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Research Paper

E-Bayesian analysis of the Gumbel type-ii distribution under type-ii censored scheme

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Abstract

This paper seeks to focus on Bayesian and E-Bayesian estimation for the unknown shape parameter of the Gumbel type-II distribution based on type-II censored samples. These estimators are obtained under symmetric loss function [squared error loss (SELF))] and various asymmetric loss functions [LINEX loss function (LLF), Degroot loss function (DLF), Quadratic loss function (QLF) and minimum expected loss function (MELF)]. Comparisons between the E-Bayesian estimators with the associated Bayesian estimators are investigated through a simulation study.

Keywords: *E-Bayesian Estimates; Gumbel Type-II Distribution; Loss Functions; Monte Carlo Simulation; Type-II Censoring.*

1. Introduction

The Gumbel type-II distribution was, firstly, introduced by Gumbel [1], and it is useful to model the extreme events like extreme earthquake, temperature, floods, etc. The Gumbel distribution used also in hydrology to analyze the variables such as quarterly and annual maximum values of daily rainfall and river discharge volumes. The Gumbel type-II distribution has probability density function (pdf) given by.

$$f(x|\alpha,\beta)=\alpha\beta x^{-(\alpha+1)}\exp\left[-\beta x^{-\alpha}\right],\quad x>0,\alpha,\beta>0 \tag{1-1}$$

And cumulative distribution function (cdf) is

$$F(x|\alpha,\beta)=1-\exp\left[-\beta x^{-\alpha}\right],\quad x>0,\alpha,\beta>0 \tag{1-2}$$

Where α and β are the scale and shape parameters respectively.

Recently, many authors have discussed the Gumbel type-II distribution. For example, Kotz and Nadarajah [2] investigated some properties of Gumbel distribution. Corsini et al [3] worked on the maximum likelihood algorithms and Cramer-Rao bounds for the parameters of the Gumbel distribution. Malinowska and Szynal [4] derived the Bayes estimates for the parameters of the Gumbel distribution based on kth record values. Feroze and Aslam [5] applied the Bayesian estimation scheme for Gumbel type-II distribution under doubly censored samples by considering various loss functions. Furthermore, Salinas et al [6] proposed goodness of fit tests for the Gumbel distribution with type-II right censored data. Abbas and Tang [7] obtained the Bayes estimators for the parameters of Gumbel distribution under different loss functions and compared these estimates with the similar performed by the maximum likelihood method. Feroze and Aslam [8] derived the Bayesian estimators of the parameters of mixture of two components of Gumbel type-II distribution.

The E-Bayesian estimation is a new technique of estimation first introduced by Han [9]. Jaheen and Okasha [10] compared the Bayesian and E-Bayesian estimators for the parameters and reliability function of the Burr type XII distribution based on type-II censoring. Wang and Chen [11] pointed out the properties of the Bayes and E-Bayes estimates of the system reliability parameter with the zero-failure data. The Bayesian and E-Bayesian estimators for the generalized half logistic distribution under progressively type-II censored samples are performed by Azimi et al [12]. Furthermore, Okasha [13] considered the E-Bayesian method for computing estimates of the unknown parameter, and some survival time parameters of the Lomax distribution based on type-II censored samples.

This article aims to produce a statistical comparison between the Bayesian and E-Bayesian methods for estimating the shape parameter of the Gumbel type-II distribution under type-II censoring. The resulting estimators are obtained based on symmetric and different asymmetric loss functions and the results obtained in this article can be generalized to use in complete sample.

The layout of the paper is as follows. In Section 2 and 3 respectively, the Bayesian and E-Bayesian estimates of the parameter β based on type-II censored samples are derived under squared, LINEX, Degroot, quadratic and minimum expected loss functions. In Section 4, the properties of the E-Bayesian estimators are discussed. Simulation study has been performed to compare the resulting estimators in Section 5. Some concluding remarks have been given in the last section.

2. Bayesian estimation

In this section, Bayes estimates of the shape parameter β of the Gumbel type-II distributions are obtained by considering SELF, LLF, DLF, QLF and MELF. Based on type-II censored samples of size r obtained from a life test of n items from the Gumbel type-II (α,β) in (1-1) and (1-2) distribution, likelihood function can be written as.

$$L(\lambda,\beta|\underline{x}) = \frac{n}{(n-r)}\prod_{i=1}^{r}\alpha\,\beta\,x_{(i)}^{-(\alpha+1)}\exp\left[-\beta x_{(i)}^{-\alpha}\right]\left[\exp\left[-\beta x_{(r)}^{-\alpha}\right]\right]^{n-r}$$

$$= \frac{n}{(n-r)}\,\frac{\alpha^{r}\;\beta^{r}}{\prod_{i=1}^{r}x_{(i)}^{\alpha+1}}\;\exp(-\beta D) \tag{2-1}$$

Where

$$D = \sum_{i=1}^{r}x_{(i)}^{-\alpha} + (n-r)x_{(r)}^{-\alpha} \tag{2-2}$$

Assuming α is known, we can use the gamma distribution as an conjugate prior distribution of β with shape and scale parameter a and b respectively and its pdf given by

$$g(\beta|a,b) = \frac{b^{a}}{\Gamma(a)}\,\beta^{a-1}\exp[-b\,\beta], \qquad \beta>0 \tag{2-3}$$

Combining (2-1) and (2-3), from Bayesian theorem the posterior density function of β can be written as

$$\pi(\beta|\underline{x}) = \frac{(D+b)^{r+a}}{\Gamma(r+a)}\,\beta^{r+a-1}\exp[-\beta(D+b)], \qquad \beta>0 \tag{2-4}$$

