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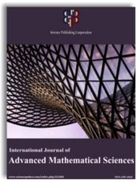

Results in Fixed Point Theorems of Ordered B-Metric Spaces for Rational Type α -Admissible Contractive Mappings

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Abstract

In this paper, we present a rational type α-Admissible contraction mappings in the context of partially ordered b-metric spaces and leverage on it to establish the existence and uniqueness of fixed point theorems for self-mappings within the framework of complete partially ordered b-metric spaces. The results of this article are applied to determine the solution to an integral equation. Through illustrative examples, we showcase the practical applicability of the results, demonstrating their effectiveness in real-world scenarios.

Keywords: *Fixed Point; α -Admissible Contraction; Partially Ordered Metric Space; b-Metric Space.*

1. Introduction

Fixed point theory stands as one of the most foundational and extensively studied areas in functional analysis and mathematics in general, with wide-ranging applications across numerous scientific and engineering disciplines. Central to this theory is the concept of a metric space, formally introduced by Fréchet in 1906 [23], which provides the essential framework for analyzing the behavior of mappings under various conditions. A landmark result in this area is the Banach Contraction Principle [1], which guarantees the existence and uniqueness of fixed points for contractive mappings in complete metric spaces, under the assumption of continuity. However, a notable limitation of this principle lies in its reliance on the continuity of the mapping, rendering it inapplicable in cases where discontinuities are present. To address this shortcoming, Kannan [2] proposed a significant generalization by establishing a fixed point theorem that does not require continuity. Further progress was made by Chatterjea [3] in 1972, who presented a result independent of both Banach's and Kannan's theorems. Building on these developments, Fisher [4] later introduced rational contractions into the framework of fixed point theory, expanding the scope of fixed point results in complete metric spaces. Motivated by the foundational role of metric spaces and the lasting impact of the Banach Contraction Principle, ongoing research continues to explore broader generalizations and alternative contractive conditions to address limitations in existing theories and to extend their applicability to a wider class of problems. Ran and Reurings [13] established fixed point results within the framework of partially ordered metric spaces, laying the groundwork for further advancements in this area. Building on their work, Nieto and Rodríguez-López [15] extended these results to non-decreasing mappings and demonstrated their applicability in solving certain types of differential equations.

On the other hand, the conceptual foundation of b-metric spaces was initially introduced by Bakhtin [8] and later formally defined by Czerwik in 1993 [5], who explored the convergence of measurable functions within this framework as a meaningful generalization of classical metric spaces. Czerwik also extended the Banach Contraction Principle to b-metric spaces. Further contributions include the work of Koleva et al. [25], who established fixed point results for Chatterjea-type inequalities in the context of b-metric spaces. Building on this, Abbas et al. [7] obtained common fixed point results for Fisher-type inequalities [4] within partially ordered b-metric spaces. For additional developments in fixed point results for partially ordered metric spaces and partially ordered b-metric spaces, readers are referred to [6,10,20–23,26] and the references therein.

More recently, in 2022, Haji et al. [11] introduced a class of generalized rational-type contraction mappings and established corresponding fixed point theorems within the framework of partially ordered b-metric spaces.

Based on the above insight, we present a class of rational-type α-admissible contraction mappings within the framework of partially ordered b-metric spaces, and utilize this framework to establish the existence and uniqueness of fixed points for self-mappings in complete partially ordered b-metric spaces. Additionally, we demonstrated the applicability of our results by investigating the existence of solutions to a certain integral equation. The validity and utility of the proposed concepts and theorems were further supported through carefully constructed illustrative examples, highlighting their practical relevance and theoretical significance.

2. Preliminaries

In this section, we begin with the basic concepts needed in the display of our major findings.

Definition 2.1: [14], [29] *Let $\mathcal{M}$ be a nonempty setS, and let $s \geq 1$ be a given real number. A function $\rho: \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ is called a b-metric if it satisfies the following conditions for all $x, y, z \in \mathcal{M}$:*

i) $\rho(x, y) = 0$ if and only if $x = y$;
ii) $\rho(x, y) = \rho(y, x)$;
iii) $\rho(x, z) \leq s[\rho(x, y) + \rho(y, z)]$.

