International Journal of Advanced Mathematical Sciences, 2 (3) (2014) 137-143
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www.sciencepubco.com/index.php/IJAMS
doi: 10.14419/ijams.v2i3.3363
Research Paper

Dynamical analysis of an epidemic model with saturated incidence rate and vaccination

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Abstract

An epidemic model with saturated incidence rate and vaccination is investigated. The model exhibits two equilibria namely disease-free and endemic equilibria. It is shown that if the basic reproduction number (R_0) is less than unity, the disease-free equilibrium is locally asymptotically stable and in such case, the endemic equilibrium does not exist. Also, it is shown that if $R_0 > 1$, the disease is persistent and the unique endemic equilibrium of the system with saturation incidence is locally asymptotically stable. Lyapunov function and Dulac's criterion plus Poincare-Bendixson theorem are applied to prove the global stability of the disease-free and endemic equilibria respectively. The effect of vaccine in the model is critically looked into.

Keywords: *Basic Reproduction Number, Dulac's Criterion, Epidemic Model, Lyapunov Function, Poincare- Bendixson Theorem, Vaccination.*

1. Introduction

Vaccinating susceptible against disease infections is an effective measure to control and prevent the spread of the infection. Kribs-Zaleta and Velasco Hernadez [1] investigated an SIS model with vaccination, standard incidence and no disease-induced diseases. Arino et al. [2] formulated an SIRS epidemic model with vaccination, standard incidence and no disease-induced deaths. Li et al. [3] studied an SIS model with vaccination, standard incidence and disease-induced deaths while Brauer [4] investigated an SIS model with vaccination, general incidence and no disease-induced death. Li and Ma [5], [6] also analyzed global behaviour of simple SIS vaccination epidemic models under the condition that the vaccine is perfectly efficient.

Adebimpe [7] investigated a SEIV epidemic model with saturated incidence rate that incorporates polynomial information on current and past states of the disease. He showed that if the basic reproduction number $R_0 < 1$, the Disease-Free Equilibrium (DFE) is locally asymptotically stable and by the use of Lyapunov function, DFE is globally asymptotically stable and in such a case, the Endemic Equilibrium (EE) is unstable.

Adebimpe et al. [8] investigated the global stability of a SEIR epidemic model with saturating incidence rate. They identified a threshold R_0 which determines the outcome of the disease. They used Dulac's criterion plus Poincare-Bendixson theorem and Lyapunov functions are used to prove the global stability of the disease-free and endemic equilibrium respectively.

Ullah et al. [9] investigated an epidemic model with a vaccination program. They determined the vaccine-induced reproduction number $R_0(\mathrm{k})$ and discussed the impact of vaccination in reducing $R_0(\mathrm{k})$. Islam et al. [10] constructed a new deterministic model and used to analyze the effect of a preventive vaccine on the transmission dynamics of an infectious disease.

In this paper, we extend the work done by Islam et al. [10] to incorporate saturated incidence rate of the form $\frac{\beta SI}{1+m_1 S+m_2 I}$ as below:

$$\begin{aligned}
\frac{dS}{dt} &= \delta - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - \gamma S - \mu S \\
\frac{dV}{dI} &= \gamma S - \mu V - c\beta VI \\
\frac{dI}{dt} &= c\beta VI - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - \mu I
\end{aligned} \tag{1}$$

Where S (t), V (t) and I (t) denote the number of the susceptible individuals, vaccinated individuals and recovered individuals respectively. All of the parameters are positive and have the following meaning: π is the recruitment of individuals (assumed susceptible) into the population, μ is the natural death rate, γ is the rate of vaccination of the susceptible, β is the effective contact rate. Since the vaccine only provides partial protection to the infection, vaccinated individual may still become infected but at the lower rate $c\beta$ than fully susceptible individuals. Here $0 << 1$ and 1-C describes vaccine efficacy, when c=0, the vaccine is perfectly effective and has no effect on the immunity of vaccinated individuals at all when c=1. $\alpha_1 and \alpha_2$ Are the parameters that measure the effects of sociological, psychological or other mechanisms?

2. Local stability of the disease-free equilibrium

The model (1) has a disease-free equilibrium given by $P_0 = \left(\frac{\delta}{\mu+\gamma}, 0, \frac{\gamma\delta}{\mu(\mu+\gamma)}\right)$. It is obvious that P_0 attracts the region (stable manifold of P_0)

$P_0 = \{(S, I, V) \in \varphi : \quad I = 0\}$

The stability of this equilibrium P_0 will be investigated using the linearization method governed by the basic reproduction number, R_0.