2.1. Bayesian estimation under squared error loss function (SELF)

A commonly used loss function is the square error loss function (SELF) introduced by Mood et al [14] as follows:

$$L_1(\hat{\beta},\beta) = k\,(\hat{\beta}-\beta)^2\;, \qquad k>0 \tag{2-5}$$

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Where $\hat{\beta}$ is an estimator of β and k is the scale of the loss function. The scale k is often taken equal to one, which has no effect upon the Bayes estimates. This loss function is symmetric in nature. i.e. it gives equal importance to both over and under estimation. We can derive the Bayes estimate of β based on SELF by using (2-5) with scale one in (2-4) to be.

$$\hat{\beta}_{BS} = \frac{r+a}{D+b} \tag{2-6}$$

2.2. Bayesian estimation under linex loss function (LLF)

Zellner [15] represents the LINEX (linear-exponential) loss function (LLF) to be.

$$L_2(\hat{\beta},\beta) = m\left\{\exp\left[s(\hat{\beta}-\beta)\right] - s(\hat{\beta}-\beta) - 1\right\}, \tag{2-7}$$

With two parameters $m > 0, s \neq 0,$ where m is the scale of the loss function and s determines its shape. Without loss of generality, we assume $m = 1$ and we can obtained the Bayes estimate of β based on LLF by using (2-7) in (2-4) to be

$$\hat{\beta}_{BL} = \left(\frac{r+a}{s}\right)\ln\left[1 + \frac{s}{D+b}\right] \tag{2-8}$$

2.3. Bayesian estimation under degroot loss function (DLF)

DeGroot [16] discussed various types of loss functions and derived the Bayes estimates under these loss functions. If $\hat{\beta}$ is an estimate of $\beta,$ then the DeGroot loss function (DLF) is defined as

$$L_3(\hat{\beta},\beta) = \left[\frac{\beta-\hat{\beta}}{\hat{\beta}}\right] \tag{2-9}$$

We can derive the Bayes estimate of β based on DLF by using (2-9) in (2-4) to be

$$\hat{\beta}_{BD} = \frac{r+a+1}{D+b} \tag{2-10}$$

2.4. Bayesian estimation under quadratic loss function (QLF)

The quadratic loss function (QLF) can be defined as:

$$L_4(\hat{\beta},\beta) = \left[\frac{\beta-\hat{\beta}}{\beta}\right]^2 \tag{2-11}$$

The Bayes estimate of β based on QLF can be obtained by using (2-11) in (2-4) to be

$$\hat{\beta}_{BQ} = \frac{r+a-1}{D+b} \tag{2-12}$$

2.5. Bayesian estimation under minimum expected loss function (MELF)

Tummala and Sathe [17] proposed minimum expected loss function (MELF) as follows:

$$L_5(\hat{\beta},\beta) = \beta^{-2}(\hat{\beta}-\beta)^2 \tag{2-13}$$

We can compute the Bayes estimate of β based on MELF by using (2-13) in (2-5) to be

$$\hat{\beta}_{BM} = \frac{r+a-2}{D+b} \tag{2-14}$$

3. E-Bayesian estimation

In this section, we obtain the E-Bayes estimates of the shape parameter β of the Gumbel type-II distribution under symmetric loss function (SELF)) and four asymmetric loss functions (LLF, DLF, QLF and MELF).
Based on Han [18], the prior parameters a and b must be choose to guarantee that $g(\beta|a,b)$ given in (2-3) is a decreasing function of β. The derivative of $g(\beta|a,b)$ with respect to β is

$$\frac{dg(\beta|a,b)}{d\beta} = \frac{b^a}{\Gamma(a)}\beta^{a-2}\left[\exp[-b\beta]\right][(a-1)-b\beta], \tag{3-1}$$

Note that $a>0, b>0$ and $\beta>0$ leads to $0<a<1, b>0$ due to $\frac{dg(\beta|a,b)}{d\beta}<0$, and therefore $g(\beta|a,b)$ is a decreasing function of β. Suppose that a and b are independent with bivariate density function

$$\pi(a,b) = \pi_1(a)\pi_2(b) \tag{3-2}$$

Then, the E-Bayesian estimate of β (expectation of the Bayesian estimate of β) can be written as

$$\hat{\beta}_{EB} = E(\beta|\underline{x}) = \iint_{\Omega}\hat{\beta}_B(a,b)\pi(a,b)\,da\,db \tag{3-3}$$

Where $\hat{\beta}_B(a,b)$ is the Bayes estimate β of given by (2-6), (2-8), (2-10), (2-12) and (2-14). For more details see Han [9, 19].

3.1. E-Bayesian estimation under squared error loss function (SELF)

E-Bayesian estimates of β are derived depending on three different distributions of the hyperparameters a and b. These distributions are used to study the impact of the different prior distributions on the E-Bayesian estimation of β. The following distributions of a and b may be used:

$$\pi_1(a,b) = \frac{2(c-b)}{c^2}, \qquad 0<a<1,\ 0<b<c \tag{3-4}$$

$$\pi_2(a,b) = \frac{1}{c}, \qquad 0<a<1,\ 0<b<c \tag{3-5}$$

$$\pi_3(a,b) = \frac{2b}{c^2}, \qquad 0<a<1,\ 0<b<c \tag{3-6}$$

We can obtained the E-Bayesian estimate of β based on $\pi_1(a,b)$ by using (2-6) and (3-4) in (3-3) to be

$$\hat{\beta}_{EBS1} = \left(\frac{2r+1}{c}\right)\left[\left(1+\frac{D}{c}\right)\ln\left(1+\frac{c}{D}\right)-1\right] \tag{3-7}$$

