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Research paper

Finding the best efficient solution for multi objectives programming problems based on the distance of objectives

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Abstract

Multi-objective programming is one of the most famous branches of operations research. Several traditional and artificial methods have been used to solve multi-objective problems in different fields. Most of these methods give a set of efficient solutions rather than an optimal solution because the objective functions are conflicting in nature. This leads to different individual solutions, or no one solution can be available for all objective functions. Therefore, they must reconcile. In such a situation, the best way is needed to find a feasible solution that is optimal for all objectives. In other words, it is the best or preferred solution that is considered the closest to the utopian point.
Despite the variety of applied methods, there is not one universal method for solving multiobjective optimization problems. Nevertheless, this paper introduces some new general mathematical models to find the best efficient solution for multi-objective programming problems. They depend on minimizing the distance of objectives from the utopian point for accurate as well as computationally fast approaches. Of course, by doing so, the required solution is directly obtained. Additionally, some illustrative numerical linear and non-linear examples demonstrate the computational details. The results are compared with the existing solutions in other researches. All results conclude that the proposed methods are very important for decision making and they can be used in a variety of problems having multiple objectives in real life. One can say that these methods are very simple to give useful insights into practical problems. Finally, this work strives to provide the best solution with stable steps, lowest time processing, flexibility, and applicability.

Keywords: *Multi-objective Optimization Problems; Optimal Solution; Utopian (Ideal) Point; Distance of Objectives; Best Efficient Solution.*

1. Introduction

Optimization is an important tool in operations research. It is concerned with minimization or maximization of a function (or functions) subject to constraints on its variable(s). The process of identifying and formulating objective(s), variable(s), and constraint(s) for a given problem is known as modeling. An appropriate model is constructed to find the problem's solution. Accordingly, there are a great many applications that can be formulated as minimization vs. maximization, local optimization vs. global optimization, single objective vs. multi-objective, unconstrained vs. constrained, linear vs. non-linear, derivative-free vs. with derivatives, static vs. dynamic, discrete (integer) vs. continuous (real) or deterministic vs. uncertain (stochastic- fuzzy- ruff) optimization problems. Figure 1 shows a diagram of optimization model classifications.
In real-world optimization problems, many problems are multi-objective (vector, multi-criteria, or multi-performance) programming. Most of these problems involve multiple conflicting objectives which should be considered simultaneously. Therefore, they called multi-objective optimization problems. They provide a set of efficient (non-dominated, Pareto-optimal, compromise, trade-off, or non-inferior) solutions. These solutions represent the optimal solutions of all objectives, and after that, the preferred or best solution is extracted. Some shapes of conflicting objectives are shown in figure 2.
It is known that the single objective optimization problem has one space for both the objective and constraint(s). On the other side, there are two spaces for the multi-objective optimization problem: The n-dimensional space of the decision variables and the k-dimensional space of objective functions. When solving such a problem, there is a mapping from the decision space into the objective space. For every solution in the decision space, there is a point in the objective space.
In addition, there is an important relationship between the optimum value of the conflicting objectives problem and its best value. In the minimum (maximum) case, the optimum value of any multi-objective programming problem (without weights) is always less (more) than or equal to the corresponding optimum value of the solved problem to get the best efficient solution. But, both the optimum value of

the original multi-objective optimization problem and its optimum value of the best solution are equal in the non-conflicting objectives optimization problem [1], [2].

Many various approaches in the literature are used to solve multi-objective problems, which may be mainly divided into classic methods (no preference methods, posteriori methods, priori methods, and interactive methods) and heuristic methods.

