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2 - Variable AQCQ - Functional equation

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Abstract

In this paper, the authors obtain the general solution and generalized Ulam - Hyers stability of a 2 - variable AQCQ functional equation

$$g(x + 2y, u + 2v) + g(x - 2y, u - 2v) = 4[g(x + y, u + v) + g(x - y, u - v)] - 6g(x, u)$$
$$+ g(2y, 2v) + g(-2y, -2v) - 4g(y, v) - 4g(-y, -v)$$

using Hyers direct method. Counter examples for non stability is also discussed.

Keywords: *Additive functional equations, quadratic functional equations, cubic functional equations, quartic functional equations, Mixed type functional equations, Ulam - Hyers stability, Ulam - Hyers - Rassias stability, Ulam - Gavruta - Rassias stability, Ulam - JRassias stability.*

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [23] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [12]. It was further generalized and excellent results obtained by number of authors [2, 7, 18, 21, 19].

Over the last seven decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations which involves only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 6, 8, 9, 11, 13, 14, 15, 22, 24, 25, 26]

The general solution and the generalized Hyers-Ulam-Rassias stability of the generalized mixed type of functional equation

$$f(x + ay) + f(x - ay) = a^2 [f(x + y) + f(x - y)] + 2(1 - a^2) f(x)$$
$$+ \frac{(a^4 - a^2)}{12} [f(2y) + f(-2y) - 4f(y) - 4f(-y)].$$

for fixed integers a with $a \neq 0, \pm 1$ having solution **additive, quadratic, cubic and quartic** was discussed by K. Ravi et. al., [20]. Its generalized Ulam-Hyers stability in multi-Banach spaces and non-Archimedean normed spaces via fixed point approach was respectively investigated by T.Z. Xu et. al [24, 26].

Recently, M. Arunkumar et al., [3] first time introduced and investigated the solution and generalized Ulam-Hyers stability of a 2 - variable AC - mixed type functional equation

$$f(2x + y, 2z + w) - f(2x - y, 2z - w) = 4[f(x + y, z + w) - f(x - y, z - w)] - 6f(y, w) \tag{1}$$

having solutions

$$f(x, y) = ax + by \tag{2}$$

and

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \tag{3}$$

in Banach space via direct and fixed point approach.

Very recently, Choonkil Park and Jung Rye Lee [17] proved the Hyers - Ulam stability of the following additive - quadratic - cubic - quartic functional equation

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \tag{4}$$

in paranormed spaces.

In this paper, the authors obtain the general solution and generalized Ulam - Hyers stability of a 2 - variable AQCQ functional equation

$$g(x + 2y, u + 2v) + g(x - 2y, u - 2v) = 4[g(x + y, u + v) + g(x - y, u - v)] - 6g(x, u)$$
$$+ g(2y, 2v) + g(-2y, -2v) - 4g(y, v) - 4g(-y, -v) \tag{5}$$

having solutions

$$g(x, y) = ax + by, \tag{6}$$
$$g(x, y) = ax^2 + bxy + cy^2, \tag{7}$$
$$g(x, y) = ax^3 + bx^2y + cxy^2 + dy^3, \tag{8}$$
$$g(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4. \tag{9}$$

In Section 2, we present the general solution of the functional equation (5). The generalized Ulam-Hyers stability using Hyers direct method for odd, even and mixed cases are respectively discussed in Sections 3, 4, and 5. In Sections 6 and 7, counter examples for non stability for odd and even case are respectively given.

2.   General solution

In this section, we present the solution of the functional equation (5). Through out this section let G and H be real vector spaces.

Lemma 2.1 *If $g : G^2 \to H$ be an odd mapping satisfying (5) and let $a : G^2 \to H$ be a mapping given by*

$$a(u, u) = g(2u, 2u) - 8g(u, u) \tag{10}$$

for all $u \in G$ then

$$a(2u, 2u) = 2a(u, u) \tag{11}$$

for all $u \in G$ such that a is additive.

Proof. Using oddness of g in (5), we arrive

$$g(x + 2y, u + 2v) + g(x - 2y, u - 2v) = 4[g(x + y, u + v) + g(x - y, u - v)] - 6g(x, u) \tag{12}$$

for all $x, y, u, v \in G$. Letting (x, y, u, v) by $(0, 0, 0, 0)$ in (12), we get

$$g(0, 0) = 0. \tag{13}$$

Replacing (x, y, z, w) by (u, u, u, u) in (3.2), we arrive

$$g(3u, 3u) = 4g(2u, 2u) - 5g(u, u) \tag{14}$$

for all $u \in G$. Again replacing (x, y, u, v) by $(2u, u, 2u, u)$ in (12) and using (13), (14), we have

$$g(4u, 4u) = 4g(3u, 3u) - 6g(2u, 2u) + 4g(u, u) \tag{15}$$

for all $u \in G$. Using (14) in (15), we get

$$g(4u, 4u) = 10g(2u, 2u) - 16g(u, u) \tag{16}$$

for all $u \in G$. From (10), we establish

$$a(2u, 2u) - 2a(u, u) = g(4u, 4u) - 10g(2u, 2u) + 16g(u, u) \tag{17}$$

for all $u \in G$. Using (16) in (17), we desired our result.

Lemma 2.2 *If $g : G^2 \to H$ be an even mapping satisfying (5) and let $c : G^2 \to H$ be a mapping given by*

$$c(u, u) = g(2u, 2u) - 2g(u, u) \tag{18}$$

for all $u \in G$ then

$$c(2u, 2u) = 8c(u, u) \tag{19}$$

for all $u \in G$ such that c is cubic.

Proof. It follows from (19) that

$$c(2u, 2u) - 8c(u, u) = g(4u, 4u) - 10g(2u, 2u) + 16g(u, u) \tag{20}$$

for all $x \in U$. Using (16) in (20), we desired our result.

Lemma 2.3 *If $g : G^2 \to H$ be an even mapping satisfying (5) and let $q_2 : G^2 \to H$ be a mapping given by*

$$q_2(u, u) = g(2u, 2u) - 16g(u, u) \tag{21}$$

for all $u \in G$ then

$$q_2(2u, 2u) = 4q_2(u, u) \tag{22}$$

for all $u \in G$ such that q_2 is quadratic.

Proof.
   Using evenness of g in (5), we arrive

$$g(x + 2y, u + 2v) + g(x - 2y, u - 2v)$$
$$= 4[g(x + y, u + v) + g(x - y, u - v)] - 6g(x, u) + 2g(2y, 2v) - 8g(y, v) \tag{23}$$

for all $x, y, u, v \in G$. Letting (x, y, u, v) by $(0, 0, 0, 0)$ in (23), we get

$$g(0, 0) = 0. \tag{24}$$

Replacing (x, y, z, w) by (u, u, u, u) in (23), we arrive

$$g(3u, 3u) = 6g(2u, 2u) - 15g(u, u) \tag{25}$$

for all $u \in G$. Again replacing (x, y, u, v) by $(2u, u, 2u, u)$ in (23) and using (24), (25), we have

$$g(4u, 4u) = 4g(3u, 3u) - 4g(2u, 2u) - 4g(u, u) \tag{26}$$

for all $u \in G$. Using (25) in (26), we get

$$g(4u, 4u) = 20g(2u, 2u) - 64g(u, u) \tag{27}$$

for all $u \in G$. From (21), we establish

$$q_2(2u, 2u) - 4q_2(u, u) = g(4u, 4u) - 20g(2u, 2u) + 64g(u, u) \tag{28}$$

for all $u \in G$. Using (27) in (28), we desired our result.

Lemma 2.4 *If $g : G^2 \to H$ be an even mapping satisfying (5) and let $q_4 : G^2 \to H$ be a mapping given by*

$$q_4(u, u) = g(2u, 2u) - 4(u, u) \tag{29}$$

for all $u \in G$ then

$$q_4(2u, 2u) = 16q_4(u, u) \tag{30}$$

for all $u \in G$ such that q_4 is quartic.

Proof. It follows from (30) that

$$q_4(2u, 2u) - 4q_4(u, u) = g(4u, 4u) - 20g(2u, 2u) + 64g(u, u) \tag{31}$$

for all $x \in U$. Using (27) in (31), we desired our result.

Remark 2.5 *If $g : G^2 \to H$ be a mapping satisfying (5) and let $a, q_2, c, q_4 : G^2 \to H$ be a mapping defined in (10), (18), (21) and (29) then*

$$g(u, u) = \frac{1}{12}(q_4(u, u) - q_2(u, u)) \tag{32}$$

and

$$g(u, u) = \frac{1}{6}(c(u, u) - a(u, u)) \tag{33}$$

for all $u \in G$.

Lemma 2.6 *If $g : G^2 \to H$ be a mapping satisfying (5) and let $f : G \to H$ be a mapping given by*

$$f(u) = g(u, u) \tag{34}$$

for all $u \in G$, then f is (4) for all $x, y, u, v \in G$.

Proof. From (5) and (34), we get

$$
\begin{aligned}
f(x &+ 2y) + f(x - 2y) \\
&= g(x + 2y, x + 2y) + g(x - 2y, x - 2y) \\
&= 4[g(x + y, x + y) + g(x - y, x - y)] - 6g(x, x) + g(2y, 2y) + g(-2y, -2y) - 4g(y, y) - 4g(-y, -y) \\
&= 4[f(x + y) + f(x - y)] - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y)
\end{aligned}
$$

for all $x, y \in G$.