Also, we can derive the E-Bayesian estimates of β based on $\pi_2(a,b)$ and $\pi_3(a,b)$ by using (2-6), (3-5) in (3-3) and (2-6), (3-6) in (3-3) respectively to be

$$\hat{\beta}_{EBS2} = \left(\frac{2r+1}{2c}\right)\left[\ln\left(1+\frac{c}{D}\right)\right] \tag{3-8}$$

And

$$\hat{\beta}_{EBS3} = \left(\frac{2r+1}{c}\right)\left[1 - \frac{D}{c}\ln\left(1+\frac{c}{D}\right)\right] \tag{3-9}$$

3.2. E-Bayesian estimation under linex loss function (LLF)

We can obtain the E-Bayesian estimate of β based on $\pi_1(a,b)$ by using (2-8) and (3-4) in (3-3) to be

$$\hat{\beta}_{EBL1} = \left(\frac{2r+1}{2}\right)\left\{\begin{array}{l}\left[\left(\frac{-(D+c)^2}{c^2 s}\right)\ln\left(1+\frac{c}{D}\right)\right] + \left[\left(\frac{(D+s+c)^2}{c^2 s}\right)\ln\left(1+\frac{c}{D+s}\right)\right] \\ + \left[\frac{1}{s}\ln\left(1+\frac{s}{D}\right)\right] - \left[\frac{1}{c}\right]\end{array}\right\} \tag{3-10}$$

By the same way, we can obtain the E-Bayesian estimates of β based on $\pi_2(a,b)$ and $\pi_3(a,b)$ by using (2-8), (3-5) in (3-3) and (2-10), (3-6) in (3-3) respectively to be

$$\hat{\beta}_{EBL2} = \left(\frac{2r+1}{2s}\right)\left\{\left[\ln\left(1+\frac{s}{D+c}\right)\right] + \left[\left(\frac{D+s}{c}\right)\ln\left(1+\frac{c}{D+s}\right)\right] - \left[\left(\frac{D}{c}\right)\ln\left(1+\frac{c}{D}\right)\right]\right\} \tag{3-11}$$

And

$$\hat{\beta}_{EBL3} = \left(\frac{2r+1}{2}\right)\left\{\begin{array}{l}\left[\left(\frac{-(D+c)^2}{c^2 s}\right)\ln\left(1+\frac{c}{D+s}\right)\right] + \left[\left(\frac{D^2}{c^2 s}\right)\ln\left(1+\frac{c}{D}\right)\right] \\ + \left[\frac{1}{s}\ln\left(1+\frac{s}{D+c}\right)\right] + \left[\frac{1}{c}\right]\end{array}\right\} \tag{3-12}$$

3.3. E-Bayesian estimation under degroot loss function (DLF)

We can compute the E-Bayesian estimate of β based on $\pi_1(a,b)$ by using (2-10) and (3-4) in (3-3) to be

$$\hat{\beta}_{EBD1} = \left(\frac{2r+3}{c}\right)\left[\left(1+\frac{D}{c}\right)\ln\left(1+\frac{c}{D}\right) - 1\right] \tag{3-13}$$

Also, we can derive the E-Bayesian estimates of β based on $\pi_2(a,b)$ and $\pi_3(a,b)$ by using (2-10), (3-5) in (3-3) and (2-10), (3-6) in (3-3) respectively to be

$$\hat{\beta}_{EBD2} = \left(\frac{2r+3}{2c}\right)\left[\ln\left(1+\frac{c}{D}\right)\right] \tag{3-14}$$

And

$$\hat{\beta}_{EBD3} = \left(\frac{2r+3}{2}\right)\left[1 - \left(\frac{D}{c}\right)\ln\left(1+\frac{c}{D}\right)\right] \tag{3-15}$$

3.4. E-Bayesian estimation under quadratic loss function (QLF)

The E-Bayesian estimate of β based on $\pi_1(a,b)$ can be computed by using (2-12) and (3-4) in (3-3) to be

$$\hat{\beta}_{EBQ1} = \left(\frac{2r-1}{c}\right)\left[\left(1+\frac{D}{c}\right)\ln\left(1+\frac{c}{D}\right) - 1\right] \tag{3-16}$$

Similarly, we can derive the E-Bayesian estimates of β based on $\pi_2(a,b)$ and $\pi_3(a,b)$ by using (2-12), (3-5) in (3-3) and (2-12), (3-6) in (3-3) respectively to be

$$\hat{\beta}_{EBQ2} = \left(\frac{2r-1}{2c}\right)\left[\ln\left(1+\frac{c}{D}\right)\right] \tag{3-17}$$

And

$$\hat{\beta}_{EBQ3} = \left(\frac{2r-1}{c}\right)\left[1-\left(\frac{D}{c}\right)\ln\left(1+\frac{c}{D}\right)\right] \tag{3-18}$$

3.5. E-Bayesian estimation under minimum expected loss Function (MELF)

The E-Bayesian estimate of β based on $\pi_1(a,b)$ can be derived by using (2-14) and (3-4) in (3-3) to be

$$\hat{\beta}_{EBM1} = \left(\frac{2r-3}{c}\right)\left[\left(1+\frac{D}{c}\right)\ln\left(1+\frac{c}{D}\right)-1\right] \tag{3-19}$$

Also we can obtain the E-Bayesian estimates of β based on $\pi_2(a,b)$ and $\pi_3(a,b)$ by using (2-14), (3-5) in (3-3) and (2-14), (3-6) in (3-3) respectively to be

$$\hat{\beta}_{EBM2} = \left(\frac{2r-3}{2c}\right)\left[\ln\left(1+\frac{c}{D}\right)\right] \tag{3-20}$$