A triplet $(\mathcal{M}, \rho, s)$ satisfying these conditions is referred to as a b-metric space.

Definition 2.2: *[19] Let $(\mathcal{M}, \preccurlyeq)$ be a partially ordered set. A sequence $\{x_n\} \subset \mathcal{M}$ It is said to be partially non-decreasing if $x_n \preccurlyeq x_{n+1}$ whenever x_n and x_{n+1} are comparable, for all $n \in \mathbb{N}$.*

Definition 2.3: *[17] Let $(\mathcal{M}, \preccurlyeq)$ be a partially ordered set, and let $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ be a mapping. Then:*

1) Elements $x, y \in \mathcal{M}$ are said to be comparable if either $x \preccurlyeq y$ or $y \preccurlyeq x$.
2) The set $\mathcal{M}$ is called well-ordered if it is nonempty and every pair of elements in $\mathcal{M}$ is comparable; that is, for all $x, y \in \mathcal{M}$, either $x \preccurlyeq y$ or $y \preccurlyeq x$ holds.
3) The mapping $\mathcal{F}$ is monotone non-decreasing (or order-preserving) concerning $\preccurlyeq$ If for all $x, y \in \mathcal{M}$, $x \preccurlyeq y$ implies $\mathcal{F}x \preccurlyeq \mathcal{F}y$.
4) The mapping $\mathcal{F}$ is monotone non-increasing (or order-reversing) concerning $\preccurlyeq$ If for all $x, y \in \mathcal{M}$, $x \preccurlyeq y$ implies $\mathcal{F}x \succcurlyeq \mathcal{F}y$.

Definition 2.4: *[12],[16],[18] A mapping $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ is called α-admissible if there exists a function $\alpha: \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ such that for all $x, y \in \mathcal{M}$ The following implication holds:*

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(\mathcal{F}x, \mathcal{F}y) \geq 1. \tag{2.1}$$

Definition 2.5: *[23], [28] Let $(\mathcal{M}, \rho)$ be a b-metric space, $x \in \mathcal{M}$, and $\{x_n\} \subset \mathcal{M}$. We define the following:*

i) The sequence $\{x_n\}$ b-converges to x if

$$\lim_{n \to \infty} \rho(x_n, x) = 0.$$

ii) The sequence $\{x_n\}$ is a b-Cauchy sequence if

$$\lim_{n,m \to \infty} \rho(x_n, x_m) = 0.$$

iii) The space $(\mathcal{M}, \rho)$ is b-complete if every b-Cauchy sequence in $\mathcal{M}$ b-converges to a limit in $\mathcal{M}$.

Definition 2.6: *[14], [27] Let $(\mathcal{M}, \rho)$ be a B-metric space with coefficient $s \geq 1$, and let $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ be a mapping. We say that $\mathcal{F}$ is continuous at $x_0 \in \mathcal{M}$ if for every sequence $\{x_n\} \subset \mathcal{M}$ such that $x_n \to x_0$ as $n \to \infty$, it follows that $\mathcal{F}x_n \to \mathcal{F}x_0$ as $n \to \infty$.*

Remark 2.7: *[14] If $\mathcal{F}$ is continuous at every point $x_0 \in \mathcal{M}$, then $\mathcal{F}$ It is said to be continuous on $\mathcal{M}$. Note that, in general, a b-metric need not be continuous.*

Definition 2.8: *[23] Let $(\mathcal{M}, \rho)$ be a complete metric space, and let $(\mathcal{M}, \preccurlyeq)$ be a partially ordered set. Then the triplet $(\mathcal{M}, \rho, \preccurlyeq)$ It is said to be a complete partially ordered metric space.*

3. Main results

We start this segment with the definition of rational type α-admissible contraction mapping.