   Hereafter, through out this paper, let we consider G be a normed space and H be a Banach space. Define a mapping $Dg : G^2 \to H$ by

$$
\begin{aligned}
Dg(x, y, u, v) = {} & g(x + 2y, u + 2v) + g(x - 2y, u - 2v) - 4[g(x + y, u + v) + g(x - y, u - v)] \\
& + 6g(x, u) - g(2y, 2v) - g(-2y, -2v) + 4g(y, v) + 4g(-y, -v)
\end{aligned}
$$

for all $x, y, u, v \in G$.

3.   Stability results: Odd case

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (5) for odd case.

Theorem 3.1 *Let $j = \pm 1$. Let $Dg : G^2 \to H$ be an odd mapping for which there exist a function $\alpha : G^4 \to [0, \infty)$ with the condition*

$$\lim_{n \to \infty} \frac{1}{2^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}u, 2^{nj}v) = 0 \tag{35}$$

such that the functional inequality

$$\|Dg(x, y, u, v)\| \leq \alpha(x, y, u, v) \tag{36}$$

for all $x, y, u, v \in G$. Then there exists a unique 2-variable additive mapping $A(u, u) : G^2 \to H$ satisfying the functional equation (5) and

$$\|g(2u, 2u) - 8g(u, u) - A(u, u)\| \leq \frac{1}{2} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}u)}{2^{kj}} \tag{37}$$

for all $u \in G$, where $\beta(2^{kj}u)$ and $A(u, u)$ are defined by

$$\beta(2^{kj}u) = 4\,\alpha(2^{kj}u, 2^{kj}u, 2^{kj}u, 2^{kj}u) + \alpha(2^{(k+1)j}u, 2^{kj}u, 2^{(k+1)j}u, 2^{kj}u) \tag{38}$$

$$A(u, u) = \lim_{n \to \infty} \frac{1}{2^{nj}} (g(2^{(n+1)j}u, 2^{(n+1)j}u) - 8g(2^{nj}u, 2^{nj}u)) \tag{39}$$

for all $u \in G$.

Proof. Assume $j = 1$. Letting (x, y, u, v) by (u, u, u, u) in (36) and using oddness of g, we obtain

$$\|g(3u, 3u) - 4g(2u, 2u) + 5g(u, u)\| \leq \alpha(u, u, u, u) \tag{40}$$

for all $u \in G$. Replacing (x, y, z, w) by $(2u, u, 2u, u)$ in (36), we get

$$\|g(4u, 4u) - 4g(3u, 3u) + 6g(2u, 2u) - 4g(u, u)\| \leq \alpha(2u, u, 2u, u) \tag{41}$$

for all $u \in G$. Now, from (40) and (41), we have

$$\begin{aligned} \|g(4u, 4u) &- 10g(2u, 2u) + 16g(u, u)\| \\ &\leq 4\,\|g(3u, 3u) - 4g(2u, 2u) + 5g(u, u)\| + \|g(4u, 4u) - 4g(3u, 3u) + 6g(2u, 2u) - 4g(u, u)\| \\ &\leq 4\alpha(u, u, u, u) + \alpha(2u, u, 2u, u) \end{aligned} \tag{42}$$

for all $u \in G$. From (42), we arrive

$$\|g(4u, 4u) - 10g(2u, 2u) + 16g(u, u)\| \leq \beta(u) \tag{43}$$

where

$$\beta(u) = 4\alpha(u, u, u, u) + \alpha(2u, u, 2u, u) \tag{44}$$

for all $u \in G$. It is easy to see from (43) that

$$\|g(4u, 4u) - 8g(2u, 2u) - 2(g(2u, 2u) - 8g(u, u))\| \leq \beta(u) \tag{45}$$

for all $u \in G$. Using (10) in (45), we obtain

$$\|a(2u, 2u) - 2a(u, u)\| \leq \beta(u) \tag{46}$$

for all $u \in G$. From (46), we arrive

$$\left\| \frac{a(2u, 2u)}{2} - a(u, u) \right\| \leq \frac{\beta(u)}{2} \tag{47}$$

for all $u \in G$. Now replacing u by $2u$ and dividing by 2 in (47), we get

$$\left\| \frac{a(2^2 u, 2^2 u)}{2^2} - \frac{a(2u, 2u)}{2} \right\| \leq \frac{\beta(2u)}{2^2} \tag{48}$$

for all $u \in G$. From (47) and (48), we obtain

$$\left\|\frac{a(2^2u, 2^2u)}{2^2} - a(u, u)\right\| \leq \left\|\frac{a(2u, 2u)}{2} - a(u, u)\right\| + \left\|\frac{a(2^2u, 2^2u)}{2^2} - \frac{a(2u, 2u)}{2}\right\|$$
$$\leq \frac{1}{2}\left[\beta(u) + \frac{\beta(2u)}{2}\right] \tag{49}$$

for all $u \in G$. Proceeding further and using induction on a positive integer n, we get

$$\left\|\frac{a(2^nu, 2^nu)}{2^n} - a(u, u)\right\| \leq \frac{1}{2}\sum_{k=0}^{n-1}\frac{\beta(2^ku)}{2^k} \tag{50}$$
$$\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\beta(2^ku)}{2^k}$$

for all $u \in G$. In order to prove the convergence of the sequence

$$\left\{\frac{a(2^nu, 2^nu)}{2^n}\right\},$$

replacing u by 2^mu and dividing by 2^m in (50), for any $m, n > 0$, we deduce

$$\left\|\frac{a(2^{n+m}u, 2^{n+m}u)}{2^{(n+m)}} - \frac{a(2^mu, 2^mu)}{2^m}\right\| = \frac{1}{2^m}\left\|\frac{a(2^n \cdot 2^mu, 2^n \cdot 2^mu)}{2^n} - a(2^mu, 2^mu)\right\|$$
$$\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{\beta(2^{k+m}u)}{2^{k+m}}$$
$$\to 0 \quad as \quad m \to \infty$$

for all $u \in G$. This shows that the sequence $\left\{\frac{a(2^nu, 2^nu)}{2^n}\right\}$ is Cauchy sequence. Since H is complete, there exists a mapping $A(u, u) : G^2 \to H$ such that

$$A(u, u) = \lim_{n \to \infty}\frac{a(2^nu, 2^nu)}{2^n} \quad \forall \ u \in G.$$

Letting $n \to \infty$ in (50) and using (10), we see that (37) holds for all $u \in G$. To show that A satisfies (5), replacing (x, y, u, v) by $(2^nx, 2^ny, 2^nu, 2^nv)$ and dividing by 2^n in (36), we obtain

$$\frac{1}{2^n}\left\|Dg(2^nx, 2^ny, 2^nu, 2^nv)\right\| \leq \frac{1}{2^n}\alpha(2^nx, 2^ny, 2^nu, 2^nv)$$

for all $x, y, u, v \in G$. Letting $n \to \infty$ in the above inequality and using the definition of $A(u, u)$, we see that

$$DA(x, y, u, v) = 0.$$

Hence A satisfies (5) for all $x, y, u, v \in G$. To prove A is unique 2-variable additive function satisfying (5), we let $B(u, u)$ be another 2-variable additive mapping satisfying (5) and (37), then

$$\|A(u, u) - B(u, u)\| = \frac{1}{2^n}\|A(2^nu, 2^nu) - B(2^nu, 2^nu)\|$$
$$\leq \frac{1}{2^n}\left\{\left\|A(2^nu, 2^nu) - g(2^{n+1}u, 2^{n+1}u) + 8g(2^nu, 2^nu)\right\|\right.$$
$$+ \left.\left\|g(2^{n+1}u, 2^{n+1}u) - 8g(2^nu, 2^nu) - B(2^nu, 2^nu)\right\|\right\}$$
$$\leq \sum_{k=0}^{\infty}\frac{\beta(2^{k+n}u)}{2^{(k+n)}}$$
$$\to 0 \quad as \quad n \to \infty$$

for all $u \in G$. Hence A is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Gavruta-Rassias and Ulam-JRassias stabilities of (5).

Corollary 3.2 *Let* $Dg : G^2 \to H$ *be a mapping and there exits real numbers* ρ *and* s *such that*

$$\|Dg(x,y,u,v)\|$$
$$\leq \begin{cases} (i) & \rho, \\ (ii) & \rho\left\{\|x\|^s + \|y\|^s + \|u\|^s + \|v\|^s\right\}, & s < 1 \quad or \quad s > 1; \\ (iii) & \rho\,\|x\|^s\|y\|^s\|u\|^s\|v\|^s, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \\ (iv) & \rho\left\{\|x\|^s\|y\|^s\|u\|^s\|v\|^s + \left\{\|x\|^{4s} + \|y\|^{4s} + \|u\|^{4s} + \|v\|^{4s}\right\}\right\}, & s < \frac{1}{4} \quad or \quad s > \frac{1}{4}; \end{cases}$$
(51)

for all $x,y,u,v \in G$, *then there exists a unique 2- variable additive function* $A(u,u) : G^2 \to H$ *such that*

$$\|g(2u,2u) - 8g(u,u) - A(u,u)\| \leq \begin{cases} (i) & 5\rho, \\ (ii) & \dfrac{(18 + 2^{s+1})\rho\|u\|^s}{|2 - 2^s|}, \\ (iii) & \dfrac{(4 + 2^{2s})\rho\|u\|^{4s}}{|2 - 2^{4s}|} \\ (iv) & \left(\dfrac{22 + 2^{2s} + 2^{4s+1}}{|2 - 2^{4s}|}\right)\rho\|u\|^{4s} \end{cases}$$
(52)

for all $u \in G$.