And

$$\hat{\beta}_{EBM3} = \left(\frac{2r-3}{c}\right)\left[1-\left(\frac{D}{c}\right)\ln\left(1+\frac{c}{D}\right)\right] \tag{3-21}$$

4. Properties of E-Bayesian estimation

In this section, we discuss the relations between the E-Bayesian estimators

$\hat{\beta}_{EBSi}, \hat{\beta}_{EBLi}, \hat{\beta}_{EBDi}, \hat{\beta}_{EBQi}$ and $\hat{\beta}_{EBMi}\ (i=1,2,3)$

4.1. Relations among $\hat{\beta}_{EBSi}\ (i=1,2,3)$

Lemma 1: *It follows from (3-7), (3-8) and (3-9) that*

i) $\hat{\beta}_{EBS1} < \hat{\beta}_{EBS2} < \hat{\beta}_{EBS3}$

ii) $\lim_{D\to\infty} \hat{\beta}_{EBS1} = \lim_{D\to\infty} \hat{\beta}_{EBS2} = \lim_{D\to\infty} \hat{\beta}_{EBS3}$

Proof: See Appendix.

4.2. Relations among $\hat{\beta}_{EBLi}\ (i=1,2,3)$

Lemma 2: *It follows from (3-10), (3-11) and (3-12) that*

i) $\hat{\beta}_{EBL3} < \hat{\beta}_{EBL1} < \hat{\beta}_{EBL2}$

ii) $\lim_{D\to\infty} \hat{\beta}_{EBL1} = \lim_{D\to\infty} \hat{\beta}_{EBL2} = \lim_{D\to\infty} \hat{\beta}_{EBL3}$

Proof: See Appendix.

4.3. Relations among $\hat{\beta}_{EBDi}\ (i = 1,2,3)$

Lemma 3: *It follows from (3-13), (3-14) and (3-15) that*

i) $\hat{\beta}_{EBD1} < \hat{\beta}_{EBD2} < \hat{\beta}_{EBD3}$

ii) $\lim_{D\to\infty} \hat{\beta}_{EBD1} = \lim_{D\to\infty} \hat{\beta}_{EBD2} = \lim_{D\to\infty} \hat{\beta}_{EBD3}$

Proof: See Appendix.

4.4. Relations among $\hat{\beta}_{EBQi}\ (i = 1,2,3)$

Lemma 4: *It follows from (3-16), (3-17) and (3-18) that*

i) $\hat{\beta}_{EBQ1} < \hat{\beta}_{EBQ2} < \hat{\beta}_{EBQ3}$

ii) $\lim_{D\to\infty} \hat{\beta}_{EBQ1} = \lim_{D\to\infty} \hat{\beta}_{EBQ2} = \lim_{D\to\infty} \hat{\beta}_{EBQ3}$

Proof: See Appendix.

4.5. Relations among $\hat{\beta}_{EBMi}\ (i = 1,2,3)$

Lemma 5: *It follows from (3-19), (3-20) and (3-21) that*

i) $\hat{\beta}_{EBM1} < \hat{\beta}_{EBM2} < \hat{\beta}_{EBM3}$

ii) $\lim_{D\to\infty} \hat{\beta}_{EBM1} = \lim_{D\to\infty} \hat{\beta}_{EBM2} = \lim_{D\to\infty} \hat{\beta}_{EBM3}$

Proof: See Appendix.

5. Monte Carlo simulation

This section conducts a simulation study to evaluate the performance of all Bayes and E-Bayes estimates discussed in the preceding sections. We considered different sample sizes $n = 25,30,35,50,70$ and different choices for _ also choose $\alpha = 0.7, s = -1$ and $c = 2$. for these cases, we genera for e from the uniform priors distributions (0, 1) and (0, c) respectively given in (3-4), (3-5) and (3-6). For given values of a and b, we generate β from the gamma prior distribution given in (2-3). Also for known values of α type-II censored samples are generated from the Gumbel type-II distribution with pdf and cdf given in (1-1) and (1-2) respectively. Based on the SELF, we computed the estimates $\hat{\beta}_{BS}, \hat{\beta}_{EBS1}, \hat{\beta}_{EBS2}$ and $\hat{\beta}_{EBS3}$ of β from (2-6), (3-7), (3-8) and (3-9) respectively. Also under the LLF, we calculated the estimates $\hat{\beta}_{BL}, \hat{\beta}_{EBL1}, \hat{\beta}_{EBL2}$ and $\hat{\beta}_{EBL3}$ of β from (2-8), (3-10), (3-11) and (3-12) respectively. Based on the DLF, we obtained the estimates $\hat{\beta}_{BD}, \hat{\beta}_{EBD1}, \hat{\beta}_{EBD2}$ and $\hat{\beta}_{EBD3}$ of β from (2-10), (3-13), (3-14) and (3-15) respectively. Under the QLF, we computed the estimates $\hat{\beta}_{BQ}, \hat{\beta}_{EBQ1}, \hat{\beta}_{EBQ2}$ and $\hat{\beta}_{EBQ3}$ of β from (2-12), (3-16), (3-17) and (3-18) respectively. Based on the MELF, we calculated the estimates $\hat{\beta}_{BM}, \hat{\beta}_{EBM1}, \hat{\beta}_{EBM2}$ and $\hat{\beta}_{EBM3}$ of β from (2-14), (3-19), (3-20) and (3-21) respectively. We repeated this process 10000 times and compute the Mean Square Error (MSE) for the estimates for different censoring schemes (different values of n, r) and given values of c, s, α where and $\hat{\beta}$ stands for an estimator of β. The simulation results are displayed in Table 1.