Now, by linearizing the system (1) about the point P_0, we have the following:

Let $S = x + S_0, V = y + V_0, I = I$

$$\begin{aligned}
\frac{dx}{dt} &= -\frac{\beta S_0 I}{1+\alpha_1 S_0} - (\gamma+\mu)x + higher\ order\ terms \\
\frac{dy}{dI} &= \gamma x - \mu y - c\beta V_0 I + higher\ order\ terms \\
\frac{dI}{dt} &= c\beta V_0 I + \frac{\beta S_0 I}{1+\alpha_1 S_0} - \mu I + higher\ order\ terms
\end{aligned} \tag{2}$$

The Jacobian matrix is

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dI}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -(\gamma+\mu) & -\frac{\beta S_0}{1+\alpha_1 S_0} & 0 \\ 0 & c\beta V_0 + \frac{\beta S_0}{1+\alpha_1 S_0} - \mu & 0 \\ \gamma & -c\beta V_0 & -\mu \end{pmatrix} \begin{pmatrix} x \\ I \\ y \end{pmatrix} + higher\ order\ terms \tag{3}$$

$$(\gamma+\mu+\lambda)\left[(c\beta V_0 + \frac{\beta S_0}{1+\alpha_1 S_0} - \mu - \lambda)(\mu+\lambda)\right] = 0 \tag{4}$$

$$\lambda_1 = -(\gamma+\mu), \quad \lambda_2 = c\beta V_0 + \frac{\beta S_0}{1+\alpha_1 S_0} - \mu, \quad \lambda_3 = -\mu$$

Since all parameters are assumed positive, so it follows that $\lambda_1 < 0$ *and* $\lambda_3 < 0$. So, the disease-free equilibrium P_0 is locally asymptotically stable if and only if $\lambda_2 = c\beta V_0 + \frac{\beta S_0}{1+\alpha_1 S_0} - \mu < 0$.

Let $R_0 = \frac{\beta\delta}{\mu(\mu+\gamma+\alpha_1\delta)} + \frac{c\beta\gamma\delta}{\mu^2(\mu+\gamma)}$

Lemma: *The disease-free equilibrium* P_0 *, of the model (1) is locally asymptotically stable (CAS) if* $R_0 < 1$, *and unstable if* $R_0 > 1$.

The quality R_0 refers to the average number of secondary cases generated by a single infectious individual in a completely susceptible population. Since λ_1 , λ_3 are negative, if $R_0 < 1$, $\lambda_2 < 0$. Therefore, the disease-free equilibrium is locally asymptotically stable

3. Local stability of the endemic equilibrium

Theorem 1: *If the endemic equilibrium* $P*(S*, I*, V*)$ *of system ( ) exists, then it is locally asymptotically stable*

Proof: According to Theorem (1), the endemic equilibrium $P*(S*, I*, V*)$ exists if and only if $R_0 > 1$. The Jacobian matrix of the system (1) is

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} -(\frac{\beta I*}{1+\alpha_1 S* + \alpha_2 I*} + \gamma + \mu) & 0 & -\frac{\beta S*}{1+\alpha_1 S* + \alpha_2 I*} \\ \gamma & -(c\beta I* + \mu) & 0 \\ \frac{\beta I*}{1+\alpha_1 S* + \alpha_2 I*} & c\beta I* & c\beta V* \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \textit{higher order terms}$$

$$\lambda^3 + (2\mu + 2\gamma + c\beta I* + \frac{\beta I*}{1+\alpha_1 S* + \alpha_2 I*} - \frac{\beta S*}{1+\alpha_1 S* + \alpha_2 I*} - c\beta V*)\lambda^2 + (c^2\beta^2 V* I* + \frac{c\beta^2 I*^2}{1+\alpha_1 S* + \alpha_2 I*} + 2c\beta I*\mu$$

$$c\beta I*\gamma + \frac{\gamma\beta I*}{1+\alpha_1 S* + \alpha_2 I*} + 2\gamma\mu + \gamma^2 + \frac{\mu\beta I*}{1+\alpha_1 S* + \alpha_2 I*} + 2\mu^2 - \frac{c\beta^2 S* I*}{1+\alpha_1 S* + \alpha_2 I*} - \frac{2\beta S*\mu}{1+\alpha_1 S* + \alpha_2 I*} - \frac{\beta S*\gamma}{1+\alpha_1 S* + \alpha_2 I*}$$ We have