Theorem 3.3 *Let* $j = \pm 1$. *Let* $Dg : G^2 \to H$ *be an odd mapping for which there exist a function* $\alpha : G^4 \to [0,\infty)$ *with the condition*

$$\lim_{n\to\infty} \frac{1}{8^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}u, 2^{nj}v) = 0$$
(53)

such that the functional inequality

$$\|Dg(x,y,u,v)\| \leq \alpha(x,y,u,v)$$
(54)

for all $x,y,u,v \in G$. *Then there exists a unique 2-variable cubic mapping* $C(u,u) : G^2 \to H$ *satisfying the functional equation* (5) *and*

$$\|g(2u,2u) - 2g(u,u) - C(u,u)\| \leq \frac{1}{8} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}u)}{8^{kj}}$$
(55)

for all $u \in G$, *where* $\beta(2^{kj}u)$ *and* $C(u,u)$ *are defined by*

$$\beta(2^{kj}u) = 4\alpha(2^{kj}u, 2^{kj}u, 2^{kj}u, 2^{kj}u) + \alpha(2^{(k+1)j}u, 2^{kj}u, 2^{(k+1)j}u, 2^{kj}u)$$
(56)

$$C(u,u) = \lim_{n\to\infty} \frac{1}{8^{nj}}(g(2^{(n+1)j}u, 2^{(n+1)j}u) - 2g(2^{nj}u, 2^{nj}u))$$
(57)

for all $u \in G$.

Proof. It is easy to see from (43) that

$$\|g(4u,4u) - 2g(2u,2u) - 8(g(2u,2u) - 2g(u,u))\| \leq \beta(u)$$
(58)

for all $u \in G$. Using (18) in (58), we obtain

$$\|c(2u,2u) - 8c(u,u)\| \leq \beta(u)$$
(59)

for all $u \in G$. The rest of proof is similar tracing to that of Theorem 3.1.

The following Corollary is an immediate consequence of Theorem 3.3 concerning the Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Gavruta-Rassias and Ulam-JRassias stabilities of (5).

Corollary 3.4 *Let* $Dg : G^2 \to H$ *be a mapping and there exits real numbers* ρ *and* s *such that*

$$\|Dg(x, y, u, v)\|$$
$$\leq \begin{cases} (i) & \rho, \\ (ii) & \rho\{||u||^s + ||y||^s + ||u||^s + ||v||^s\}, & s < 3 \quad or \quad s > 3; \\ (iii) & \rho\,||u||^s||y||^s||u||^s||v||^s, & s < \frac{3}{4} \quad or \quad s > \frac{3}{4}; \\ (iv) & \rho\{||u||^s||y||^s||u||^s||v||^s + \{||u||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\}\}, & s < \frac{3}{4} \quad or \quad s > \frac{3}{4}; \end{cases} \tag{60}$$

for all $x, y, u, v \in G$, *then there exists a unique 2- variable cubic function* $C(u, u) : G^2 \to H$ *such that*

$$\|g(2u, 2u) - 2g(u, u) - C(u, u)\| \leq \begin{cases} (i) & \dfrac{5\rho}{7}, \\ (ii) & \dfrac{(18 + 2^{s+1})\rho||u||^s}{|8 - 2^s|}, \\ (iii) & \dfrac{(4 + 2^{2s})\rho||u||^{4s}}{|8 - 2^{4s}|} \\ (iv) & \left(\dfrac{22 + 2^{2s} + 2^{4s+1}}{|8 - 2^{4s}|}\right)\rho||u||^{4s} \end{cases} \tag{61}$$

for all $u \in G$.

Theorem 3.5 *Let* $j = \pm 1$. *Let* $Dg : G^2 \to H$ *be a mapping for which there exist a function* $\alpha : G^4 \to [0, \infty)$ *with the condition given in* *(35)* *and* *(53)* *respectively, such that the functional inequality*

$$\|Dg(x, y, u, v)\| \leq \alpha(x, y, u, v) \tag{62}$$

for all $x, y, u, v \in G$. *Then there exists a unique 2-variable additive mapping* $A(u, u) : G^2 \to H$ *and a unique 2-variable cubic mapping* $C(u, u) : G^2 \to H$ *satisfying the functional equation* *(5)* *and*

$$\|g(u, u) - A(u, u) - C(u, u)\| \leq \frac{1}{6}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}u)}{2^{kj}} + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}u)}{8^{kj}}\right\} \tag{63}$$

for all $u \in G$, *where* $\beta(2^{kj}u)$, $A(u, u)$ *and* $C(u, u)$ *are respectively defined in* *(38)*, *(39)* *and* *(57)* *for all* $u \in G$.

Proof. By Theorems 3.1 and 3.3, there exists a unique 2-variable additive function $A_1(u, u) : G^2 \to H$ and a unique 2-variable cubic function $C_1(u, u) : G^2 \to H$ such that

$$\|g(2u, 2u) - 8g(u, u) - A_1(u, u)\| \leq \frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{2^{kj}} \tag{64}$$

and

$$\|g(2u, 2u) - 2g(u, u) - C_1(u, u)\| \leq \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{8^{kj}} \tag{65}$$

for all $u \in G$. Now from (64) and (65), one can see that

$$\left\|g(u, u) + \frac{1}{6}A_1(u, u) - \frac{1}{6}C_1(u, u)\right\|$$
$$= \left\|\left\{-\frac{g(2u, 2u)}{6} + \frac{8g(u, u)}{6} + \frac{A_1(u, u)}{6}\right\} + \left\{\frac{g(2u, 2u)}{6} - \frac{2g(u, u)}{6} - \frac{C_1(u, u)}{6}\right\}\right\|$$
$$\leq \frac{1}{6}\{\|g(2u, 2u) - 8g(u, u) - A_1(u, u)\| + \|g(2u, 2u) - 2g(u, u) - C_1(u, u)\|\}$$
$$\leq \frac{1}{6}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{2^{kj}} + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}x)}{8^{kj}}\right\}$$

for all $u \in U$. Thus we obtain (65) by defining $A(u, u) = \frac{-1}{6}A_1(u, u)$ and $C(u, u) = \frac{1}{6}C_1(u, u)$, where $\beta(2^{kj}u)$, $A(u, u)$ and $C(u, u)$ are respectively defined in (38), (39) and (57) for all $u \in G$.

The following corollary is the immediate consequence of Theorem 3.5, using Corollaries 3.2 and 3.4 concerning the Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Gavruta-Rassias and Ulam-JRassias stabilities of (5).

Corollary 3.6 *Let $Dg : G^2 \to H$ be a mapping and there exits real numbers ρ and s such that*

$$\|Dg(x, y, u, v)\|$$
$$\leq \begin{cases} (i) & \rho, \\ (ii) & \rho \left\{ ||x||^s + ||y||^s + ||u||^s + ||v||^s \right\}, & s \neq 1, 3; \\ (iii) & \rho \, ||x||^s ||y||^s ||u||^s ||v||^s, & s \neq \frac{1}{4}, \frac{3}{4}; \\ (iv) & \rho \left\{ ||x||^s ||y||^s ||u||^s ||v||^s + \left\{ ||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s} \right\} \right\}, & s \neq \frac{1}{4}, \frac{3}{4}; \end{cases} \tag{66}$$

for all $x, y, u, v \in G$, then there exists a unique 2-variable additive mapping $A(u, u) : G^2 \to H$ and a unique 2-variable cubic mapping $C(u, u) : G^2 \to H$ such that

$$\|g(u, u) - A(u, u) - C(u, u)\|$$
$$\leq \begin{cases} (i) & \dfrac{5\rho}{6} \left(1 + \dfrac{1}{7} \right), \\[2mm] (ii) & \dfrac{(18 + 2^{s+1})}{6} \left(\dfrac{1}{|2 - 2^s|} + \dfrac{1}{|8 - 2^s|} \right) \rho ||u||^s, \\[2mm] (iii) & \dfrac{(4 + 2^{2s})}{6} \left(\dfrac{1}{|2 - 2^{4s}|} + \dfrac{1}{|8 - 2^{4s}|} \right) \rho ||u||^{4s} \\[2mm] (iv) & \dfrac{(22 + 2^{2s} + 2^{4s+1})}{6} \left(\dfrac{1}{|2 - 2^{4s}|} + \dfrac{1}{|8 - 2^{4s}|} \right) \rho ||u||^{4s} \end{cases} \tag{67}$$

for all $u \in G$.

4. Stability results: Even case

In this section, we present the generalized Ulam-Hyers stability of the functional equation (5) for even case.

Theorem 4.1 *Let $j = \pm 1$. Let $Dg : G^2 \to H$ be an even mapping for which there exist a function $\alpha : G^4 \to [0, \infty)$ with the condition*

$$\lim_{n \to \infty} \frac{1}{4^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}u, 2^{nj}v) = 0 \tag{68}$$

such that the functional inequality

$$\|Dg(x, y, u, v)\| \leq \alpha(x, y, u, v) \tag{69}$$

for all $x, y, u, v \in G$. Then there exists a unique 2-variable quadratic mapping $Q_2(u, u) : G^2 \to H$ satisfying the functional equation (5) and

$$\|g(2u, 2u) - 16g(u, u) - Q_2(u, u)\| \leq \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}u)}{4^{kj}} \tag{70}$$

for all $u \in G$, where $\beta(2^{kj}u)$ and $Q(u, u)$ are defined by

$$\beta(2^{kj}u) = 4 \, \alpha(2^{kj}u, 2^{kj}u, 2^{kj}u, 2^{kj}u) + \alpha(2^{(k+1)j}u, 2^{kj}u, 2^{(k+1)j}u, 2^{kj}u) \tag{71}$$

$$Q_2(u, u) = \lim_{n \to \infty} \frac{1}{4^{nj}} (g(2^{(n+1)j}u, 2^{(n+1)j}u) - 16g(2^{nj}u, 2^{nj}u)) \tag{72}$$

for all $u \in G$.

