dot{+} Fx$ is a subalgebra of L, $Rad\,(L)$ is a solvable ideal of B and $B/Rad\,(L) \approx F$ is abelian. Therefore B is a solvable ideal by Proposition 3.3.
Again we use [3, Corollary 4.4] to see that the representation of V is nilpotent on $[B, B]$ and hence that $r_{[y,x]}$ is nilpotent.

Since $T \subseteq Rad(L)$ and $[y, x] \in [y, Rad\,(L)] \subseteq Rad\,(L)$, we have
$[[y, x], T] \subseteq [Rad\,(L), T] \subseteq T$ and $[[y, x], T] \subseteq [Rad\,(L), T] \subseteq T$.
Finally the preceding lemma show that the representation of V is nilpotent on the subspace $T \oplus F[x, y]$. This contradicts the maximality of T.

We conclude that T must be equal to $\{Rad(L), L\}$, so the representation of V is nilpotent on $\{Rad(L), L\}$. $\square$

Applying the precedent proposition to the adjoint representation (Ad, ad, L) of the Leibniz algebra L and using Engel's Theorem [2], we get the:

Corollary 4.3. *The ideal $\{Rad(L), L\}$ is nilpotent. In particular, x is ad-nilpotent for every x in $\{Rad(L), L\}$.*

Corollary 4.4. *Let L be a Leibniz algebra and D a derivation of L.*
Then $D(Rad(L)) \subseteq Nil(L)$. In particular $Nil(L)$ is a characteristical ideal.

Proof. For a derivation D of L, define the Leibniz algebra $\tilde{L} = L \times|_D F$ with the bracket $[(x,t),(y,l)] = (lD(x) - tD(y) + [x,y], 0)$. Then, $(D(Rad(L)),0) = [(Rad(L),0)(0,1)] \subseteq (L,0) \cap \left[Rad(\tilde{L}), \tilde{L}\right] \subseteq \tilde{L} \cap Nil\left(\tilde{L}\right) \subseteq Nil\left(\tilde{L}\right) = (nil(L),0)$. So $D(Rad(L)) \subseteq Nil(L)$. $\square$

5. Main theorem

We deal in this section with Leibniz algebras which sastify equation

$\forall x, y \in L, tr\left(ad_x \circ ad_y\right)_{|Ess(L)} = 0$

Call such Leibniz algebras: Killing Leibniz Algebras.

A bilinear form $(-,-) : L \times L \longrightarrow F$ is called invariant if $([x,y],z) + (y,[x,z]) = 0$ for all x, y, z in L. Notice that if $(-,-)$ is an invariant form, and $\imath$ is an ideal, then its orthogonal $\imath^{\perp}$ is again an ideal.

One way of producing invariant forms is from representations: if (l, r, V) is a representation of L, then $(x,y)_r = tr(r_x \circ r_y)$ is invariant. Indeed,

$$\begin{aligned}([x,y],z)_r + (y,[x,z])_r &= tr\left((r_y \circ r_x - r_x \circ r_y) \circ r_z + r_y \circ (r_z \circ r_x - r_x \circ r_z)\right)\\ &= tr\left((r_y \circ r_z) \circ r_x - r_x \circ (r_y \circ r_z)\right) = 0.\end{aligned}$$

In particular, if we take $l = Ad$, $r = ad$, $V = L$ the corresponding bilinear form is called the Killing form and will be denoted by $\mathfrak{K} = (-,-)_{\mathfrak{K}}$.

Remark 5.1. *for all x in $\mathrm{Ess}(L)$, y, z in L we have: $(ad_x \circ ad_y)(z) = (ad_x)([z,y]) = [[z,y],x] = 0$.*
Then $ad_x \circ ad_y \equiv 0$ and $(x,y)_{\mathfrak{K}} = tr(ad_x \circ ad_y) = 0$, so $\mathrm{Ess}(L) \subseteq \ker(\mathfrak{K})$.

Theorem 5.2. *Let L be a leibniz algebra of a class Killing Leibniz Algebras and $\ker(\mathfrak{K})$ the kernel of its Killing form.*

$$\ker(\mathfrak{K}) = \mathrm{Ess}(L) \Leftrightarrow L \text{ is semisimple.}$$

Proof. $\Leftarrow$ Suppose that L is semisimple. Let us show that the kernel of the Killing form is $Ess(g)$.

So let $\mathfrak{W} = L^{\perp} = \{x \in L,\ tr\left(ad_x \circ ad_y\right) = 0 \text{ for all } y \in L\}$. If $x \in \mathfrak{W}$, $y, z \in L$ then

$$\begin{aligned}tr\left(ad_{[x,z]} \circ ad_y\right) &= tr\left(ad_x \circ ad_z \circ ad_y - ad_z \circ ad_x \circ ad_y\right) = tr\left(ad_x \circ (ad_z \circ ad_y - ad_y \circ ad_z)\right)\\ &= tr\left(ad_x \circ ad_{[z,y]}\right) = 0,\end{aligned}$$

And so on, we have also $tr\left(ad_{[z,x]} \circ ad_y\right) = 0$.