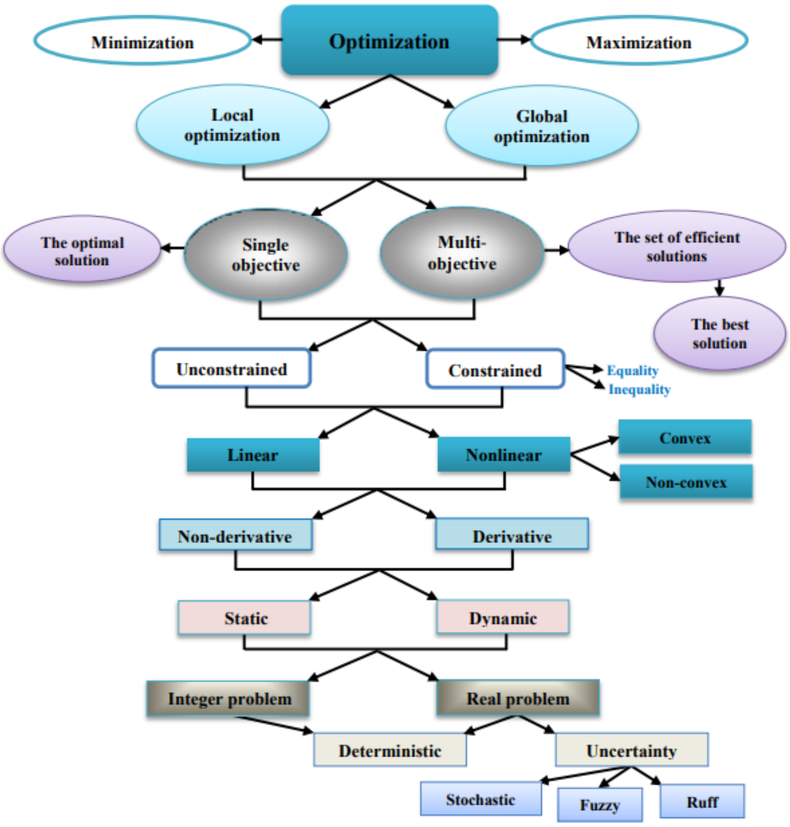

Fig. 1: A Diagram of Optimization Model Classification.

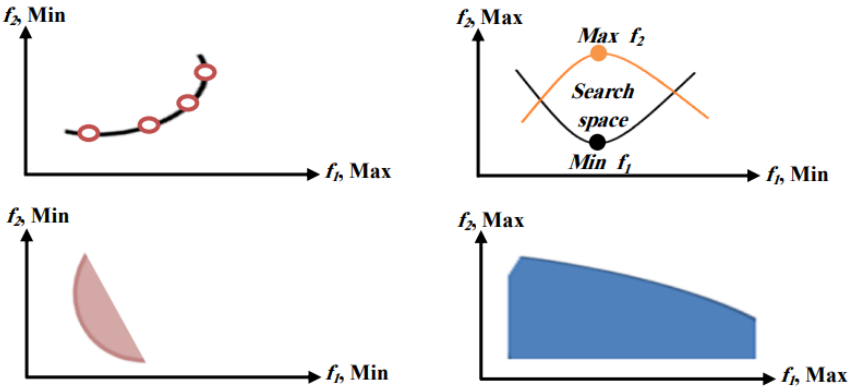

Fig. 2: Some Shapes of Conflicting Objectives.

In figure 3, the general classification approaches in multi-objective optimization problems are considered according to different criteria. Most of the classic methods convert multi-objective into single-objective models such as the weighted sum method, ε-constraints method, hybrid method, weighted metric method, global criterion approach, and goal programming. However, goal programming has a different treatment from the multi-objective, where the decision maker determines the relevant preferences before beginning to solve the problem.

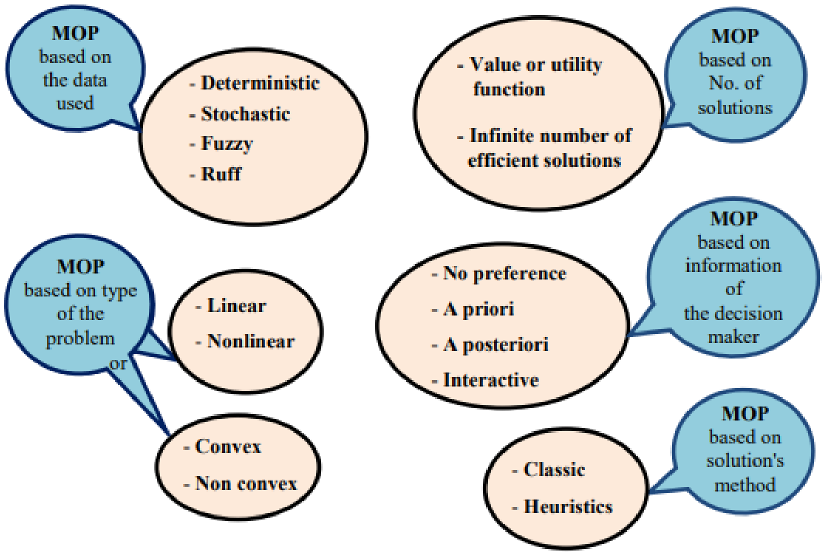

Fig. 3: A Summary of General Classification Approaches in Multi-objective Optimization Problems.