Proof. Assume $j = 1$. Letting (x, y, u, v) by (u, u, u, u) in (69) and using evenness of g, we obtain

$$\|g(3u, 3u) - 6g(2u, 2u) + 15g(u, u)\| \leq \alpha(u, u, u, u) \tag{73}$$

for all $u \in G$. Replacing (x, y, z, w) by $(2u, u, 2u, u)$ in (69), we get

$$\|g(4u, 4u) - 4g(3u, 3u) + 4g(2u, 2u) + 4g(u, u)\| \leq \alpha(2u, u, 2u, u) \tag{74}$$

for all $u \in G$. Now, from (73) and (74), we have

$$
\begin{aligned}
\|g(4u,4u) &- 20g(2u,2u) + 64g(u,u)\| \\
&\leq 4 \|g(3u,3u) - 6g(2u,2u) + 15g(u,u)\| + \|g(4u,4u) - 4g(3u,3u) + 4g(2u,2u) + 4g(u,u)\| \\
&\leq 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)
\end{aligned}
\tag{75}
$$

for all $u \in G$. From (75), we arrive

$$
\|g(4u,4u) - 20g(2u,2u) + 64g(u,u)\| \leq \beta(u)
\tag{76}
$$

where

$$
\beta(u) = 4\alpha(u,u,u,u) + \alpha(2u,u,2u,u)
\tag{77}
$$

for all $u \in G$. It is easy to see from (76) that

$$
\|g(4u,4u) - 16g(2u,2u) - 4(g(2u,2u) - 16g(u,u))\| \leq \beta(u)
\tag{78}
$$

for all $u \in G$. Using (21) in (78), we obtain

$$
\|q_2(2u,2u) - 4q_2(u,u)\| \leq \beta(u)
\tag{79}
$$

for all $u \in G$. The rest of the proof is similar lines to that of Theorem 3.1.

The following Corollary is an immediate consequence of Theorem 4.1 concerning the Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Gavruta-Rassias and Ulam-JRassias stabilities of (5).

Corollary 4.2 *Let $Dg : G^2 \to H$ be a mapping and there exits real numbers ρ and s such that*

$$
\begin{aligned}
&\|Dg(x,y,u,v)\| \\
&\leq \begin{cases}
(i) & \rho, \\
(ii) & \rho\left\{||x||^s + ||y||^s + ||u||^s + ||v||^s\right\}, & s < 2 \quad or \quad s > 2; \\
(iii) & \rho\, ||x||^s ||y||^s ||u||^s ||v||^s, & s < \frac{1}{2} \quad or \quad s > \frac{1}{2}; \\
(iv) & \rho\left\{||x||^s ||y||^s ||u||^s ||v||^s + \left\{||x||^{4s} + ||y||^{4s} + ||u||^{4s} + ||v||^{4s}\right\}\right\}, & s < \frac{1}{2} \quad or \quad s > \frac{1}{2};
\end{cases}
\end{aligned}
\tag{80}
$$

for all $x, y, u, v \in G$, then there exists a unique 2- variable quadratic function $Q_2(u,u) : G^2 \to H$ such that

$$
\|g(2u,2u) - 8g(u,u) - Q_2(u,u)\| \leq \begin{cases}
(i) & \dfrac{5\rho}{4}, \\[2mm]
(ii) & \dfrac{(18 + 2^{s+1})\rho||u||^s}{|4 - 2^s|}, \\[2mm]
(iii) & \dfrac{(4 + 2^{2s})\rho||u||^{4s}}{|4 - 2^{4s}|} \\[2mm]
(iv) & \left(\dfrac{22 + 2^{2s} + 2^{4s+1}}{|4 - 2^{4s}|}\right)\rho||u||^{4s}
\end{cases}
\tag{81}
$$

for all $u \in G$.

Theorem 4.3 *Let $j = \pm 1$. Let $Dg : G^2 \to H$ be an even mapping for which there exist a function $\alpha : G^4 \to [0,\infty)$ with the condition*

$$
\lim_{n \to \infty} \frac{1}{16^{nj}} \alpha(2^{nj}x, 2^{nj}y, 2^{nj}u, 2^{nj}v) = 0
\tag{82}
$$

such that the functional inequality

$$
\|Dg(x,y,u,v)\| \leq \alpha(x,y,u,v)
\tag{83}
$$

for all $x, y, u, v \in G$. Then there exists a unique 2-variable quartic mapping $Q_4(u,u) : G^2 \to H$ satisfying the functional equation (5) and

$$
\|g(2u,2u) - 4g(u,u) - Q_4(u,u)\| \leq \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}u)}{16^{kj}}
\tag{84}
$$

for all $u \in G$, where $\beta(2^{kj}u)$ and $Q_4(u,u)$ are defined by

$$\beta(2^{kj}u) = 4\alpha(2^{kj}u, 2^{kj}u, 2^{kj}u, 2^{kj}u) + \alpha(2^{(k+1)j}u, 2^{kj}u, 2^{(k+1)j}u, 2^{kj}u) \tag{85}$$

$$Q_4(u,u) = \lim_{n\to\infty} \frac{1}{16^{nj}}(g(2^{(n+1)j}u, 2^{(n+1)j}u) - 4g(2^{nj}u, 2^{nj}u)) \tag{86}$$

for all $u \in G$.

Proof. It is easy to see from (76) that

$$\|g(4u,4u) - 4g(2u,2u) - 16(g(2u,2u) - 4g(u,u))\| \le \beta(u) \tag{87}$$

for all $u \in G$. Using (18) in (87), we obtain

$$\|q_4(2u,2u) - 16q_4(u,u)\| \le \beta(u) \tag{88}$$

for all $u \in G$. The rest of proof is similar tracing to that of Theorem 3.1.

The following Corollary is an immediate consequence of Theorem 4.3 concerning the Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Gavruta-Rassias and Ulam-JRassias stabilities of (5).

Corollary 4.4 *Let $Dg : G^2 \to H$ be a mapping and there exits real numbers ρ and s such that*

$$\|Dg(x,y,u,v)\|$$
$$\le \begin{cases} (i) & \rho, \\ (ii) & \rho\{\|x\|^s + \|y\|^s + \|u\|^s + \|v\|^s\}, & s < 4 \quad or \quad s > 4; \\ (iii) & \rho\,\|x\|^s\|y\|^s\|u\|^s\|v\|^s, & s < 1 \quad or \quad s > 1; \\ (iv) & \rho\{\|x\|^s\|y\|^s\|u\|^s\|v\|^s + \{\|u\|^{4s} + \|y\|^{4s} + \|u\|^{4s} + \|v\|^{4s}\}\}, & s < 1 \quad or \quad s > 1; \end{cases} \tag{89}$$

for all $x,y,u,v \in G$, then there exists a unique 2- variable quartic function $Q_4(u,u) : G^2 \to H$ such that

$$\|g(2u,2u) - 4cg(u,u) - Q_4(u,u)\| \le \begin{cases} (i) & \dfrac{5\rho}{15}, \\[2mm] (ii) & \dfrac{(18 + 2^{s+1})\rho\|u\|^s}{|16 - 2^s|}, \\[2mm] (iii) & \dfrac{(4 + 2^{2s})\rho\|u\|^{4s}}{|16 - 2^{4s}|} \\[2mm] (iv) & \left(\dfrac{22 + 2^{2s} + 2^{4s+1}}{|16 - 2^{4s}|}\right)\rho\|u\|^{4s} \end{cases} \tag{90}$$

for all $u \in G$.

Theorem 4.5 *Let $j = \pm 1$. Let $Dg : G^2 \to H$ be a mapping for which there exist a function $\alpha : G^4 \to [0,\infty)$ with the condition given in (68) and (82) respectively, such that the functional inequality*

$$\|Dg(x,y,u,v)\| \le \alpha(x,y,u,v) \tag{91}$$

for all $x,y,u,v \in G$. Then there exists a unique 2-variable quadratic mapping $Q_2(u,u) : G^2 \to H$ and a unique 2-variable quartic mapping $Q_4(u,u) : G^2 \to H$ satisfying the functional equation (5) and

$$\|g(u,u) - Q_2(u,u) - Q_4(u,u)\| \le \frac{1}{12}\left\{ \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}u)}{4^{kj}} + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}u)}{16^{kj}} \right\} \tag{92}$$

for all $u \in G$, where $\beta(2^{kj}u)$, $Q_2(u,u)$ and $Q_4(u,u)$ are respectively defined in (71), (72) and (86) for all $u \in G$.

Proof. By Theorems 4.1 and 4.3, there exists a unique 2-variable quadratic function $Q_{2_1}(u,u) : G^2 \to H$ and a unique 2-variable quartic function $Q_{4_1}(u,u) : G^2 \to H$ such that

$$\|g(2u,2u) - 16g(u,u) - Q_{2_1}(u,u)\| \le \frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}x)}{4^{kj}} \tag{93}$$

and

$$\|g(2u, 2u) - 4g(u, u) - Q_{4_1}(u, u)\| \leq \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}x)}{16^{kj}} \tag{94}$$

for all $u \in G$. Now from (93) and (94), one can see that

$$
\begin{aligned}
&\left\| g(u, u) + \frac{1}{12} Q_{2_1}(u, u) - \frac{1}{12} Q_{4_1}(u, u) \right\| \\
&= \left\| \left\{ -\frac{g(2u, 2u)}{12} + \frac{16g(u, u)}{12} + \frac{Q_{2_1}(u, u)}{12} \right\} + \left\{ \frac{g(2u, 2u)}{12} - \frac{4g(u, u)}{12} - \frac{Q_{4_1}(u, u)}{12} \right\} \right\| \\
&\leq \frac{1}{12} \left\{ \|g(2u, 2u) - 16g(u, u) - Q_{2_1}(u, u)\| + \|g(2u, 2u) - 4g(u, u) - Q_{4_1}(u, u)\| \right\} \\
&\leq \frac{1}{12} \left\{ \frac{1}{4} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}x)}{4^{kj}} + \frac{1}{16} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(2^{kj}x)}{16^{kj}} \right\}
\end{aligned}
$$

for all $u \in U$. Thus we obtain (94) by defining $Q_2(u, u) = \frac{-1}{12} Q_{2_1}(u, u)$ and $Q_4(u, u) = \frac{1}{12} Q_{4_1}(u, u)$, where $\beta(2^{kj}u)$, $Q_2(u, u)$ and $Q_4(u, u)$ are respectively defined in (71), (72) and (86) for all $u \in G$.