Definition 3.1: *Let $\mathcal{M}$ be a nonempty set, $(\mathcal{M}, \rho, \preccurlyeq)$ a partially ordered b-metric space with $s \geq 1$ as the coefficient and $\alpha: \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$.* The mapping $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ is referred to as a rational type α-admissible contraction mapping if there exist nonnegative constants $a_{11}, a_{12}, a_{13}, a_{14} \in [0,1)$ such that $a_{11}s + (2s + s^2)a_{12} + a_{13}s + a_{14}s < 1$ and for all $x, y \in \mathcal{M}$ with $x \preccurlyeq y$,

$$\alpha(x, y)\rho(\mathcal{F}x, \mathcal{F}y) \leq a_{11}\rho(x, y) + a_{12}[\rho(x, \mathcal{F}x) + \rho(x, \mathcal{F}y)] + a_{13}\frac{\rho(x,\mathcal{F}x)\rho(y,\mathcal{F}y)}{\rho(x,y)+\rho(x,\mathcal{F}y)+\rho(y,\mathcal{F}x)} + a_{14}\frac{\rho(x,\mathcal{F}x)\rho(x,\mathcal{F}y)+\rho(y,\mathcal{F}x)\rho(y,\mathcal{F}y)}{\rho(y,\mathcal{F}x)+\rho(x,\mathcal{F}y)} \tag{3.1}$$

Theorem 3.2: *Let $(\mathcal{M}, \rho, \preccurlyeq)$ be a complete, partially ordered b-metric space with $s \geq 1$ as the coefficient, $\alpha: \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ and let $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ be a rational type α-admissible contraction mapping satisfying the conditions:*

i) $\mathcal{F}$ is α-admissible,
ii) There exists $x_0 \in \mathcal{M}$ such that $\alpha(x_0, \mathcal{F}x_0) \geq 1$ and $x_0 \preccurlyeq \mathcal{F}x_0$,
iii) $\mathcal{F}$ is a non-decreasing mapping concerning $\preccurlyeq$ and is continuous.

Then $\mathcal{F}$ has a fixed point $x^* \in \mathcal{M}$.

Proof Let $x_0 \in \mathcal{M}$ such that $\alpha(x_0, \mathcal{F}x_0) \geq 1$ and $x_0 \preccurlyeq \mathcal{F}x_0$. The sequence $\{x_n\}$ in $\mathcal{M}$ is define by $x_{n+1} = \mathcal{F}x_n$ for all $n \geq 0$. If there exists $n \geq 0$ for which $\mathcal{F}x_n = x_{n+1}$, then $\mathcal{F}x_n = x_n$. This proves that x_n is a fixed point of $\mathcal{F}$.

Now, we suppose that $x_{n+1} \neq x_n$ for every $n \geq 0$. It follows from condition (i) and (ii), that

$$\alpha(x_0, x_1) = \alpha(x_0, \mathcal{F}x_0) \geq 1 \Rightarrow \alpha(\mathcal{F}x_0, \mathcal{F}x_1) = \alpha(x_1, x_2) \geq 1. \tag{3.2}$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

Since $\mathcal{F}$ has a non-decreasing property, we have

$$x_0 \preccurlyeq \mathcal{F}x_0 = x_1 \preccurlyeq \mathcal{F}x_1 = x_2 \preccurlyeq \mathcal{F}x_2 \ldots x_n \preccurlyeq \mathcal{F}x_n = x_{n+1} \preccurlyeq \cdots \tag{3.4}$$

With (3.1), we have

$$\rho(x_n, x_{n+1}) = \rho(\mathcal{F}x_{n-1}, \mathcal{F}x_n)$$

$$\leq \alpha(x_{n-1}, x_n)\rho(\mathcal{F}x_{n-1}, \mathcal{F}x_n) \leq a_{11}\rho(x_{n-1}, x_n) + a_{12}[\rho(x_{n-1}, \mathcal{F}x_{n-1}) + \rho(x_{n-1}, \mathcal{F}x_n)] + a_{13}\frac{\rho(x_{n-1},\mathcal{F}x_{n-1})\rho(x_n,\mathcal{F}x_n)}{\rho(x_{n-1},x_n)+\rho(x_{n-1},\mathcal{F}x_n)+\rho(x_n,\mathcal{F}x_{n-1})} + a_{14}\frac{\rho(x_{n-1},\mathcal{F}x_{n-1})\rho(x_{n-1},\mathcal{F}x_n)+\rho(x_n,\mathcal{F}x_{n-1})\rho(x_n,\mathcal{F}x_n)}{\rho(x_n,\mathcal{F}x_{n-1})+\rho(x_{n-1},\mathcal{F}x_n)}$$