On the other hand, heuristic optimization algorithms are artificial intelligence search methods that can be utilized to find optimal decisions in various complex systems. They include genetic algorithms, neural networks, particle swarm, simulated annealing, tabu search, ant colony, bee colony, bacterial foraging optimization, harmony search, artificial immune systems, corona algorithm, and other related topics. These approaches provide a discrete picture of the Pareto front in the objectives space [2 - 16].

Some of these methods have been utilized to achieve the preferred or best solution, such as Petr [14], Ahmad Abubaker et al. [17], Majid et al. [18], Murshid et al. [11], Josip et al. [7], and Alia [1], [2], [19], [20]. Several existing methods in the literature involve complex computational procedures that are difficult to understand. Thus, to improve the performance of such approaches easily, this research can significantly provide some new optimization methods. These methods mainly focus on finding the best compromise solution for multi-objective programming problems based on the weighted Euclidean distance of objectives from the ideal point of objective functions. Their performances are compared to the previous works. The results show that the proposed methods present the best solution directly in general.

The rest of this paper is organized as follows: In section 2, some basic definitions of multi-objective optimization programming are presented. In section 3, the proposed mathematical models are introduced, defined, and discussed. In section 4, some illustrative numerical examples are provided to clarify the idea of the proposed methods. In section 5, conclusions and future work are given.

2. Basic definitions related to multi-objective optimization programming

In this section, some basic definitions that related to the topic are presented as follows:

2.1. Multi-objective optimization problem

A multi-objective optimization problem (MOP) can be formulated as follows:

(MOP): Minimize $F(x) = (f_1(x), f_2(x), \dots, f_k(x)),\ k \geq 2,$

$$\text{Subject to } M = \{x \in R^n / g_r(x) \leq 0,\ r = 1, 2, \dots, m\}. \tag{1}$$

Where:
$F(x) = (f_1(x), f_2(x), \dots, f_k(x))$ is a vector of k objective functions, and k is used to identify the number of objective functions.
$g_r(x)$ for all $r = 1, 2, \dots, m$ is a set of constraints.
x is an n-vector of decision variables.
The set M is a non-empty and feasible region included in R^n that is determined by the constraints on the multi-objective problem.
Assume that

$f_i(x^*) = \text{Minimize: } f_i(x),\ i = 1, 2, \dots, k,$

$$\text{Subject to: } x \in M. \tag{2}$$

The goals of a multi-objective optimization problem can be summarized as follows:
1) Finding a set of solutions as close as possible to the Pareto-optimal front.
2) Finding a set of solutions as diverse as possible feasible objective space [9].
3) Finding the best feasible solution from the Pareto-optimal set.

The following figure 4 shows a graphical representation of these goals of a multi-objective optimization problem.

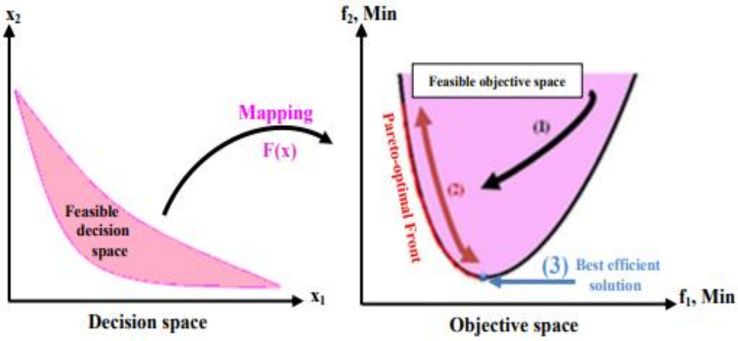

Fig. 4: A Graphical Representation of the Goals of A Multi-objective Optimization Problem.

2.2. Optimal solution

A feasible solution that achieves the minimum (or maximum) value of an objective function with constraint(s) is called an optimal solution.

2.3. Efficient solution

A decision vector $x^* \in M$ is said to be an efficient solution, if there is not exist another decision vector $x \in M$ such that $f_i(x) \leq f_i(x^*)$ for $i = 1, 2, \dots, k$ and $f_j(x) < f_j(x^*)$ for at least one index j [8].