The following corollary is the immediate consequence of Theorem 4.5, using Corollaries 4.2 and 4.4 concerning the the Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Gavruta-Rassias and Ulam-JRassias stabilities of (5).

Corollary 4.6 *Let $Dg : G^2 \to H$ be a mapping and there exits real numbers ρ and s such that*

$$
\|Dg(x, y, u, v)\|
\leq
\begin{cases}
(i) & \rho, \\
(ii) & \rho \left\{ \|x\|^s + \|y\|^s + \|u\|^s + \|v\|^s \right\}, & s \neq 2, 4; \\
(iii) & \rho \, \|x\|^s \|y\|^s \|u\|^s \|v\|^s, & s \neq \frac{1}{2}, 1; \\
(iv) & \rho \left\{ \|x\|^s \|y\|^s \|u\|^s \|v\|^s + \left\{ \|x\|^{4s} + \|y\|^{4s} + \|u\|^{4s} + \|v\|^{4s} \right\} \right\}, & s \neq \frac{1}{2}, 1;
\end{cases}
\tag{95}
$$

for all $x, y, u, v \in G$, then there exists a unique 2-variable quadratic mapping $Q_2(u, u) : G^2 \to H$ and a unique 2-variable quartic mapping $Q_4(u, u) : G^2 \to H$ such that

$$
\|g(u, u) - Q_2(u, u) - Q_4(u, u)\|
\leq
\begin{cases}
(i) & \dfrac{5\rho}{12} \left(\dfrac{1}{3} + \dfrac{1}{15} \right), \\[2ex]
(ii) & \dfrac{(18 + 2^{s+1})}{12} \left(\dfrac{1}{|4 - 2^s|} + \dfrac{1}{|16 - 2^s|} \right) \rho \|u\|^s, \\[2ex]
(iii) & \dfrac{(4 + 2^{2s})}{12} \left(\dfrac{1}{|4 - 2^{4s}|} + \dfrac{1}{|16 - 2^{4s}|} \right) \rho \|u\|^{4s} \\[2ex]
(iv) & \dfrac{(22 + 2^{2s} + 2^{4s+1})}{12} \left(\dfrac{1}{|4 - 2^{4s}|} + \dfrac{1}{|16 - 2^{4s}|} \right) \rho \|u\|^{4s}
\end{cases}
\tag{96}
$$

for all $u \in G$.

5. Stability results: Mixed case

Theorem 5.1 *Let $j = \pm 1$. Let $Dg : G^2 \to H$ be a mapping for which there exist a function $\alpha : G^4 \to [0, \infty)$ with the condition given in (35), (53), (68) and (82) respectively, such that the functional inequality*

$$\|Dg(x, y, u, v)\| \leq \alpha(x, y, u, v) \tag{97}$$

for all $x, y, u, v \in G$. Then there exists a unique 2-variable additive mapping $A(u, u) : G^2 \to H$, a unique 2-variable quadratic mapping $Q_2(u, u) : G^2 \to H$, a unique 2-variable cubic mapping $C(u, u) : G^2 \to H$ and a unique 2-variable

quartic mapping $Q_4(u,u) : G^2 \to H$ *satisfying the functional equation* (5) *and*

$$\|g(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\|$$

$$\leq \frac{1}{12}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{2^{kj}} + \frac{\beta(-2^{kj}u)}{2^{kj}}\right) + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{8^{kj}} + \frac{\beta(-2^{kj}u)}{8^{kj}}\right)\right\}$$

$$+ \frac{1}{24}\left\{\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{4^{kj}} + \frac{\beta(-2^{kj}u)}{4^{kj}}\right) + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{16^{kj}} + \frac{\beta(-2^{kj}u)}{16^{kj}}\right)\right\} \tag{98}$$

for all $u \in G$, *where* $\beta(2^{kj}u)$, $A(u,u)$, $C(u,u)$, $Q_2(u,u)$ *and* $Q_4(u,u)$ *are respectively defined in* (38), (39), (57), (72) *and* (86) *for all* $u \in G$.

Proof. Let $g_o(u,u) = \dfrac{g(u,u) - g(-u,-u)}{2}$ for all $u \in G$. Then $g_o(0,0) = 0$ and $g_o(-u,-u) = -g_o(u,u)$ for all $u \in G$. Hence

$$\|Dg_o(x,y,u,v)\| \leq \frac{1}{2}\left\{\alpha(x,y,u,v) + \alpha(-x,-y,-u,-v)\right\} \tag{99}$$

for all $x,y,u,v \in G$. By Theorem 3.5, there exists a unique 2-variable additive function $A(u,u) : G^2 \to H$ and a unique 2-variable cubic function $C(u,u) : G^2 \to H$ such that

$$\|g_o(u,u) - A(u,u) - C(u,u)\|$$

$$\leq \frac{1}{12}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{2^{kj}} + \frac{\beta(-2^{kj}u)}{2^{kj}}\right) + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{8^{kj}} + \frac{\beta(-2^{kj}u)}{8^{kj}}\right)\right\} \tag{100}$$

for all $u \in G$. Also, let $g_e(u,u) = \dfrac{g(u,u) + g(-u,-u)}{2}$ for all $u \in G$. Then $g_e(0,0) = 0$ and $g_e(-u,-u) = g_e(u,u)$ for all $u \in G$. Hence

$$\|Dg_e(x,y,u,v)\| \leq \frac{1}{2}\left\{\alpha(x,y,u,v) + \alpha(-x,-y,-u,-v)\right\} \tag{101}$$

for all $x,y,u,v \in G$. By Theorem 4.5, there exists a unique 2-variable quadratic mapping $Q_2(u,u) : G^2 \to H$ and a unique 2-variable quartic mapping $Q_4(u,u) : G^2 \to H$ such that

$$\|g_e(u,u) - Q_2(u,u) - Q_4(u,u)\|$$

$$\leq \frac{1}{24}\left\{\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{4^{kj}} + \frac{\beta(-2^{kj}u)}{4^{kj}}\right) + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{16^{kj}} + \frac{\beta(-2^{kj}u)}{16^{kj}}\right)\right\} \tag{102}$$

for all $u \in G$. Define

$$g(u,u) = g_o(u,u) + g_e(u,u) \tag{103}$$

for all $u \in G$. Now from (103), (102) and (100)

$$\|g(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\|$$

$$= \|g_o(u,u) + g_e(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\|$$

$$\leq \|g_o(u,u) - A(u,u) - C(u,u)\| + \|g_e(u,u) - Q_2(u,u) - Q_4(u,u)\|$$

$$\leq \frac{1}{12}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{2^{kj}} + \frac{\beta(-2^{kj}u)}{2^{kj}}\right) + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{8^{kj}} + \frac{\beta(-2^{kj}u)}{8^{kj}}\right)\right\}$$

$$+ \frac{1}{24}\left\{\frac{1}{4}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{4^{kj}} + \frac{\beta(-2^{kj}u)}{4^{kj}}\right) + \frac{1}{16}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(2^{kj}u)}{16^{kj}} + \frac{\beta(-2^{kj}u)}{16^{kj}}\right)\right\} \tag{104}$$

for all $u \in G$, where $\beta(2^{kj}u)$, $A(u,u)$, $C(u,u)$, $Q_2(u,u)$ and $Q_4(u,u)$ are respectively defined in (38), (39), (57), (72) and (86) for all $u \in G$.

The following corollary is the immediate consequence of Theorem 5.1, using Corollaries 3.6 and 4.6 concerning the Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Gavruta-Rassias and Ulam-JRassias stabilities of (5).