$$= a_{11}\rho(x_{n-1}, x_n) + a_{12}[\rho(x_{n-1}, x_n) + \rho(x_{n-1}, x_{n+1})] + a_{13}\frac{\rho(x_{n-1},x_n)\rho(x_n,x_{n+1})}{\rho(x_{n-1},x_n)+\rho(x_{n-1},x_{n+1})+\rho(x_n,x_n)} + a_{14}\frac{\rho(x_{n-1},x_n)\rho(x_{n-1},x_{n+1})+\rho(x_n,x_n)\rho(x_n,x_{n+1})}{\rho(x_n,x_n)+\rho(x_{n-1},x_{n+1})}$$

Since $\rho(x_n, x_n) = 0$, we have

$$\leq a_{11}\rho(x_{n-1}, x_n) + a_{12}\rho(x_{n-1}, x_n) + a_{12}\rho(x_{n-1}, x_{n+1}) + a_{13}\rho(x_{n-1}, x_n) + a_{14}\rho(x_{n-1}, x_n)$$

Implies,

$$\rho(x_n, x_{n+1}) \leq a_{11}\rho(x_{n-1}, x_n) + a_{12}\rho(x_{n-1}, x_n) + a_{12}s\rho(x_{n-1}, x_n) + a_{12}s\rho(x_n, x_{n+1}) + a_{13}\rho(x_{n-1}, x_n) + a_{14}\rho(x_{n-1}, x_n)$$

$$(1 - a_{12}s)\rho(x_n, x_{n+1}) \leq (a_{11} + (1 + s)a_{12} + a_{13} + a_{14})\rho(x_{n-1}, x_n),$$

$$\rho(x_n, x_{n+1}) \leq \frac{(a_{11}+(1+s)a_{12}+a_{13}+a_{14})}{(1-a_{12}s)}\rho(x_{n-1}, x_n) \tag{3.5}$$

Let $\frac{(a_{11}+(1+s)a_{12}+a_{13}+a_{14})}{(1-a_{12}s)} = \mu \in \left[0, \frac{1}{s}\right)$, then we have (3.5) as

$$\rho(x_n, x_{n+1}) \leq \mu\rho(x_{n-1}, x_n).$$

Similarly,

$$\rho(x_{n-1}, x_n) \leq \mu\rho(x_{n-2}, x_{n-1}).$$

So, by induction we get

$$\rho(x_n, x_{n+1}) \leq \mu^n\rho(x_0, x_1), \forall n \geq 1.$$

Since $\left[0, \frac{1}{s}\right)$, $\mu^n\rho(x_0, x_1) \to 0$ as $n \to \infty$. Hence, $\rho(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

That is,

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0. \tag{3.6}$$

Which shows that the sequence $\{x_n\}$ is Cauchy in $\mathcal{M}$.
Thus, to show that the sequence $\{x_n\}$ is b-Cauchy. For $m, n \in N$ with $m > n$,we apply triangular inequality to get,

$$\rho(x_n, x_m) \leq s[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_m)]$$

$$\leq s\rho(x_n, x_{n+1}) + s^2[\rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{m-1}, x_m)]$$

$$\leq s\rho(x_n, x_{n+1}) + s^2\rho(x_{n+1}, x_{n+2}) + \cdots + s^m\rho(x_{m-1}, x_m)$$

$$\leq s\mu^n\rho(x_0, x_1) + s^2\mu^{n+1}\rho(x_0, x_1) + \cdots + s^m\mu^{m-1}\rho(x_0, x_1)$$

$$= s\mu^n\rho(x_0, x_1)\left[1 + (s\mu) + (s\mu)^2 + \cdots + (s\mu)^{m-(n+1)}\right]$$