So $\mathfrak{W}$ is an ideal and clearly $Ess(L) \subseteq \mathfrak{W}$.

$ad_{\mathfrak{W}}$ is a solvable a Lie subalgebra of $End(V)$ by Cartan's criterion. Thanks to Proposition 3.1, $\mathfrak{W}$ is solvable and hence $\mathfrak{W} = Rad(L) = Ess(L)$.

$\Rightarrow$ suppose L is not semisimple and so has a solvable ideal such that $a \supsetneq Ess(L) \supseteq a^2$ by Remark 2.8. Let us show that $(x,y)_K = 0$ for all x in a, y in L and then $a \subset \ker(\mathfrak{K})$.

Let $\sigma = ad_x \circ ad_y$.

By assumption $tr(\sigma_{|Ess(L)}) = 0$.

And since σ maps L to a, a to a^2 and $a^2 \subseteq Ess(L)$, we have that

$\sigma^2(L) \subseteq \sigma(a) \subseteq a^2 \subseteq Ess(L)$.

Write $L = Ess(L) \oplus L_2$. Then we have by Lemma 2.18, that $tr(\sigma) = tr(\sigma_{|Ess(L)}) = 0$. Hence if L is not semisimple then the kernel of its Killing form satisfies $Ess(L) \subsetneq \ker(\mathfrak{K})$.

$\square$

Remark 5.3. • *I. Demir et al. give another proof of* $\Rightarrow$*. (see [4, Theorem 5.8]).*

- *In Lie algebras case, Theorem 5.2 is the well known Cartan's criterion for semisimplicity.*

6. Conclusion

Let us cite an example of Leibniz algebra which is solvable and the kernel of it's Killing form is $Ess(L)$.

Example 6.1. *[6]*
Let $L = \mathbb{C}x + \mathbb{C}y$ *be the two dimensional complex Leibniz algebra which generators satisfy* $[x, y] = x$*;* $[x, x] = [y, y] = [y, x] = 0$.

Let us find the kernel of the Killing form of the non lie leibniz algebra $L = Fx \oplus Fy$ defined in Example 6.1. Let $a = a_{11}x + a_{12}y$ and $b = a_{21}x + a_{22}y$ be two elements of algebra. The matrix of the endomorphism ad_a is $\begin{pmatrix} a_{12} & 0 \\ 0 & 0 \end{pmatrix}$ and the matrix of the endomorphism ad_b is $\begin{pmatrix} a_{22} & 0 \\ 0 & 0 \end{pmatrix}$.

Then the Killing form is defined by $(a, b)_{\mathfrak{K}} = a_{12}a_{22}$ for all a, b in L.

Since $Ess(L) = \{0\}$ for any Lie algebra; Lie algebras are Killing Leibniz algebras and the Theorem 5.2 is knowned for Lie algebras (cf. [5]).
"Left central Leibniz" are also Killing Leibniz algebras.
Example 6.1 is an algebra not in a class of Killing Leibniz algebras.

We claim that

Claim: The class of Killing Leibniz Algebras is a widest class wich satisfies Theorem 5.2.

In [6], the authors call an algebra that is both a left and right Leibniz algebra a symmetric Leibniz algebra. they call L a left central Leibniz algebra if it is a left Leibniz algebra that also satisfies $[[a, a], b] = 0, a \in L, b \in L$. There is a hierarchy of algebras
$\{leftLeibniz\} \supsetneq \{leftcentralLeibniz\} \supsetneq \{symmetricLeibniz\} \supsetneq \{Lie\}$.

We call a right central Leibniz algebra if it is a right Leibniz algebra that also satisfies $[b, [a, a]] = 0, a \in L, b \in L$ and there is a hierarchy of algebras

$$\{rightLeibniz\} \supsetneq \{rightcentralLeibniz\} \supsetneq \{symmetricLeibniz\} \supsetneq \{Lie\}.$$

So we can complete the hierarchy of Leibniz algebras as

$$\{rightLeibniz\} \supsetneq \{rightKillingLeibniz\} \supsetneq \{rightcentralLeibniz\} \supsetneq \{symmetricLeibniz\} \supsetneq \{Lie\}.$$

and

$$\{leftLeibniz\} \supsetneq \{leftKillingLeibniz\} \supsetneq \{leftcentralLeibniz\} \supsetneq \{symmetricLeibniz\} \supsetneq \{Lie\}.$$

Questions:

- Can we prove the Weyl's theorem on complete reducibility for Killing Leibniz Algebras?
- In [6], the authors show that "left central Leibniz algebras" satisfy a version of the Malcev theorem. Do the Killing Leibniz Algebras also satisfy this theorem?

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