Corollary 5.2 *Let* $Dg : G^2 \to H$ *be a mapping and there exits real numbers* ρ *and* s *such that*

$$
\|Dg(x,y,u,v)\|
$$
$$
\leq \begin{cases}
(i) & \rho, \\
(ii) & \rho\left\{\|x\|^s + \|y\|^s + \|u\|^s + \|v\|^s\right\}, & s \neq 1,2,3,4; \\
(iii) & \rho\,\|x\|^s\|y\|^s\|u\|^s\|v\|^s, & s \neq \frac{1}{4},\frac{1}{2},\frac{3}{4},1; \\
(iv) & \rho\left\{\|x\|^s\|y\|^s\|u\|^s\|v\|^s + \left\{\|x\|^{4s} + \|y\|^{4s} + \|u\|^{4s} + \|v\|^{4s}\right\}\right\}, & s \neq \frac{1}{4},\frac{1}{2},\frac{3}{4},1;
\end{cases}
\tag{105}
$$

for all $x,y,u,v \in G$, *then there exists a unique 2-variable additive mapping* $A(u,u) : G^2 \to H$, *a unique 2-variable quadratic mapping* $Q_2(u,u) : G^2 \to H$, *a unique 2-variable cubic mapping* $C(u,u) : G^2 \to H$ *and a unique 2-variable quartic mapping* $Q_4(u,u) : G^2 \to H$ *such that*

$$
\|g(u,u) - A(u,u) - Q_2(u,u) - C(u,u) - Q_4(u,u)\|
$$
$$
\leq \begin{cases}
(i) & \dfrac{5\rho}{6}\left(1 + \dfrac{1}{7} + \dfrac{1}{2\cdot 3} + \dfrac{1}{2\cdot 15}\right), \\[2mm]
(ii) & \dfrac{(18 + 2^{s+1})}{6}\left(\dfrac{1}{|2 - 2^s|} + \dfrac{1}{|8 - 2^s|} + \dfrac{1}{2|4 - 2^s|} + \dfrac{1}{2|16 - 2^s|}\right)\rho\|u\|^s, \\[2mm]
(iii) & \dfrac{(4 + 2^{2s})}{6}\left(\dfrac{1}{|2 - 2^{4s}|} + \dfrac{1}{|8 - 2^{4s}|} + \dfrac{1}{2|4 - 2^{4s}|} + \dfrac{1}{2|16 - 2^{4s}|}\right)\rho\|u\|^{4s} \\[2mm]
(iv) & \dfrac{(22 + 2^{2s} + 2^{4s+1})}{6}\left(\dfrac{1}{|2 - 2^{4s}|} + \dfrac{1}{|8 - 2^{4s}|} + \dfrac{1}{2|4 - 2^{4s}|} + \dfrac{1}{2|16 - 2^{4s}|}\right)\rho\|u\|^{4s}
\end{cases}
\tag{106}
$$

for all $u \in G$.

6. Counter examples for non stable cases: Odd case

Now the author provide an example to illustrate that the functional equation (5) is not stable for $s = 1$ in condition (ii) of Corollary 3.2.

Example 6.1 *Let* $\alpha : R^4 \to R$ *be a function defined by*

$$
\alpha(u) = \begin{cases} \mu u, & \text{if } |u| < 1 \\ \mu, & \text{otherwise} \end{cases}
$$

where $\mu > 0$ *is a constant, and define a function* $g : R^2 \to R$ *by*

$$
g(u,u) = \sum_{n=0}^{\infty} \frac{\alpha(2^n u)}{2^n} \qquad \text{for all} \quad u \in R.
$$

Then g *satisfies the functional inequality*

$$
|Dg(x,y,u,v)| \leq 52\,\mu\,(|x| + |y| + |u| + |v|)
\tag{107}
$$

for all $x,y,u,v \in R$. *Then there do not exist a additive mapping* $A : R^2 \to R$ *and a constant* $\delta > 0$ *such that*

$$
|g(2u,2u) - 8g(u,u) - A(u,u)| \leq \delta|u| \qquad \text{for all} \quad u \in R.
\tag{108}
$$

Proof. Now

$$
|g(x,x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n u)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\,\mu.
$$

Therefore we see that g is bounded. We are going to prove that g satisfies (107).

If $x = y = u = v = 0$ then (107) is trivial. If $|x| + |y| + |u| + |v| \geq \frac{1}{2}$ then the left hand side of (107) is less than 52μ. Now suppose that $0 < |x| + |y| + |u| + |v| < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^k} \leq |x| + |y| + |u| + |v| < \frac{1}{2^{k-1}}, \tag{109}$$

so that $2^{k-1}x < \frac{1}{2}$, $2^{k-1}y < \frac{1}{2}$, $2^{k-1}u < \frac{1}{2}$, $2^{k-1}v < \frac{1}{2}$ and consequently

$$2^{k-1}(y, v), 2^{k-1}(-y, -v), 2^{k-1}(2y, 2v), 2^{k-1}(-2y, -2v), 2^{k-1}(x, u), 2^{k-1}(x + y, u + v),$$
$$2^{k-1}(x - y, u - v), 2^{k-1}(x + 2y, u + 2v), 2^{k-1}(x - 2y, u - 2v) \in (-1, 1).$$

Therefore for each $n = 0, 1, \ldots, k - 1$, we have

$$2^n(y, v), 2^n(-y, -v), 2^n(2y, 2v), 2^n(-2y, -2v), 2^n(x, u), 2^n(x + y, u + v),$$
$$2^n(x - y, u - v), 2^n(x + 2y, u + 2v), 2^n(x - 2y, u - 2v) \in (-1, 1).$$

and

$$\alpha(2^n(x + 2y, u + 2v)) + \alpha(2^n(x - 2y, u - 2v)) - 4\alpha(2^n(x + y, u + v))$$
$$- 4\alpha(2^n(x - y, u - v)) + 6\alpha(2^n(x, u)) - \alpha(2^n(2y, 2v)) - \alpha(2^n(-2y, -2v))$$
$$+ 4\alpha(2^n(y, v)) + 4\alpha(2^n(-y, -v)) = 0$$

for $n = 0, 1, \ldots, k - 1$. From the definition of f and (109), we obtain that

$$\left| g(x + 2y, u + 2v) + g(x - 2y, u - 2v) - 4[g(x + y, u + v) + g(x - y, u - v)] \right.$$
$$\left. + 6g(x, u) - g(2y, 2v) - g(-2y, -2v) + 4g(y, v) + 4g(-y, -v) \right|$$
$$\leq \sum_{n=0}^{\infty} \frac{1}{2^n} \left| \alpha(2^n(x + 2y, u + 2v)) + \alpha(2^n(x - 2y, u - 2v)) - 4\alpha(2^n(x + y, u + v)) \right.$$
$$- 4\alpha(2^n(x - y, u - v)) - 6\alpha(2^n(x, u)) - \alpha(2^n(2y, 2v)) - \alpha(2^n(-2y, -2v))$$
$$\left. + 4\alpha(2^n(y, v)) + 4\alpha(2^n(-y, -v)) \right|$$
$$\leq \sum_{n=k}^{\infty} \frac{1}{2^n} \left| \alpha(2^n(x + 2y, u + 2v)) + \alpha(2^n(x - 2y, u - 2v)) - 4\alpha(2^n(x + y, u + v)) \right.$$
$$- 4\alpha(2^n(x - y, u - v)) - 6\alpha(2^n(x, u)) - \alpha(2^n(2y, 2v)) - \alpha(2^n(-2y, -2v))$$
$$\left. + 4\alpha(2^n(y, v)) + 4\alpha(2^n(-y, -v)) \right|$$
$$\leq \sum_{n=k}^{\infty} \frac{1}{2^n} 26\mu = 26 \, \mu \times \frac{2}{2^k} = 52 \, \mu \left(|x| + |y| + |u| + |v| \right).$$

Thus g satisfies (107) for all $x, y, u, v \in R$ with $0 < |x| + |y| + |u| + |v| < \frac{1}{2}$.

We claim that the additive functional equation (5) is not stable for $s = 1$ in condition (ii) Corollary 3.2. Suppose on the contrary that there exist a additive mapping $A : R^2 \to R$ and a constant $\delta > 0$ satisfying (108). Since g is bounded and continuous for all $u \in R$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(u, u) = cu$ for any u in R. Thus we obtain that

$$|g(2u, 2u) - 8g(u, u) - A(u, u)| \leq (\delta + |c|) |u|. \tag{110}$$

But we can choose a positive integer m with $m\mu > \delta + |c|$.

If $u \in \left(0, \frac{1}{2^{m-1}}\right)$, then $2^n u \in (0, 1)$ for all $n = 0, 1, \ldots, m - 1$. For this u, we get

$$g(2u, 2u) - 8g(u, u) = \sum_{n=0}^{\infty} \frac{\alpha(2^n u)}{2^n} \geq \sum_{n=0}^{m-1} \frac{\mu(2^n u)}{2^n} = m\mu u > (\beta + |c|) u$$

which contradicts (110). Therefore the additive functional equation (5) is not stable in sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality condition (ii) of (51).

A counter example to illustrate the non stability in condition (iii) of Corollary 3.2 is given in the following example.

Example 6.2 *Let s be such that $0 < s < \dfrac{1}{4}$. Then there is a function $g : R^2 \to R$ and a constant $\lambda > 0$ satisfying*

$$|Dg(x,y,u,v)| \le \lambda |x|^{\frac{s}{4}} \ |y|^{\frac{s}{4}} \ |u|^{\frac{s}{4}} \ |v|^{\frac{1-3s}{4}} \tag{111}$$

for all $x, y, u, v \in R$ and

$$\sup_{u \ne 0} \frac{|g(2u, 2u) - 8g(u, u) - A(u, u)|}{|u|} = +\infty \tag{112}$$

for every additive mapping $A(u, u) : R^2 \to R$.