$$\leq s\mu^n\rho(x_0, x_1)[1 + (s\mu) + (s\mu)^2 + \cdots]$$

$$= \frac{s\mu^n}{1 - s\mu}\rho(x_0, x_1).$$

Implies,

$$\rho(x_n, x_m) \leq \frac{s\mu^n}{1 - s\mu}\rho(x_0, x_1).$$

Since $0 \leq \mu < \frac{1}{s}$, we have

$$\frac{s\mu^n}{1-s\mu}\rho(x_0, x_1) \to 0 \text{ as } n \to \infty. \text{ Hence, } \rho(x_n, x_m) \to 0 \text{ as } m, n \to \infty.$$

That is,

$$\lim_{m,n\to\infty} \rho(x_n, x_m) = 0. \quad (3.6)$$

Which shows that the sequence $\{x_n\}$ is b-Cauchy in $\mathcal{M}$. Using the fact that $\mathcal{M}$ is b-complete, there exists $x \in \mathcal{M}$ satisfying $x_n \to x$ implying that

$$\lim_{n\to\infty} x_n = x. \quad (3.7)$$

Using condition (iii) $\mathcal{F}$ is continuous, then

$$x = \lim_{n\to\infty} x_{n+1}$$

$$= \lim_{n\to\infty} \mathcal{F}x_n$$

$$= \mathcal{F}\left(\lim_{n\to\infty} x_n\right)$$

$$= \mathcal{F}x.$$

Thus, $\mathcal{F}x = x$.

The next Theorem is given by replacing the continuity of $\mathcal{F}$ With the continuity of ρ.

Theorem 3.3: *Let $(\mathcal{M}, \rho, \preccurlyeq)$ be a complete, partially ordered b-metric space with $s \geq 1$ as the coefficient, $\alpha: \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ and let $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ be a rational type α-admissible contraction mapping satisfying the conditions:*

1) $\mathcal{F}$ is α-admissible,
2) There exists $x_0 \in \mathcal{M}$ such that $\alpha(x_0, \mathcal{F}x_0) \geq 1$ and $x_0 \preccurlyeq \mathcal{F}x_0$,
3) There exists a non-decreasing sequence $x_n \to x$ in $\mathcal{M}$ with $x_n \preccurlyeq x$ and for all $n \geq 0$, $\alpha(x_n, x) \geq 1$
4) $\mathcal{F}$ is a non-decreasing mapping concerning $\preccurlyeq$,
5) ρ is continuous.

Then $\mathcal{F}$ has a fixed point $x^* \in \mathcal{M}$.

Proof Let $x_0 \in \mathcal{M}$ such that $\alpha(x_0, \mathcal{F}x_0) \geq 1$ and $x_0 \preccurlyeq \mathcal{F}x_0$. The sequence $\{x_n\}$ in $\mathcal{M}$ is define by $\mathcal{F}^{n+1}x_0 = \mathcal{F}(\mathcal{F}^n x_0)$ for all $n \geq 0$. Since $\mathcal{F}$ has a non-decreasing property, we have

$$x_0 \preccurlyeq \mathcal{F}x_0 = x_1 \preccurlyeq \mathcal{F}^2 x_0 = x_2 \preccurlyeq \mathcal{F}^3 x_0 \ldots x_n \preccurlyeq \mathcal{F}^{n+1}x_0 = x_{n+1} \preccurlyeq \cdots \quad (3.8)$$

Following the same approach as in the proof of Theorem 3.2, we show that the sequence $\{x_n\}$ in $\mathcal{M}$ is Cauchy, and it converges to x.
To show the existence of a fixed point, we let $x \neq \mathcal{F}x$. By (iii), we have the existence of a sequence $\{x_n\}$ in $\mathcal{M}$ such that for all $n \geq 0$, $\alpha(x_n, x) \geq 1$.
With (3.1), we have

$$\rho(x_{n+1}, \mathcal{F}x_n) \leq a_{11}\rho(x_n, x) + a_{12}[\rho(x_n, \mathcal{F}x_n) + \rho(x_n, \mathcal{F}x)] + a_{13}\frac{\rho(x_n,\mathcal{F}x_n)\rho(x,\mathcal{F}x)}{\rho(x_n,x)+\rho(x_n,\mathcal{F}x)+\rho(x,\mathcal{F}x_n)} + a_{14}\frac{\rho(x_n,\mathcal{F}x_n)\rho(x_n,\mathcal{F}x)+\rho(x,\mathcal{F}x_n)\rho(x,\mathcal{F}x)}{\rho(x,\mathcal{F}x_n)+\rho(x_n,\mathcal{F}x)} \quad (3.9)$$