Table 1: Averaged Values of MSE for Estimates of the Parameter β

n	r	$\hat{\beta}_{BS}$	$\hat{\beta}_{EBS1}$	$\hat{\beta}_{BL}$	$\hat{\beta}_{EBL}$	$\hat{\beta}_{BD}$	$\hat{\beta}_{EBD}$	$\hat{\beta}_{BQ}$	$\hat{\beta}_{EBQ}$	$\hat{\beta}_{BM}$	$\hat{\beta}_{EBM}$
25	20	1.0056	0.9972	0.9933	0.9851	1.0546	1.0456	0.9578	0.9500	0.9112	0.9039
			1.0056		0.9932		1.0546		0.9578		0.9111
			1.0140		0.2775		1.0636		0.9656		0.9184
	25	1.1502	1.1407	1.1363	1.1272	1.1982	1.1881	1.1032	1.0943	1.0572	1.0488
			1.1501		1.1363		1.1981		1.1031		1.0571
			1.1596		0.2750		1.2081		1.1120		1.0654
30	20	0.9843	0.9763	0.9726	0.9649	1.0317	1.0231	0.9380	0.9306	0.8929	0.8860
			0.9842		0.9726		1.0316		0.9380		0.8928
			0.9922		0.2773		1.0402		0.9453		0.8996
	25	1.0029	0.9960	0.9930	0.9864	1.0418	1.0347	0.9644	0.9581	0.9269	0.9209
			1.0027		0.9929		1.0418		0.9643		0.9268
			1.0093		0.2742		1.0488		0.9706		0.9326
	30	1.1350	1.1273	1.1238	1.1164	1.1742	1.1662	1.0964	1.0891	1.0585	1.0516
			1.1349		1.1237		1.1741		1.0963		1.0584
			1.1424		0.2726		1.1821		1.1035		1.0652
35	30	0.9971	0.9916	0.9891	0.9837	1.0294	1.0237	0.9653	0.9601	0.9341	0.9291
			0.9970		0.9890		1.0293		0.9652		0.9340
			1.0025		0.2720		1.0350		0.9704		0.9389
	35	1.1151	1.1089	1.1060	1.0999	1.1479	1.1414	1.1083	1.0768	1.0509	1.0452
			1.1150		1.1059		1.1478		1.0827		1.0509
			1.1212		0.2709		1.1542		1.0886		1.0565
50	40	0.9462	0.9426	0.9410	0.9374	0.9686	0.9649	0.9241	0.9206	0.9022	0.8989
			0.9461		0.9409		0.9685		0.9240		0.9022
			0.9496		0.2690		0.9722		0.9274		0.9054
	45	1.0019	0.9982	0.9965	0.9929	1.0237	1.0198	0.9804	0.9768	0.9592	0.9557
			1.0018		0.99644		1.02367		0.9803		0.9597
			1.0055		0.2684		1.0273		0.9839		0.9625
	50	1.0912	1.0871	1.0852	1.0812	1.1135	1.1093	1.0692	1.0652	1.0474	1.0435
			1.0912		1.0852		1.1134		1.0691		1.0473
			1.0952		0.2679		1.1176		1.0731		1.0511
70	60	0.9129	0.9103	0.9091	0.9065	0.9299	0.9272	0.8962	0.8936	0.8795	0.8771
			0.9129		0.9090		0.9298		0.8961		0.8795
			0.9154		0.2673		0.9325		0.8986		0.8819
	65	1.0109	1.0083	1.0071	1.0045	1.0262	1.0235	0.9957	0.9932	0.9807	0.9782
			1.0108		1.0070		1.0261		0.9957		0.9806
			1.0133752		0.2661		1.0287		0.9983		0.9831
	70	1.0778	1.0749	1.0736	1.0708	1.0934	1.0905	1.0623	1.0595	1.0469	1.0442
			1.0777		1.0736		1.0933		1.0622		1.0468
			1.0805		0.2659		1.0962		1.0650		1.0495

6. Conclusion remarks

- We can concluded based on the results shown in Table 1,that the E-Bayes estimates $\hat{\beta}_{EBi}\,(i=1,2)$ of β under SELF, DLF, QLF and MELF have smaller MSE as compared with the associated Bayes estimates $\hat{\beta}_{Bi}\,(i=1,2)$ in all cases. On the other hand, the E-Bayesian estimates $\hat{\beta}_{EBi}\,(i=1,2,3)$ of β based on LLF have smaller MSE as compared with the corresponding Bayes estimates $\hat{\beta}_{Bi}\,(i=1,2,3)$ in all cases.
- In comparing the different E-Bayesian estimates, we can deducted from the given in Table 1, that the efficiency of the E-Bayesian estimates $\hat{\beta}_{EBi}\,(i=1,2)$ of β under SELF, LLF, DLF, QLF and MELF can be ordered due to smaller MSE to be

$$\hat{\beta}_{EBMi} > \hat{\beta}_{EBQi} > \hat{\beta}_{EBLi} > \hat{\beta}_{EBSi} > \hat{\beta}_{EBDi}, \qquad i=1,2$$

On the other hand, the efficiency of the E-Bayesian estimates $\hat{\beta}_{EB3}$ of β under SELF, LLF, DLF, QLF and MELF can be ordered due to smaller MSE to be
$\hat{\beta}_{EBL3} > \hat{\beta}_{EBM3} > \hat{\beta}_{EBQ3} > \hat{\beta}_{EBS3} > \hat{\beta}_{EBD3}$