Proof. If we take

$$g(u, u) = \begin{cases} (u, u) \ln |u, u| & if \quad u \ne 0, \\ 0, & if \quad u = 0. \end{cases}$$

Then from the relation (112), it follows that

$$\sup_{u \ne 0} \frac{|g(2u, 2u) - 8g(u, u) - A(u, u)|}{|u|} \ge \sup_{\substack{n \in N \\ n \ne 0}} \frac{|g(2n, 2n) - 8g(n, n) - A(n, n)|}{|n|}$$

$$= \sup_{\substack{n \in N \\ n \ne 0}} \frac{|n(2, 2) \ln |2n, 2n| - 8n(1, 1) \ln |n, n| - n \ A(1, 1)|}{|n|}$$

$$= \sup_{\substack{n \in N \\ n \ne 0}} |(2, 2) \ln |2n, 2n| - 8(1, 1) \ln |n, n| - A(1, 1)| = \infty.$$

We have to prove (111) is true.
Case (i):   If $x, y, u, v > 0$ in (111) then,

$$\left| g(x + 2y, u + 2v) + g(x - 2y, u - 2v) - 4[g(x + y, u + v) + g(x - y, u - v)] \right.$$

$$\left. + 6g(x, u) - g(2y, 2v) - g(-2y, -2v) + 4g(y, v) + 4g(-y, -v) \right|$$

$$= |(x + 2y, u + 2v) \ln |x + 2y, u + 2v| + (x - 2y, u - 2v) \ln |x - 2y, u - 2v|$$

$$- 4[(x + y, u + v) \ln |x + y, u + v| + (x - y, u - v) \ln |x - y, u - v|] + 6(x, u) \ln |x, u|$$

$$- (2y, 2v) \ln |2y, 2v| - (-2y, -2v) \ln |-2y, -2v|$$

$$+ 4[(y, v) \ln |y, v| + (-y, -v) \ln |-y, -v|]|.$$

Set $x = v_1, y = v_2, u = v_3, v = v_4$ it follows that

$$\left| g(x + 2y, u + 2v) + g(x - 2y, u - 2v) - 4[g(x + y, u + v) + g(x - y, u - v)] \right.$$

$$\left. + 6g(x, u) - g(2y, 2v) - g(-2y, -2v) + 4g(y, v) + 4g(-y, -v) \right|$$

$$= |(v_1 + 2v_2, v_3 + 2v_4) \ln |v_1 + 2v_2, v_3 + 2v_4| + (v_1 - 2v_2, v_3 - 2v_4) \ln |v_1 - 2v_2, v_3 - 2v_4|$$

$$- 4[(v_1 + v_2, v_3 + v_4) \ln |v_1 + v_2, v_3 + v_4| + (v_1 - v_2, v_3 - v_4) \ln |v_1 - v_2, v_3 - v_4|]$$

$$+ 6(v_1, v_3) \ln |v_1, v_3| - (2v_2, 2v_4) \ln |2v_2, 2v_4| - (-2v_2, -2v_4) \ln |-2v_2, -2v_4|$$

$$+ 4[(v_2, v_4) \ln |v_2, v_4| + (-v_2, -v_4) \ln |-v_2, -v_4|]|.$$

$$= \left| g(v_1 + 2v_2, v_3 + 2v_4) + g(v_1 - 2v_2, v_3 - 2v_4) - 4[g(v_1 + v_2, v_3 + v_4) + g(v_1 - v_2, v_3 - v_4)] \right.$$

$$\left. + 6g(v_1, v_3) - g(2v_2, 2v_4) - g(-2v_2, -2v_4) + 4g(v_2, v_4) + 4g(-v_2, -v_4) \right|$$

$$\le \lambda |v_1|^{\frac{s}{4}} \ |v_2|^{\frac{s}{4}} \ |v_3|^{\frac{s}{4}} \ |v_4|^{\frac{1-3s}{4}}$$

$$= \lambda |x|^{\frac{s}{4}} \ |y|^{\frac{s}{4}} \ |u|^{\frac{s}{4}} \ |v|^{\frac{1-3s}{4}}.$$

For the cases (ii) $x, y, u, v < 0$, (iii) $x, z > 0, u, v < 0$, (iv) $x, z > 0, u, v < 0$ and (v) $x = y = u = v = 0$, the proof is similar tracing to that of Case (i).

Now we will provide an example to illustrate that the functional equation (5) is not stable for $s = \dfrac{1}{4}$ in condition (iv) of Corollary 3.2.

Example 6.3 *Let $\alpha : R \to R$ be a function defined by*

$$\alpha(u) = \begin{cases} \mu u, & if \ |u| < \dfrac{1}{4} \\ \dfrac{\mu}{4}, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $g : R^2 \to R$ by

$$g(u, u) = \sum_{n=0}^{\infty} \frac{\alpha(2^n u)}{2^n} \qquad for \ all \quad u \in R.$$

Then g satisfies the functional inequality

$$|Dg(x, y, u, v)| \le 13\mu \left(|x|^{\frac{1}{4}} |y|^{\frac{1}{4}} |u|^{\frac{1}{4}} |v|^{\frac{1}{4}} + \{|x| + |y| + |u| + |v|\} \right) \tag{113}$$

for all $x, y, u, v \in R$. Then there do not exist a additive mapping $A : R^2 \to R$ and a constant $\delta > 0$ such that

$$|g(2u, 2u) - 8g(u, u) - A(u, u)| \le \delta|u| \qquad for \ all \quad u \in R. \tag{114}$$

Proof. Now

$$|g(u, u)| \le \sum_{n=0}^{\infty} \frac{|\alpha(2^n u)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{1}{2^n} \times \frac{\mu}{4} = \frac{\mu}{2}.$$

Therefore we see that g is bounded. We are going to prove that g satisfies (113).

If $x = y = u = v = 0$ then (113) is trivial.

If $|x|^{\frac{1}{4}} |y|^{\frac{1}{4}} |u|^{\frac{1}{4}} |v|^{\frac{1}{4}} + \{|x| + |y| + |u| + |v|\} \ge \dfrac{1}{2}$ then the left hand side of (113) is less than 13μ. The rest of the proof is similar tracing to that of Example 6.1.

Now the author provide an example to illustrate that the functional equation (5) is not stable for $s = 3$ in condition (ii) of Corollary 3.4.

Example 6.4 *Let $\alpha : R^4 \to R$ be a function defined by*

$$\alpha(u) = \begin{cases} \mu u^3, & if \ |u| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $g : R^2 \to R$ by

$$g(u, u) = \sum_{n=0}^{\infty} \frac{\alpha(2^n u)}{8^n} \qquad for \ all \quad u \in R.$$

Then g satisfies the functional inequality

$$|Dg(x, y, u, v)| \le \frac{52 \times 8^3}{7}\mu \left(|x|^3 + |y|^3 + |u|^3 + |v|^3 \right) \tag{115}$$

for all $x, y, u, v \in R$. Then there do not exist a cubic mapping $C : R^2 \to R$ and a constant $\delta > 0$ such that

$$|g(2u, 2u) - 2g(u, u) - C(u, u)| \le \delta|u|^3 \qquad for \ all \quad u \in R. \tag{116}$$

Proof. Now

$$|g(x, x)| \le \sum_{n=0}^{\infty} \frac{|\alpha(2^n u)|}{|8^n|} = \sum_{n=0}^{\infty} \frac{\mu}{8^n} = \frac{8}{7}\mu.$$

Therefore we see that g is bounded. We are going to prove that g satisfies (115).

If $x = y = u = v = 0$ then (115) is trivial. If $|x| + |y| + |u| + |v| \geq \dfrac{1}{8}$ then the left hand side of (115) is less than $\dfrac{52 \times 8}{7}\mu$. The rest of the proof is similar tracing to that of Example 6.1.   A counter example to illustrate the non stability in condition (iii) of Corollary 3.4 is given in the following example.

Example 6.5 *Let s be such that $0 < s < \dfrac{3}{4}$. Then there is a function $g : R^2 \to R$ and a constant $\lambda > 0$ satisfying*

$$|Dg(x,y,u,v)| \leq \lambda |x|^{\frac{s}{4}} \ |y|^{\frac{s}{4}} \ |u|^{\frac{s}{4}} \ |v|^{\frac{3-3s}{4}} \tag{117}$$

for all $x, y, u, v \in R$ and

$$\sup_{u \neq 0} \frac{|g(2u,2u) - 2g(u,u) - C(u,u)|}{|u|^3} = +\infty \tag{118}$$

for every cubic mapping $C(u,u) : R^2 \to R$.

Proof. If we take

$$g(u,u) = \begin{cases} (u,u)^3 \ln|u,u| & if \ \ u \neq 0, \\ 0, & if \ \ u = 0. \end{cases}$$

The rest of the proof is similar tracing to that of Example 6.2.

Now we will provide an example to illustrate that the functional equation (5) is not stable for $s = \dfrac{3}{4}$ in condition (iv) of Corollary 3.4.

Example 6.6 *Let $\alpha : R \to R$ be a function defined by*

$$\alpha(u) = \begin{cases} \mu u^3, & if \ |u| < \dfrac{3}{4} \\ \dfrac{3\mu}{4}, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $g : R^2 \to R$ by

$$g(u,u) = \sum_{n=0}^{\infty} \frac{\alpha(2^n u)}{8^n} \qquad for \ all \ \ u \in R.$$

Then g satisfies the functional inequality

$$|Dg(x,y,u,v)| \leq \frac{156 \times 8^2 \mu}{7} \left(|x|^{\frac{3}{4}} \ |y|^{\frac{3}{4}} \ |u|^{\frac{3}{4}} \ |v|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |u|^3 + |v|^3 \right\} \right) \tag{119}$$

for all $x, y, u, v \in R$. Then there do not exist a cubic mapping $C : R^2 \to R$ and a constant $\delta > 0$ such that

$$|g(2u,2u) - 2g(u,u) - C(u,u)| \leq \delta |u|^3 \qquad for \ all \ \ u \in R. \tag{120}$$

Proof. Now

$$|g(u,u)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n u)|}{|2^n|} = \sum_{n=0}^{\infty} \frac{1}{8^n} \times \frac{3\mu}{4} = \frac{6\mu}{7}.$$

Therefore we see that g is bounded. We are going to prove that g satisfies (113).

If $x = y = u = v = 0$ then (113) is trivial.