On letting $n \to 0$ in (3.9), we have

$$\rho(x, \mathcal{F}x) \leq a_{12}\, \rho(x, \mathcal{F}x)$$

Implies
$(1 - a_{12})\, \rho(x, \mathcal{F}x) \leq 0$. Since $(1 - a_{12}) > 0$, we have $\rho(x, \mathcal{F}x) = 0$, that is, $x = \mathcal{F}x$. Thus, x is the fixed point of $\mathcal{F}$.

The following Theorem provides the uniqueness of fixed point theorem for Theorem 3.2 and Theorem 3.3.

Theorem 3.4. If, in addition to the conditions of Theorem 3.2 and Theorem 3.3, we have for every $x, x^* \in \mathcal{M}$ there exists $z \in \mathcal{M}$ which is comparable to x and x^*. Then $\mathcal{F}$ has a unique fixed point in $\mathcal{M}$.
Proof Let x and x^* There are two fixed points of $\mathcal{F}$Then we claim that $x = x^*$. Suppose that $x \neq x^*$. From (3.1), we have
$\rho(x, x^*) = \rho(\mathcal{F}x, \mathcal{F}x^*)$

$$\leq \alpha(x, x^*)\rho(\mathcal{F}x, \mathcal{F}x^*) \leq a_{11}\rho(x, x^*) + a_{12}[\rho(x, \mathcal{F}x) + \rho(x, \mathcal{F}x^*)] + a_{13}\frac{\rho(x,\mathcal{F}x)\rho(x^*,\mathcal{F}x^*)}{\rho(x,x^*)+\rho(x,\mathcal{F}x^*)+\rho(x^*,\mathcal{F}x)} + a_{14}\frac{\rho(x,\mathcal{F}x)\rho(x,\mathcal{F}x^*)+\rho(x^*,\mathcal{F}x)\rho(x^*,\mathcal{F}x^*)}{\rho(y,\mathcal{F}x)+\rho(x,\mathcal{F}x^*)}$$

$$= a_{11}\rho(x, x^*) + a_{12}[\rho(x, x) + \rho(x, x^*)] + a_{13}\frac{\rho(x,x)\rho(x^*,x^*)}{\rho(x,x^*)+\rho(x,x^*)+\rho(x^*,x)} + a_{14}\frac{\rho(x,x)\rho(x,x^*)+\rho(x^*,x)\rho(x^*,x^*)}{\rho(x^*,x)+\rho(x,x^*)}$$

$$= (a_{11} + a_{12})\rho(x, x^*) < \rho(x, x^*),$$

Since $a_{11} + a_{12} < 1$, a contradiction, hence $\rho(x, x^*) = 0$, that is, $x = x^*$. Thus, $\mathcal{F}$ has a unique fixed point.

Example 3.4. Consider $\mathcal{M} = [0,1]$ with the usual order $\leq$ and b-metric $\rho: \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ which is define $s = 2$ by

$$\rho(x, y) = \frac{1}{16}|x - y|^2.$$

Define $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{F}x = \frac{x}{4}, \text{if } x \in \left[0, \frac{1}{4}\right]$$

Also, define $\alpha: \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ by

$$\alpha(x, y) = \begin{cases} 2, \text{if } x, y \in \left[0, \frac{1}{4}\right] \\ 0, \text{otherwise} \end{cases}$$