$$-c^2\beta^2 V* I* - c\beta V*\mu - \frac{c\beta^2 V* I*}{1+\alpha_1 S* + \alpha_2 I*} - c\beta V*\mu - c\beta V*\gamma)\lambda + \frac{c\beta^2 I*^2\mu}{1+\alpha_1 S* + \alpha_2 I*} + c\beta I*\mu^2 + c\beta I*\mu\gamma + \beta I*\mu^2 + \mu^3$$

$$\mu^2\gamma - \frac{\beta S*\mu^2}{1+\alpha_1 S* + \alpha_2 I*} - \frac{c\beta^2 S* I*^2\mu}{1+\alpha_1 S* + \alpha_2 I*} - \frac{c\beta^2 V* I*\mu}{1+\alpha_1 S* + \alpha_2 I*} - \frac{\beta S*\mu\gamma}{1+\alpha_1 S* + \alpha_2 I*} - c\beta V*\mu^2 - c\beta V*\mu\gamma$$

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

Where

$$a_0 = 1,\ a_1 = 2\mu + 2\gamma + c\beta I* + \frac{\beta I*}{1+\alpha_1 S* + \alpha_2 I*} - \frac{\beta S*}{1+\alpha_1 S* + \alpha_2 I*} - c\beta V*,\ a_2 =$$

$$a_2 = c^2\beta^2 V* I* + \frac{c\beta^2 I*^2}{1+\alpha_1 S* + \alpha_2 I*} + 2c\beta I*\mu c\beta I*\gamma + \frac{\gamma\beta I*}{1+\alpha_1 S* + \alpha_2 I*} + 2\gamma\mu + \gamma^2 + \frac{\mu\beta I*}{1+\alpha_1 S* + \alpha_2 I*} + 2\mu^2 - \frac{c\beta^2 S* I*}{1+\alpha_1 S* + \alpha_2 I*}$$

$$- \frac{2\beta S*\mu}{1+\alpha_1 S* + \alpha_2 I*} - \frac{\beta S*\gamma}{1+\alpha_1 S* + \alpha_2 I*} - c^2\beta^2 V* I* - c\beta V*\mu - \frac{c\beta^2 V* I*}{1+\alpha_1 S* + \alpha_2 I*} - c\beta V*\mu - c\beta V*\gamma$$

$a_3 = \frac{c\beta^2 I_*^2 \mu}{1+\alpha_1 S_* + \alpha_2 I_*} + c\beta I_* \mu^2 + c\beta I_* \mu\gamma + \beta I_* \mu^2 + \mu^3 \mu^2 \gamma - \frac{\beta S_* \mu^2}{1+\alpha_1 S_* + \alpha_2 I_*} - \frac{c\beta^2 S_* I_*^2 \mu}{1+\alpha_1 S_* + \alpha_2 I_*} - \frac{c\beta^2 V_* I_* \mu}{1+\alpha_1 S_* + \alpha_2 I_*}$ If $a_1 a_2 > a_0 a_3$, by $- \frac{\beta S_* \mu\gamma}{1+\alpha_1 S_* + \alpha_2 I_*} - c\beta V_* \mu^2 - c\beta V_* \mu\gamma$

Routh Hurwitz criterion, the endemic equilibrium is locally asymptotically stable.

4. Global stability of disease-free equilibrium

In order to prove the global stability of the endemic equilibrium P_* of the equation (1) we apply Dulac's criterion plus Poincare-Bendixson Theorem.

Theorem 2: *(Dulac's Criterion)*

Consider the following general nonlinear autonomous system of de

$x(t) = f(x), x \in E$ (*)

Let $f = C^1(E)$ where E is a simple connected region in R^2. If the exists a function it $H \in C^1(E)$ such that $\nabla.(Hf)$ is not identically zero and does not change sign in E, the system (*) has no close orbit lying entirely in E. if A is an annular region contained in E on which $\nabla.(Hf)$ does not change sign, then there is at most one limit cycle of the system (*) in A.

Theorem 3:*(The Poincare-Bendixson Theorem):Suppose that $f \in C^1(E)$ where E is an open subset of R^n and that the system (*) has a rejecting Γ contained in a compact subset f of E. assume that the system (*) has only one unique equilibrium point x_0 in f, then one of the following possibilities holds.*

a) $w(\Gamma)$ is the equilibrium point x_0
b) $w(\Gamma)$ is a periodic orbit
c) $w(\Gamma)$ is a graphic

Theorem 4: *Let P_* be the unique positive equilibrium point of the system (1). If R_0>1, then P_* of the system (*) is globally asymptotically stable.*

Proof: We use Dulac's criterion plus Poincare'-Bendixson Theorem to analyze the system (1). Consider.