Appendix

Proof of Lemma 1.
(ii) From (3-7), (3-8) and (3-9), we get

$$\hat{\beta}_{EBS2} - \hat{\beta}_{EBS1} = \hat{\beta}_{EBS3} - \hat{\beta}_{EBS2} = \left(\frac{2r+1}{2c}\right)\left[2 - \left(1 + \frac{2D}{c}\right)\ln\left(1 + \frac{c}{D}\right)\right] \qquad (A.1)$$

For $-1 < x < 1$, we have:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}.$$

Assuming $x = \frac{c}{D}$ when $0 < c < D$, $0 < \frac{c}{D} < 1$, we get

$$2 - \left(1 + \frac{2D}{c}\right)\ln\left(1 + \frac{c}{D}\right) = 2 - \left(1 + \frac{2D}{c}\right)\left[\begin{array}{l}\frac{c}{D} - \frac{1}{2}\left(\frac{c^2}{D^2}\right) + \frac{1}{3}\left(\frac{c^3}{D^3}\right) - \frac{1}{4}\left(\frac{c^4}{D^4}\right) \\ + \frac{1}{5}\left(\frac{c^5}{D^5}\right) - \frac{1}{6}\left(\frac{c^6}{D^6}\right) + \ldots\end{array}\right]$$

$$= 2 - \left[\begin{array}{l}\frac{c}{D} - \frac{1}{2}\left(\frac{c^2}{D^2}\right) + \frac{1}{3}\left(\frac{c^3}{D^3}\right) - \frac{1}{4}\left(\frac{c^4}{D^4}\right) + \frac{1}{5}\left(\frac{c^5}{D^5}\right) - \ldots \\ +2 - \frac{c}{D} + \frac{2}{3}\left(\frac{c^2}{D^2}\right) - \frac{2}{4}\left(\frac{c^3}{D^3}\right) + \frac{2}{5}\left(\frac{c^4}{D^4}\right) - \frac{2}{6}\left(\frac{c^5}{D^5}\right) + \ldots\end{array}\right]$$

$$= -\frac{1}{2}\left(\frac{c^2}{D^2}\right) - \frac{1}{3}\left(\frac{c^3}{D^3}\right) + \frac{1}{4}\left(\frac{c^4}{D^4}\right) - \frac{1}{5}\left(\frac{c^5}{D^5}\right) + \ldots$$
$$-\frac{2}{3}\left(\frac{c^2}{D^2}\right) + \frac{2}{4}\left(\frac{c^3}{D^3}\right) - \frac{2}{5}\left(\frac{c^4}{D^4}\right) + \frac{2}{6}\left(\frac{c^5}{D^5}\right) - \ldots$$

$$= \left(\frac{1}{2} - \frac{2}{3}\right)\left(\frac{c^2}{D^2}\right) + \left(\frac{2}{4} - \frac{1}{3}\right)\left(\frac{c^3}{D^3}\right) + \left(\frac{1}{4} - \frac{2}{5}\right)\left(\frac{c^4}{D^4}\right) + \left(\frac{2}{6} - \frac{1}{5}\right)\left(\frac{c^5}{D^5}\right) - \ldots$$

$$= \frac{-1}{6}\left(\frac{c^2}{D^2}\right) + \frac{1}{6}\left(\frac{c^3}{D^3}\right) - \frac{3}{20}\left(\frac{c^4}{D^4}\right) + \frac{2}{15}\left(\frac{c^5}{D^5}\right) - \ldots$$
$$= \frac{c^2}{6D^2}\left(1 - \frac{c}{D}\right) + \frac{c^4}{D^4}\left(\frac{2c}{15D} - \frac{3}{20}\right) + \ldots$$
$$= \frac{c^2}{6D^2}\left(1 - \frac{c}{D}\right) + \frac{c^4}{60D^4}\left(\frac{8c}{D} - 9\right) + \ldots \qquad (A.2)$$

According to (A.1) and (A.2), we have

$$\hat{\beta}_{EBS2} - \hat{\beta}_{EBS1} = \hat{\beta}_{EBS3} - \hat{\beta}_{EBS2} > 0$$

That is

$$\hat{\beta}_{EBS1} < \hat{\beta}_{EBS2} < \hat{\beta}_{EBS3}$$

(ii) From (A.1) and (A.2), we get

$$\lim_{D\to\infty}\left(\hat{\beta}_{EBS2} - \hat{\beta}_{EBS1}\right) = \lim_{D\to\infty}\left(\hat{\beta}_{EBS3} - \hat{\beta}_{EBS2}\right)$$

$$= \lim_{D\to\infty}\left[\frac{c^2}{6D^2}\left(1-\frac{c}{D}\right) + \frac{c^4}{60D^4}\left(\frac{8c}{D}-9\right)+\ldots\right] = 0$$

That is

$$\lim_{D\to\infty}\hat{\beta}_{EBS1} = \lim_{D\to\infty}\hat{\beta}_{EBS2} = \lim_{D\to\infty}\hat{\beta}_{EBS3}$$

Thus, the proof is complete

Proof of Lemma 2

(i) From (3-10), (3-11) and (3-12), we get

$$\hat{\beta}_{EBS1} - \hat{\beta}_{EBS3} = \hat{\beta}_{EBS2} - \hat{\beta}_{EBS1} = \left(\frac{2r+1}{2sc}\right)\left\{\begin{array}{l}\left[\left(\frac{(D+s)^2}{c} + (D+s)\right)\ln\left(1+\frac{c}{D+s}\right)\right] \\ -\left(\frac{D^2}{c}+D\right)\ln\left(1+\frac{c}{D}\right) - s\end{array}\right\} \tag{A.3}$$

For $-1 < x < 1$, we have:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}.$$