2.4. Weak efficient solution

A point $x^* \in M$ is weakly efficient solution, if there is not exist another decision vector $x \in M$ such that $f_i(x) < f_i(x^*)$ for all $i = 1, 2, \dots, k$ [4], [8].

2.5. Strong efficient solution

A point $x^* \in M$ is strongly efficient solution, if there is not exist another point $x \in M$ such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, 2, \dots, k$ and for at least one value of i, $f_i(x) < f_i(x^*)$ [4].
The following figure shows strong and weak efficient solutions in a bi-objective optimization problem.

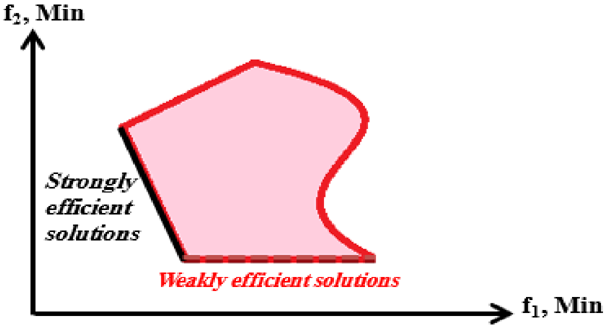

Fig. 5: Strongly and Weakly Efficient Solutions for A Bi-objective Optimization Problem.

2.6. Utopian (Ideal) point

The point $(f_1(x_1^*), f_2(x_2^*), \dots, f_k(x_k^*))$ in the objective space is called utopian (ideal) point [19].

2.7. Euclidean distance of objectives

Euclidean distance of objectives in the k-dimension space is calculated by the following formula:

$$\sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}\,. \tag{3}$$

2.8. Pareto-optimal front

In a multi-objective optimization problem, the Pareto-optimal front (Pareto curve, trade-off surface or efficient frontier) is the set of all the non-dominated solutions [1], [19].

2.9. Best efficient solution

The best compromise solution on the efficient front is a feasible solution that has the shortest distance to the utopian point [19]. The following figure shows a diagram of the best solution for a multi-objective optimization problem.

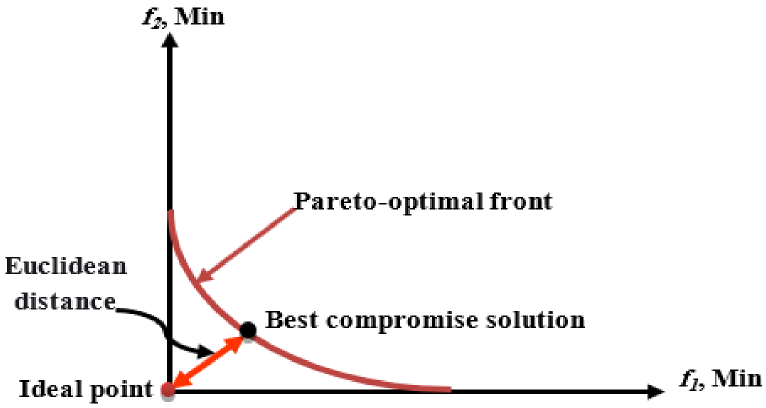

Fig. 6: A Representation of the Best Solution with Minimum Euclidean Distance for A Multi-objective Optimization Problem.

2.10. The sum weighted method

This method combines all objectives together using different weighting coefficients for each objective. This means that the multi-objective problem is transformed into a single objective optimization problem. Its form is as follows:

$$\text{Min: } \sum_{i=1}^{k} w_i\, f_i(x). \tag{4}$$

Where: $w_i \geq 0$, i=1, 2, …, k are the weighting coefficients representing the relative importance of the objectives. It is usually assumed that

$$\sum_{i=1}^{k} w_i = 1. \tag{5}$$

In this case, it is possible to find various efficient solutions instead of an optimal solution for such a problem [4], [8], [11].