$H(S,V,I) = \frac{1}{SVI}$

Where S>0, V>0, I>0. Then,

$\nabla.(Hf) = \frac{\partial}{\partial S}(H.f_1) + \frac{\partial}{\partial V}(H.f_2) + \frac{\partial}{\partial I}(H.f_3)$

$= \frac{\partial}{\partial S}[(\frac{1}{SVI}(\delta - \frac{\beta SI}{1+\alpha_1 S + \alpha_2 I} - \gamma S - \mu S) + \frac{\partial}{\partial V}[(\frac{1}{SVI}(\gamma S - \mu V - c\beta VI) + \frac{\partial}{\partial I}[(\frac{1}{SVI}(c\beta VI + \frac{\beta SI}{1+\alpha_1 S + \alpha_2 I} - \gamma I)]$

$= -\frac{\delta}{S^2 VI} + \frac{\beta\alpha_1}{(1+\alpha_1 S + \alpha_2 I)^2} - \frac{\gamma}{V^2 I} - \frac{\beta\alpha_2}{(1+\alpha_1 S + \alpha_2 I)^2}$

If $\alpha_1 = \alpha_2$, $\nabla.(Hf) < 0$. Hence, by the Dulac's criterion, there is no closed orbit in the first quadrant. Therefore, the endemic equilibrium is globally asymptotically stable.

5. Numerical simulations

In this section, we perform numerical calculations to support our theoretical analysis of this paper. Below are the graphs emanated from the numerical computations?

6. Discussion of results

Simulation was carried out with different values of the parameters and stability analysis and values of the threshold were obtained.

From figures 1-4 above, $R_0 < 1$ and different values of c were used. It was discovered that as c increases, the more stability the models become. This means that the vaccine has a lot to do with the eradication of the disease.

From figure 5, we have $R_0 > 1$ and different values were used and it was discovered that there is endemic and that means that disease may break out and become endemic.

We established that when $R_0 < 1$, the disease may die out at the long run and when $R_0 > 1$, there may be endemic.

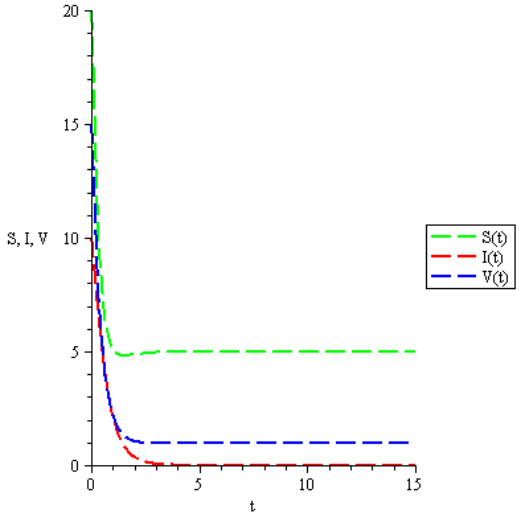

Fig. 1: Graph Of S (T), I (T) And V (T) Against Time (T) When δ=15, β =0.5, α_1 =0.5, α_2 = 0.3, γ =0.5, μ = 2.5, c =0.1 and R_0 <1

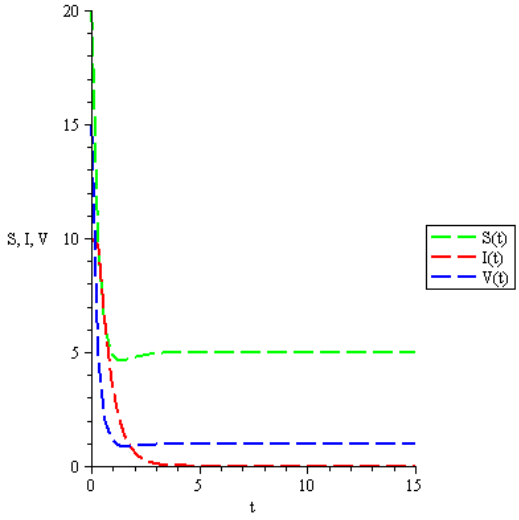

Fig. 2: Graph of S (T), I (T) and V (T) Against Time (T) When δ=15, β =0.5, α_1 =0.5, α_2 = 0.3, γ =0.5, μ = 2.5, c =0.4 and R_0 <1