Assuming $x = \frac{c}{D}$ when $0 < c < D,\ 0 < \frac{c}{D} < 1,$ we obtain

$$\left[\left(\frac{(D+s)^2}{c} + (D+s)\right)\ln\left(1+\frac{c}{D+s}\right)\right] - \left[\left(\frac{D^2}{c}+D\right)\ln\left(1+\frac{c}{D}\right)\right] - s$$

$$= \left(\frac{(D+s)^2}{c} + (D+s)\right)\left[\frac{c}{D+s} - \frac{c^2}{2(D+s)^2} + \frac{c^3}{3(D+s)^3} - \frac{c^4}{4(D+s)^4} + \frac{c^5}{5(D+s)^5} - \frac{c^6}{6(D+s)^6} + \ldots\right]$$
$$-\left(\frac{D^2}{c}+D\right)\left[\frac{c}{D} - \frac{c^2}{2D^2} + \frac{c^3}{3D^3} - \frac{c^4}{4D^4} + \frac{c^5}{5D^5} - \frac{c^6}{6D^6} + \ldots\right] - s$$

$$= \left[(D+s) - \frac{c}{2} + \frac{c^2}{3(D+s)} - \frac{c^3}{4(D+s)^2} + \frac{c^4}{5(D+s)^3} - \frac{c^5}{6(D+s)^4}\ldots\right]$$
$$+\left[c - \frac{c^2}{2(D+s)} + \frac{c^3}{3(D+s)^2} - \frac{c^4}{4(D+s)^3} + \frac{c^5}{5(D+s)^4} - \ldots\right]$$
$$-\left[D - \frac{c}{2} + \frac{c^2}{3D} - \frac{c^3}{4D^2} + \frac{c^4}{5D^3} - \frac{c^5}{6D^4}\ldots\right] - \left[c - \frac{c^2}{2D} + \frac{c^3}{3D^2} - \frac{c^4}{4D^3} + \frac{c^5}{5D^4} - \frac{c^6}{6D^5} + \ldots\right] - s$$

$$= \left[D + s + \frac{c}{2} + \left(\frac{1}{3}-\frac{1}{2}\right)\frac{c^2}{(D+s)} + \left(\frac{1}{3}-\frac{1}{4}\right)\frac{c^3}{(D+s)^2} + \left(\frac{1}{5}-\frac{1}{4}\right)\frac{c^4}{(D+s)^3} + \left(\frac{1}{5}-\frac{1}{6}\right)\frac{c^5}{(D+s)^4} + \ldots\right]$$
$$-\left[D + \frac{c}{2} + \left(\frac{1}{3}-\frac{1}{2}\right)\frac{c^2}{D} + \left(\frac{1}{3}-\frac{1}{4}\right)\frac{c^3}{D^2} + \left(\frac{1}{5}-\frac{1}{4}\right)\frac{c^4}{D^3} + \left(\frac{1}{5}-\frac{1}{6}\right)\frac{c^5}{D^4} - \ldots\right] - s$$

$$= D + s + \frac{c}{2} - \frac{c^2}{6(D+s)} + \frac{c^3}{12(D+s)^2} - \frac{c^4}{20(D+s)^3} - \frac{c^5}{30(D+s)^4} + \ldots$$
$$- D - \frac{c}{2} + \frac{c^2}{6D} - \frac{c^3}{12D^2} + \frac{c^4}{20D^3} - \frac{c^5}{30D^4} + \ldots - s$$

$$= \frac{c^2}{2D}\left[\frac{1}{3} - \frac{c}{6D} + \frac{c^2}{10D^2} - \frac{c^3}{15D^3} + \ldots\right]$$
$$- \frac{c^2}{2(D+s)}\left[\frac{1}{3} - \frac{c}{6(D+s)} + \frac{c^2}{10(D+s)^2} - \frac{c^3}{15(D+s)^3} + \ldots\right] \tag{A.4}$$

According to (A.3) and (A.4), we have

$$\hat{\beta}_{EBL1} - \hat{\beta}_{EBL3} = \hat{\beta}_{EBL2} - \hat{\beta}_{EBL1} > 0$$

That is $\hat{\beta}_{EBL3} < \hat{\beta}_{EBL1} < \hat{\beta}_{EBL2}$

(ii) From (A.3) and (A.4), we get

$$\lim_{D\to\infty}\left(\hat{\beta}_{EBL1} - \hat{\beta}_{EBL3}\right) = \lim_{D\to\infty}\left(\hat{\beta}_{EBL2} - \hat{\beta}_{EBL1}\right)$$
$$= \lim_{D\to\infty}\left\{\frac{c^2}{2D}\left[\frac{1}{3} - \frac{c}{6D} + \frac{c^2}{10D^2} - \frac{c^3}{15D^3} + \ldots\right]\right\}$$
$$- \lim_{D\to\infty}\left\{\frac{c^2}{2(D+s)}\left[\frac{1}{3} - \frac{c}{6(D+s)} + \frac{c^2}{10(D+s)^2} - \frac{c^3}{15(D+s)^3} + \ldots\right]\right\} = 0$$

That is

$$\lim_{D\to\infty}\hat{\beta}_{EBL1} = \lim_{D\to\infty}\hat{\beta}_{EBL2} = \lim_{D\to\infty}\hat{\beta}_{EBL3}$$

Thus, the proof is complete

Proof of Lemma 3
(i) From (3-13), (3-14) and (3-15), we obtain

$$\hat{\beta}_{EBD2} - \hat{\beta}_{EBD1} = \hat{\beta}_{EBD3} - \hat{\beta}_{EBD2} = \left(\frac{2r+3}{2c}\right)\left[2 - \left(1 + \frac{2D}{c}\right)\ln\left(1 + \frac{c}{D}\right)\right] \tag{A.5}$$