3. The proposed mathematical methods

In this section, a brief description of the proposed methods is presented. The main idea is to use the weighted Euclidean distance from the ideal point for optimizing a multi-objective problem. These methods are designed for dealing with linear and nonlinear multi-objective optimization problems. They can be considered an improvement on the modified Alia's methods [20] to get the best efficient solution. However, there are two differences: the first one is the way for scalarizing a multi-objective optimization problem, which depends basically on minimizing the weighted Euclidean distance of objectives from the utopian point only. Secondly, the search direction is not calculated. It is noteworthy that all proposed methods use the weighted sum method to ensure the accuracy of the solution. Therefore, these new methods have certain advantages over the other multi-objective optimization methods. Relatively, the best point can be found directly instead of achieving the mentioned goals through several stages. In addition, it has short computation times involved with the problem at hand.
This study assumes that the optimization problem is a minimization type. To streamline calculations, a maximization problem can be converted to a minimization optimization type.
These proposed methods are described as follows:

3.1. The steps of the proposed methods

These methods consist of four steps. The general description of the presented steps is as follows:

1) Each function with constraint (s) (if any) of the multi-objective problem is calculated to get its optimal solution and then the utopian point.
2) The multi-objective problem is transformed into a single objective by minimizing the weighted Euclidean distance of objectives from the utopian point (using any one of the three proposed methods) with equal weights (or without weights) at the first solution process.
3) An exact solver is used to obtain an efficient solution.
4) If the best point is found, stop the algorithm. Otherwise, change the values of a weighting vector and go to step 3.

3.2. Mathematical methods formulation

In this section, the proposed methods for solving a multi-objective programming problem can be formulated as follows:

3.2.1. The first distance method (FDM) formulation

$$\text{(FDM): Minimize } F(x) = \sqrt{\sum_{i=1}^{k} w_i (f_i(x) - f_i^*)^2} + Ɗ,\ k \geq 2,$$

Subject to:

$$\sqrt{\sum_{i=1}^{k} \overline{w}_i (f_i(x) - f_i^*)^2} - Ɗ \leq 0,$$

$M = \{x \in R^n / g_r(x) \leq 0, r = 1, 2, \dots, m\}$. (6)

Where:
$F(x) = (f_1(x), f_2(x), \dots, f_k(x))$ is a vector of k objective functions, and k is used to identify the number of objective functions.
$x = (x_1, x_2, \dots, x_n)^T$ is a vector of the decision variables.
n is a number of the decision variables.
$w_1, w_2, \dots, w_k$ are weights of the objectives $f_i(x)$ in the objective function, $w_i > 0$, i= 1, 2, ... , k, $\sum_{i=1}^{k} w_i = 1$.
$\overline{w}_1, \overline{w}_2, \dots, \overline{w}_k$ are weights of the objectives $f_i(x)$ in the constraints, $\overline{w}_i > 0$, i= 1, 2, ... , k, $\sum_{i=1}^{k} \overline{w}_i = 1$.
f_i^*, i = 1, 2, 3, …, k is the individual optimal of the objectives.
Ɗ (variable) is the distance of objectives.
M is the feasible space.
$g_r(x)$ for all r = 1, 2, ... , m is a set of constraints.

Theorem 1: *A solution x* ∈ M is an efficient solution for (MOP) if and only if x* is an optimal solution of (FDM).*
Proof:
This proof can be consisting of the following two parts:
a) If the optimal solution of (FDM) is unique, then solution x* ∈ M is the efficient solution for (MOP),
b) If the optimal solution of (FDM) is not unique, then there exists one that is an efficient solution for (MOP).
To proof (a), let x* be an optimal solution of the problem (FDM) such that:

$$\sqrt{\sum_{i=1}^{k} \overline{w}_i (f_i^*(x^*) - f_i^*)^2} - Ɗ = 0. \quad (7)$$

Then, this equality must hold. If it does not hold, say if

$$\sqrt{\sum_{i=1}^{k} \overline{w}_i (f_i^*(x^*) - f_i^*)^2} - Ɗ < 0. \quad (8)$$

Then there exists another solution $\overline{x} \in M$, which is also an optimal point of (FDM). That contradicts the assumption of uniqueness. Therefore, x* is the efficient point of (MOP).
To proof (b), let x* be an optimal solution of (FDM) and M is the set of efficient solutions for (MOP). If the equality (7) holds for each point ∈ M, then x^* is an efficient solution of (MOP). Suppose that x^* is not efficient. This means that there exists another point y ∈ M and it does not satisfy the equality (7). Then, $f_i(y) < f_i(x^*)$, i= 1, 2, … , k. Accordingly,

$$\sqrt{\sum_{i=1}^{k} \overline{w}_i (f_i(y) - f_i^*)^2} < \sqrt{\sum_{i=1}^{k} \overline{w}_i (f_i(x^*) - f_i^*)^2}. \quad (9)$$

This is a contradiction to the assumption that x* is an optimal solution of (FDM). Thus, x* has to be an efficient solution of (MOP). This completes the proof.