Obviously, $\mathcal{F}$ is a non-decreasing mapping and α-admissible mapping.
Also, for $x_0 = 0$, we get

$$\alpha(x_0, \mathcal{F}x_0) = \alpha(0, \mathcal{F}0) = 2 \geq 1.$$

With (3.1),we let $s = 2, a_{11} = \frac{1}{32}, a_{12} = \frac{1}{64}, a_{13} = \frac{1}{32}, a_{14} = \frac{1}{16}$ with $a_{11}s + (2s + s^2)a_{12} + a_{13}s + a_{14}s < 1$.
Now, if $x, y \in \left[0, \frac{1}{4}\right]$, we get

$\alpha(x, y)\rho(\mathcal{F}x, \mathcal{F}y)$

$$= 2\left(\frac{1}{16}\right)\left|\frac{x}{4} - \frac{y}{4}\right|^2 = \frac{1}{128}|x - y|^2 \leq a_{11}|x - y|^2 + a_{12}\left[\left|x - \frac{x}{4}\right|^2 + \left|x - \frac{y}{4}\right|^2\right] + a_{13}\frac{\left|x-\frac{x}{4}\right|^2\left|y-\frac{y}{4}\right|^2}{|x-y|^2+\left|x-\frac{y}{4}\right|^2+\left|y-\frac{x}{4}\right|^2} + a_{14}\frac{\left|x-\frac{x}{4}\right|^2\left|x-\frac{y}{4}\right|^2+\left|y-\frac{x}{4}\right|^2\left|y-\frac{y}{4}\right|^2}{\left|y-\frac{x}{4}\right|^2+\left|x-\frac{y}{4}\right|^2}$$

If $x, y \in \left[0, \frac{1}{4}\right]$, we have the inequality in (3.1) holds. Then $\mathcal{F}$ satisfies the hypothesis of Theorem 3.2 and $0,1 \in \mathcal{M}$ are fixed points of $\mathcal{F}$.

Corollary 3.6 Let $(\mathcal{M}, \rho, \preccurlyeq)$ be a complete partially ordered b-metric space with $s \geq 1$ as the coefficient, and let $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ be a non-decreasing mapping with respect to $\preccurlyeq$, if there exist nonnegative constants $a_{11}, a_{12}, a_{13}, a_{14} \in [0,1)$ such that $a_{11}s + (2s + s^2)a_{12} + a_{13}s + a_{14}s < 1$ and for all $x, y \in \mathcal{M}$ with $x \preccurlyeq y$, satisfying the following condition

$$\rho(\mathcal{F}x, \mathcal{F}y) \leq a_{11}\rho(x, y) + a_{12}[\rho(x, \mathcal{F}x) + \rho(x, \mathcal{F}y)] + a_{13}\frac{\rho(x,\mathcal{F}x)\rho(y,\mathcal{F}y)}{\rho(x,y)+\rho(x,\mathcal{F}y)+\rho(y,\mathcal{F}x)} + a_{14}\frac{\rho(x,\mathcal{F}x)\rho(x,\mathcal{F}y)+\rho(y,\mathcal{F}x)\rho(y,\mathcal{F}y)}{\rho(y,\mathcal{F}x)+\rho(x,\mathcal{F}y)} \quad (3.10)$$

Then $\mathcal{F}$ is a unique fixed point $x^* \in \mathcal{M}$.
Proof The proof of the corollary follows from Theorem 3.2 by taking $\alpha(x, y) = 1$ for all $x, y \in \mathcal{M}$.

4. Application

Here, we demonstrate applicability of the results developed in the previous sections to explore the existence of solution to an integral equation,

$$x(u) = \int_0^1 \mathcal{L}(u, v, x(v))dv. \quad (4.1)$$

Consider $\mathcal{M} = C[0,1]$ be the collections of real continuous functions define on interval [0,1]. Define a partial order $\preccurlyeq$ on $\mathcal{M}$ by $x \preccurlyeq y$ if and only if $x(u) \preccurlyeq y(u)$ for all $u \in [0,1]$ with

$$\rho(x, y) = \max_{u\in[0,1]}(|x(u) - y(u)|)^m \quad (4.2)$$

For all $x, y \in \mathcal{M}$ and $m \geq 1$. Clearly, $(\mathcal{M}, \rho, \preccurlyeq)$ is a complete partial ordered b-metric space with $s = 2^{m-1}$. Let $\mathcal{F}: \mathcal{M} \to \mathcal{M}$ be a self-mapping define by $\mathcal{F}x(u) = \int_0^1 \mathcal{L}(u, v, x(v))dv$.