If $|x|^{\frac{3}{4}} \ |y|^{\frac{3}{4}} \ |u|^{\frac{3}{4}} \ |v|^{\frac{3}{4}} + \left\{ |x|^3 + |y|^3 + |u|^3 + |v|^3 \right\} \geq \dfrac{1}{8}$ then the left hand side of (113) is less than $\dfrac{156\mu}{7}$. The rest of the proof is similar tracing to that of Example 6.1.

7. Counter examples for non stable cases: Even case

Now the author provide an example to illustrate that the functional equation (5) is not stable for $s = 2$ in condition (*ii*) of Corollary 4.2.

Example 7.1 *Let $\alpha : R^4 \to R$ be a function defined by*

$$\alpha(u) = \begin{cases} \mu u^2, & if \ |u| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $g : R^2 \to R$ by

$$g(u, u) = \sum_{n=0}^{\infty} \frac{\alpha(2^n u)}{4^n} \qquad for \ all \quad u \in R.$$

Then g satisfies the functional inequality

$$|Dg(x, y, u, v)| \leq \frac{26 \times 4}{3} \mu \left(|x|^2 + |y|^2 + |u|^2 + |v|^2 \right) \tag{121}$$

for all $x, y, u, v \in R$. Then there do not exist a quadratic mapping $Q_2 : R^2 \to R$ and a constant $\delta > 0$ such that

$$|g(2u, 2u) - 16g(u, u) - Q_2(u, u)| \leq \delta |u|^2 \qquad for \ all \quad u \in R. \tag{122}$$

Proof. Now

$$|g(u, u)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n u)|}{|4^n|} = \sum_{n=0}^{\infty} \frac{\mu}{4^n} = \frac{4}{3} \mu.$$

Therefore we see that g is bounded. We are going to prove that g satisfies (121).

If $x = y = u = v = 0$ then (121) is trivial. If $|x| + |y| + |u| + |v| \geq \frac{1}{4}$ then the left hand side of (121) is less than $\frac{26 \times 4}{3} \mu$. The rest of the proof is similar tracing to that of Example 6.1. A counter example to illustrate the non stability in condition (*iii*) of Corollary 4.2 is given in the following example.

Example 7.2 *Let s be such that $0 < s < \frac{1}{2}$. Then there is a function $g : R^2 \to R$ and a constant $\lambda > 0$ satisfying*

$$|Dg(x, y, u, v)| \leq \lambda |x|^{\frac{s}{2}} \ |y|^{\frac{s}{2}} \ |u|^{\frac{s}{2}} \ |v|^{\frac{1-3s}{2}} \tag{123}$$

for all $x, y, u, v \in R$ and

$$\sup_{u \neq 0} \frac{|g(2u, 2u) - 16g(u, u) - Q_2(u, u)|}{|u|^2} = +\infty \tag{124}$$

for every quadratic mapping $Q_2(u, u) : R^2 \to R$.

Proof. If we take

$$g(u, u) = \begin{cases} (u, u)^2 \ln |u, u| & if \ \ u \neq 0, \\ 0, & if \ \ u = 0. \end{cases}$$

The rest of the proof is similar tracing to that of Example 6.2.

Now we will provide an example to illustrate that the functional equation (5) is not stable for $s = \frac{1}{2}$ in condition (*iv*) of Corollary 4.2.

Example 7.3 *Let $\alpha : R \to R$ be a function defined by*

$$\alpha(u) = \begin{cases} \mu u^2, & if \ |u| < \frac{1}{2} \\ \dfrac{\mu}{2}, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $g : R^2 \to R$ by

$$g(u, u) = \sum_{n=0}^{\infty} \frac{\alpha(2^n u)}{4^n} \qquad for \ all \quad u \in R.$$

Then g satisfies the functional inequality

$$|Dg(x, y, u, v)| \le \frac{26 \times 4\mu}{6} \left(|x|^{\frac{1}{2}} \ |y|^{\frac{1}{2}} \ |u|^{\frac{1}{2}} \ |v|^{\frac{1}{2}} + \left\{ |x|^2 + |y|^2 + |u|^2 + |v|^2 \right\} \right) \tag{125}$$

for all $x, y, u, v \in R$. Then there do not exist a quadratic mapping $Q_2 : R^2 \to R$ and a constant $\delta > 0$ such that

$$|g(2u, 2u) - 16g(u, u) - Q_2(u, u)| \le \delta |u| \qquad for \ all \quad u \in R. \tag{126}$$

Proof. Now

$$|g(u, u)| \le \sum_{n=0}^{\infty} \frac{|\alpha(2^n u)|}{|4^n|} = \sum_{n=0}^{\infty} \frac{1}{4^n} \times \frac{1\mu}{2} = \frac{4\mu}{6}.$$

Therefore we see that g is bounded. We are going to prove that g satisfies (113).

If $x = y = u = v = 0$ then (113) is trivial.

If $|x|^{\frac{1}{2}} \ |y|^{\frac{1}{2}} \ |u|^{\frac{1}{2}} \ |v|^{\frac{1}{2}} + \left\{ |x|^2 + |y|^2 + |u|^2 + |v|^2 \right\} \ge \dfrac{1}{4}$ then the left hand side of (113) is less than $\dfrac{26 \times 4\mu}{6}$. The rest of the proof is similar tracing to that of Example 6.1.

Now the author provide an example to illustrate that the functional equation (5) is not stable for $s = 4$ in condition (ii) of Corollary 4.4.

Example 7.4 *Let $\alpha : R^4 \to R$ be a function defined by*

$$\alpha(u) = \begin{cases} \mu u^4, & if \ |u| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $g : R^2 \to R$ by

$$g(u, u) = \sum_{n=0}^{\infty} \frac{\alpha(2^n u)}{16^n} \qquad for \ all \quad u \in R.$$

Then g satisfies the functional inequality

$$|Dg(x, y, u, v)| \le \frac{26 \times 16\mu}{15} \mu \left(|x|^4 + |y|^4 + |u|^4 + |v|^4 \right) \tag{127}$$

for all $x, y, u, v \in R$. Then there do not exist a quartic mapping $Q_4 : R^2 \to R$ and a constant $\delta > 0$ such that

$$|g(2u, 2u) - 4g(u, u) - Q_4(u, u)| \le \delta |u|^4 \qquad for \ all \quad u \in R. \tag{128}$$

Proof. Now

$$|g(u, u)| \le \sum_{n=0}^{\infty} \frac{|\alpha(2^n u)|}{|16^n|} = \sum_{n=0}^{\infty} \frac{\mu}{16^n} = \frac{16}{15} \mu.$$

Therefore we see that g is bounded. We are going to prove that g satisfies (127).

If $x = y = u = v = 0$ then (127) is trivial. If $|x| + |y| + |u| + |v| \ge \dfrac{1}{16}$ then the left hand side of (127) is less than $\dfrac{26 \times 16}{15} \mu$. The rest of the proof is similar tracing to that of Example 6.1.   A counter example to illustrate the non stability in condition (iii) of Corollary 4.4 is given in the following example.

Example 7.5 *Let s be such that $0 < s < 1$. Then there is a function $g : R^2 \to R$ and a constant $\lambda > 0$ satisfying*

$$|Dg(x, y, u, v)| \le \lambda |x|^s \ |y|^s \ |u|^s \ |v|^{1-3s} \tag{129}$$

for all $x, y, u, v \in R$ and

$$\sup_{u \ne 0} \frac{|g(2u, 2u) - 4g(u, u) - Q_4(u, u)|}{|u|^4} = +\infty \tag{130}$$

for every quartic mapping $Q_u(u, u) : R^2 \to R$.

Proof. If we take

$$g(u,u) = \begin{cases} (u,u)^4 \ln|u,u| & if \quad u \neq 0, \\ 0, & if \quad u = 0. \end{cases}$$

The rest of the proof is similar tracing to that of Example 6.2.

Now we will provide an example to illustrate that the functional equation (5) is not stable for $s = 1$ in condition (iv) of Corollary 4.4.

Example 7.6 *Let $\alpha : R \to R$ be a function defined by*

$$\alpha(u) = \begin{cases} \mu u^4, & if \ |u| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $g : R^2 \to R$ by

$$g(u,u) = \sum_{n=0}^{\infty} \frac{\alpha(2^n u)}{16^n} \qquad for \ all \quad u \in R.$$

Then g satisfies the functional inequality

$$|Dg(x,y,u,v)| \leq \frac{26 \times 16\mu}{15}\left(|x|^{\frac{1}{4}} \ |y|^{\frac{1}{4}} \ |u|^{\frac{1}{4}} \ |v|^{\frac{1}{4}} + \{|x| + |y| + |u| + |v|\}\right) \tag{131}$$

for all $x, y, u, v \in R$. Then there do not exist a quartic mapping $Q_4 : R^2 \to R$ and a constant $\delta > 0$ such that

$$|g(2u,2u) - 4g(u,u) - Q_4(u,u)| \leq \delta|u| \qquad for \ all \quad u \in R. \tag{132}$$

Proof. Now

$$|g(u,u)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(2^n u)|}{|16^n|} = \sum_{n=0}^{\infty} \frac{1}{16^n}\mu = \frac{16\mu}{15}.$$

Therefore we see that g is bounded. We are going to prove that g satisfies (113).

If $x = y = u = v = 0$ then (113) is trivial.

If $|x|^{\frac{1}{4}} \ |y|^{\frac{1}{4}} \ |u|^{\frac{1}{4}} \ |v|^{\frac{1}{4}} + \{|x| + |y| + |u| + |v|\} \geq \frac{1}{4}$ then the left hand side of (131) is less than $\frac{26 \times 16\mu}{15}$. The rest of the proof is similar tracing to that of Example 6.1.

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