Substituting from (A.2) in (A.5), we get

$$2 - \left(1 + \frac{2D}{c}\right)\ln\left(1 + \frac{c}{D}\right) = \frac{c^2}{6D^2}\left(1 - \frac{c}{D}\right) + \frac{c^4}{60D^4}\left(\frac{8c}{D} - 9\right) + \ldots \tag{A.6}$$

According to (A.5) and (A.6), we have

$$\hat{\beta}_{EBD2} - \hat{\beta}_{EBD1} = \hat{\beta}_{EBD3} - \hat{\beta}_{EBD2} > 0$$

That is

$$\hat{\beta}_{EBD1} < \hat{\beta}_{EBD2} < \hat{\beta}_{EBD3}$$

(ii) From (A.5) and (A.6), we get

$$\lim_{D\to\infty}\left(\hat{\beta}_{EBD2} - \hat{\beta}_{EBD1}\right) = \lim_{D\to\infty}\left(\hat{\beta}_{EBD3} - \hat{\beta}_{EBD2}\right)$$

$$= \lim_{D\to\infty}\left[\frac{c^2}{6D^2}\left(1-\frac{c}{D}\right)+\frac{c^4}{60D^4}\left(\frac{8c}{D}-9\right)+\ldots\right]=0$$

That is

$$\lim_{D\to\infty}\hat{\beta}_{EBD1}=\lim_{D\to\infty}\hat{\beta}_{EBD2}=\lim_{D\to\infty}\hat{\beta}_{EBD3}$$

Thus, the proof is complete

Proof of Lemma 4

(i) From (3-16), (3-17) and (3-18) that

$$\hat{\beta}_{EBQ2}-\hat{\beta}_{EBQ1}=\hat{\beta}_{EBQ3}-\hat{\beta}_{EBQ2}=\left(\frac{2r-1}{2c}\right)\left[2-\left(1+\frac{2D}{c}\right)\ln\left(1+\frac{c}{D}\right)\right] \tag{A.7}$$

Substituting from (A.2) in (A.7), we get

$$2-\left(1+\frac{2D}{c}\right)\ln\left(1+\frac{c}{D}\right)=\frac{c^2}{6D^2}\left(1-\frac{c}{D}\right)+\frac{c^4}{60D^4}\left(\frac{8c}{D}-9\right)+\ldots \tag{A.8}$$

According to (A.7) and (A.8), we have

$$\hat{\beta}_{EBQ2}-\hat{\beta}_{EBQ1}=\hat{\beta}_{EBQ3}-\hat{\beta}_{EBQ2}>0$$

That is

$$\hat{\beta}_{EBQ1}<\hat{\beta}_{EBQ2}<\hat{\beta}_{EBQ3}$$

(ii) From (A.7) and (A.8), we get

$$\lim_{D\to\infty}\left(\hat{\beta}_{EBQ2}-\hat{\beta}_{EBQ1}\right)=\lim_{D\to\infty}\left(\hat{\beta}_{EBQ3}-\hat{\beta}_{EBQ2}\right)$$

$$=\lim_{D\to\infty}\left[\frac{c^2}{6D^2}\left(1-\frac{c}{D}\right)+\frac{c^4}{60D^4}\left(\frac{8c}{D}-9\right)+\ldots\right]=0$$

That is

$$\lim_{D\to\infty}\hat{\beta}_{EBQ1}=\lim_{D\to\infty}\hat{\beta}_{EBQ2}=\lim_{D\to\infty}\hat{\beta}_{EBQ3}$$

Thus, the proof is complete

Proof of Lemma 5

(i) From (3-19), (3-20) and (3-21) that

$$\hat{\beta}_{EBM\,2}-\hat{\beta}_{EBM\,1}=\hat{\beta}_{EBM\,3}-\hat{\beta}_{EBM\,2}=\left(\frac{2r-3}{2c}\right)\left[2-\left(1+\frac{2D}{c}\right)\ln\left(1+\frac{c}{D}\right)\right] \tag{A.9}$$

Substituting from (A.2) in (A.9), we get

$$2-\left(1+\frac{2D}{c}\right)\ln\left(1+\frac{c}{D}\right)=\frac{c^2}{6D^2}\left(1-\frac{c}{D}\right)+\frac{c^4}{60D^4}\left(\frac{8c}{D}-9\right)+\ldots \tag{A.10}$$

According to (A.9) and (A.10), we have

$$\hat{\beta}_{EBM\,2} - \hat{\beta}_{EBM\,1} = \hat{\beta}_{EBM\,3} - \hat{\beta}_{EBM\,2} > 0$$

That is

$$\hat{\beta}_{EBM\,1} < \hat{\beta}_{EBM\,2} < \hat{\beta}_{EBM\,3}$$

(ii) From (A.9) and (A.10), we get

$$\lim_{D\to\infty}\left(\hat{\beta}_{EBM\,2} - \hat{\beta}_{EBM\,1}\right) = \lim_{D\to\infty}\left(\hat{\beta}_{EBM\,3} - \hat{\beta}_{EBM\,2}\right)$$

$$= \lim_{D\to\infty}\left[\frac{c^2}{6D^2}\left(1-\frac{c}{D}\right) + \frac{c^4}{60D^4}\left(\frac{8c}{D}-9\right) + \dots\right] = 0$$

That is

$$\lim_{D\to\infty}\hat{\beta}_{EBM\,1} = \lim_{D\to\infty}\hat{\beta}_{EBM\,2} = \lim_{D\to\infty}\hat{\beta}_{EBM\,3}$$

Thus, the proof is complete

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