Theorem 4.1: *Consider the integral equation* (4.1). *Suppose the following conditions hold:*

1) $\mathcal{L}: [0,1] \times [0,1] \to \mathbb{R} \times \mathbb{R}^+$ satisfies continuous function;
2) $\mu: [0,1] \times [0,1] \to \mathbb{R}^+$ satisfies continuous function such that

$$\int_0^1 \mu(u, v)dv \leq 1, \text{for all } u \in [0,1];$$

3) for all $x, y \in \mathcal{M}$ and for all $(u, v) \in [0,1]^2$, there exists a constant $\beta \in [0,1)$ satisfying

$$\left|\mathcal{L}(u, v, x(v)) - \mathcal{L}(u, v, y(v))\right| \leq \beta^{\frac{1}{m}}\mu(u, v)|x(v) - y(v)| \tag{4.3}$$

Then, (4.1) has a unique solution $x \in \mathcal{M}$.
Proof Let $x, y \in \mathcal{M}$, by the hypothesis of (ii) and (iii) for all $u \in [0,1]$,

$$\rho(\mathcal{F}x(u), \mathcal{F}y(u)) = (|\mathcal{F}x(u) - \mathcal{F}y(u)|)^m$$

$$= \left(\left|\int_0^1 \mathcal{L}(u, v, x(v))dr - \int_0^1 \mathcal{L}(u, v, y(v))dv\right|\right)^m$$

$$= \left(\left|\int_0^1 \left(\mathcal{L}(u, v, x(v)) - \mathcal{L}(u, v, y(v))\right) dv\right|\right)^m$$

$$\leq \left(\int_0^1 \left|\mathcal{L}(u, v, x(v)) - \mathcal{L}(u, v, y(v))\right| dv\right)^m$$

$$\leq \left(\int_0^1 \beta^{\frac{1}{m}}\mu(u, v)|x(v) - y(v)| \, dv\right)^m$$

$$= \left(\int_0^1 \beta^{\frac{1}{m}}\mu(u, v)(|x(v) - y(v)|^m)^{\frac{1}{m}} \, dv\right)^m$$

$$\leq \left(\int_0^1 \beta^{\frac{1}{m}}\mu(u, v)\left(\rho(x(u), y(u))\right)^{\frac{1}{m}} dv\right)^m$$

$$= \beta\rho(x, y)\left(\int_0^1 \mu(u, v) \, dv\right)^m$$

$$\leq \beta\rho(x, y)$$

$$\leq a_{11}\rho(x, y) + a_{12}[\rho(x, \mathcal{F}x) + \rho(x, \mathcal{F}y)] + a_{13}\frac{\rho(x, \mathcal{F}x)\rho(y, \mathcal{F}y)}{\rho(x, y) + \rho(x, \mathcal{F}y) + \rho(y, \mathcal{F}x)} + a_{14}\frac{\rho(x, \mathcal{F}x)\rho(x, \mathcal{F}y) + \rho(y, \mathcal{F}x)\rho(y, \mathcal{F}y)}{\rho(y, \mathcal{F}x) + \rho(x, \mathcal{F}y)}.$$

Thus, all the hypotheses of Corollary 3.6 are satisfied, and therefore it is the solution to (4.1).

5. Conclusion

This study demonstrates the applicability of partially ordered b-metric spaces in establishing the existence and uniqueness of fixed points for rational-type α-admissible contraction mappings. The findings contribute significantly to the understanding of partially ordered metric spaces. Through illustrative examples, we highlighted the practical relevance of the theoretical framework and applied it to demonstrate the existence of a solution to an integral equation. Future research may extend these results to differential equation in b-multiplicative metric spaces, further enriching the scope and applicability of fixed point theory in various mathematical contexts.

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