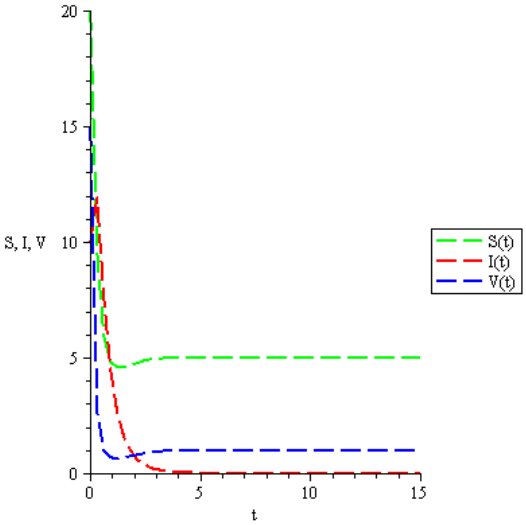

Fig. 3: Graph of S (T), I (T) And V (T) Against Time (T) When δ=15, β =0.5, α_1 =0.5, α_2 = 0.3, γ =0.5, μ = 2.5, c =0.7 and R_0 <1

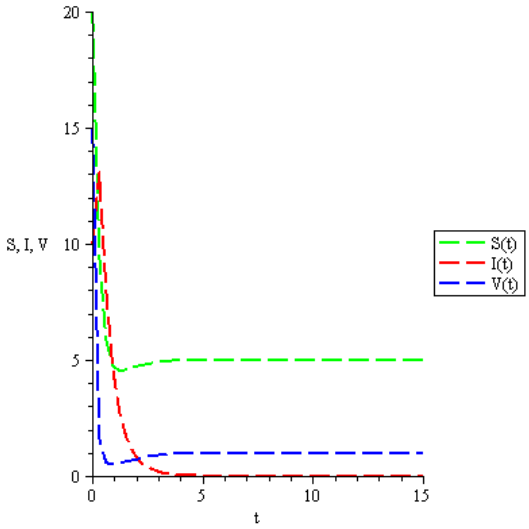

Fig. 4: Graph of S (T), I (T) and V (T) Against Time (T) when δ=15, β =0.5, α_1 =0.5, α_2 = 0.3, γ =0.5, μ = 2.5, c =0.95 and R_0 <1

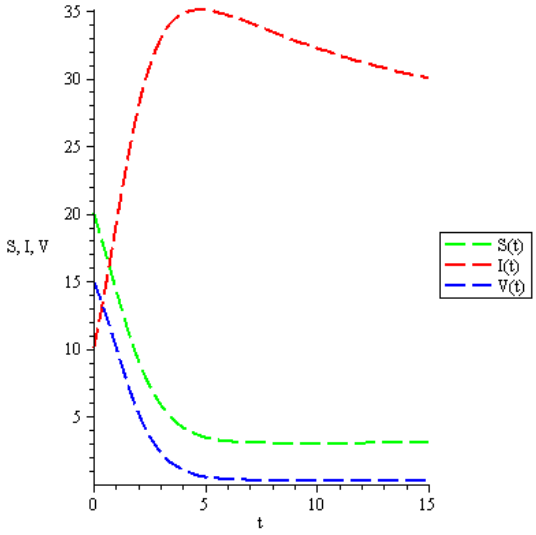

Fig. 5: Graph of S (T), I (T) and V (T) Against Time (T) when δ=15, β =0.3, α_1 =0.5, α_2 = 0.3, γ =0.1, μ = 0.1, c =0.1 and R_0 >1

7. Conclusion

In this paper, an SVI deterministic model with saturating incidence rate is investigated. Some of the main findings of this study are:

i) The model has a globally-asymptotically stable disease-free equilibrium whenever the associated reproduction number is less than unity;

ii) The model has a unique endemic equilibrium under certain conditions. Under these conditions, the endemic equilibrium is locally-asymptotically stable whenever the associated reproduction number exceeds unity. The endemic equilibrium is shown to be globally-asymptotically stable using Dulac's criterion plus Poincare's – Bendixson theorem.

iii) Using Dulac's criterion plus Poincare's –Bendixson theorem and $\alpha_1 = \alpha_2$, the model is globally asymptotically stable.

iv) Numerical Simulations illustrate that the parameter 1-c which describes the vaccine efficacy has a lot to do in disease eradication. As c increases when R_0 <1, there is stability.

This study shows that the disease being considered can be eliminate from the population whenever the basic reproduction number R_0 is less than unity. The disease persists in the community whenever the basic reproduction number R_0 exceeds unity.

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