Theorem 2: *For equal weights of objectives or some of them ($w_i > 0$), a solution $x^* \in M$ is said to be the best efficient solution in the objective space of a multi-objective programming problem if and only if there is no vector $x \in M$ with the characteristics: $f_i(x) \leq f_i(x^*)$, $\forall i = 1, 2, \dots, k$, and $f_i(x) < f_i(x^*)$ for at least one i.*

Proof:
Assume the solution x^* is not best efficient in the objective space of a multi-objective programming problem. That is, there exists another feasible solution x in the objective space, which achieves the following inequality:
$f_i(x) \leq f_i(x^*)$, $\forall i = 1, 2, \dots, k$. (10)
This inequality means that either:
1) x^* is dominated by x:

$f_i(x) < f_i(x^*)$, $\forall i = 1, 2, \dots, k$, or (11)

2) It is equal to x:

$f_i(x) = f_i(x^*)$, $\forall i = 1, 2, \dots, k$. (12)

In other words, x^* is either a non-efficient or non-unique efficient solution. This conclusion is contrary to the promise, therefore x^* has a minimum distance from the utopian point. Thus, the theorem is proved.

3.2.2. The second distance method (SDM) formulation

(SDM): Minimize $F(x) = \sqrt{\sum_{i=1}^{k} w_i (f_i(x) - f_i^*)^2}$, $k \geq 2$,

Subject to:

$M = \{x \in R^n / g_r(x) \leq 0, r = 1, 2, \dots, m\}$,

$w_i > 0$, $\sum_{i=1}^{k} w_i = 1$. (13- a)

This second formulation is considered as shortcut to the first formulation.

Theorem 3: *A solution $x^* \in M$ is an efficient solution for (MOP) if and only if x^* is an optimal solution of (SDM).*

Proof:
Based on optimization theory for (SDM) and the contradiction that x^* cannot be an efficient solution of (MOP), the proof is obtained.

Theorem 4: *For equal weights of objectives or some of them ($w_i > 0$), a solution $x^* \in M$ is the best efficient solution for (MOP) if and only if x^* is an optimal solution of (SDM).*

Proof:
Since the objective function optimizes the weighting Euclidian distance of objectives from the utopian point, then based on optimization theory, the theorem is proved.
Note that:
In solving some multi-objective optimization programming problems, to ensure an accurate best solution is obtained by (SDM), Ɗ is added in its objective function. Thus, the formulation becomes as follows:

(SDM): Minimize $F(x) = \sqrt{\sum_{i=1}^{k} w_i(f_i(x) - f_i^*)^2} + Ɗ$, $k \geq 2$,

Subject to: $M = \{x \in R^n / g_r(x) \leq 0, r = 1, 2, \dots, m\}$,

$w_i > 0, \sum_{i=1}^{k} w_i$. (13- b)

To explain, the variable $Ɗ \in R$ is a co-factor in this model. Of course, its value is always equal to zero.

3.2.3. The third distance method (TDM) formulation

(TDM): Minimize Ɗ,
Subject to:

$$\sqrt{\sum_{i=1}^{k} w_i(f_i(x) - f_i^*)^2} - Ɗ \leq B, \quad k \geq 2,$$

$M = \{x \in R^n / g_r(x) \leq 0, r = 1, 2, \dots, m\}$,

$w_i > 0, \sum_{i=1}^{k} w_i$. (14)

Where:

$Ɗ \in R$ is the deviational variable for the distance = $\sqrt{\sum_{i=1}^{k} w_i(f_i(x) - f_i^*)^2}$.
B (Constant) is zero value or/ and the negative of the absolute optimum value of the total objective functions (without weights).

Theorem 5: *A solution $x^* \in M$ is an efficient solution for (MOP) if and only if x^* is an optimal solution of (TDM).*

Proof:
Based on the contrary, supposing that x^* is not an efficient solution of (MOP), and the optimization theory for (TDM), the proof is obtained.

Theorem 6: *A solution $x^* \in M$ is the best efficient solution of (MOP) for equal weights of objectives or some of them ($0 < w \in R^k$), if and only if x^* is an optimal solution of (TDM).*

Proof:
Let $x^* \in M$ be the best efficient solution of (MOP) for equal weighting vector or some of them ($0 < w \in R^k$). Assume that it is not the best solution. In this case, there exists a point $\bar{x} \in M$ such that:
$f_i(\bar{x}) \leq f_i(x^*)$ for all i= 1, …, k. It means that:

$$\sqrt{\sum_{i=1}^{k} w_i(f_i(\bar{\boldsymbol{x}}) - f_i^*)^2} \leq \sqrt{\sum_{i=1}^{k} w_i(f_i(\boldsymbol{x}^*) - f_i^*)^2} \leq B + \mathcal{D} \text{ for all i.} \tag{15}$$

Thus, x^* cannot be the best efficient solution to the problem (TDM). Here, this contradiction completes the proof.
Corollary:
All solutions of the proposed methods are strong Pareto-optimal solutions because their distances can be near to the distance of the best solution.
Remarks:
1) Multi-objective optimization problems usually conflict with each other.
2) The best compromise solution is a single optimal solution that achieves all objectives with minimum distance from the ideal point.

3) The proposed methods do not need to calculate the search direction before using a software solver. Only the individual optimum for each objective is calculated. Besides, the optimal of total objectives (without weights) is calculated in the third method.
4) For mixed problems (min-max), all the objectives are converted into min type by minimizing their negative maximum.
5) The presented theorems reveal that when the best point is given for the problem (MOP), it's considered an optimal solution for any of the proposed methods.
6) The term without weights means that: $w_1, w_2, \dots, w_k$= 1.
7) It is worth noting that the second and third formulations can be considered as shortcuts to the first formulation. Generally, each of the proposed methods may be considered a universal optimization tool to solve multi-objective problems.

3.3. The salient features of the proposed methods

The salient features of the proposed methods can be presented as follows:

1) These methods can handle any kind of objective function and any kind of constraint (e.g. linear and/or nonlinear) defined on discrete, continuous, or mixed searching space for multi-objective optimization problems. Therefore, they are considered powerful tools with rapid time for solving such problems.
2) These methods save the operation time, because instead of calculating the distance after an efficient solution has been obtained by any other method for optimizing a multi-objective problem; they obtain an accurate efficient solution and its distance at once.
3) The resulting best point by these methods can easily be indicated in most problems without weights of objectives or when the weights are equal ($w_i > 0$).
4) The solution of the proposed methods is optimal, and at the same time, it is considered the efficient solution for the given multi-objective programming problem.
5) (FDM), (SDM), and (TDM) reformulate (MOP) and solve the formulated problem with any nonlinear solver even if (MOP) is linear.
6) Throughout this work, a robust set of efficient solutions closest to the utopian point is obtained. Therefore, it helps the decision-maker to make a better and more reliable decision.
7) In the first and third methods, the value of Ɗ is equal to the Euclidean distance of objectives when the objectives have equal weights or without using weights, and B= 0.
8) On solving most multi-objective problems by the proposed methods, the best point is achieved when the weights of objectives ($w_i > 0$) are equal, and B= 0.
9) On solving some two objectives problems using the first proposed method, the total weights of every objective in both the objective function and the distance constraint are equal to one. However, the value of D is equal to the distance of objectives when the equal weights = w_i = 1 (or without weighting) with B= 0.
10) If B= the total of objectives in the third method, the resulting distance of objectives =
((The resulted D) – | the total of objectives "B" |) without weighting of objectives or with equal weights = 1.
11) The value of B in (TDM) may be a positive value of the total objectives (without weights) that are near zero for some multi-objective optimization programming problems.
12) The weighted sum method is used in these new formulations to give an accurate best solution for a problem at hand, where the second and third methods produce the best point without weighting of objectives in most optimization problems directly.
13) In the first method (FDM), the weights of objective functions may be equal or different from the weights in the distance constraint ($w_i > 0$). But, every group of weights must be equal.
14) When the number of variables increases in the linear multi-objective problem, the first model (FDM) may provide its best solution with better accuracy than the other two models. Slightly, in the non-linear multi-objective problems, the third proposed method (TDM) gives an accurate best point more than (FDM) and (SDM), because they contain more mathematics in the objective function.

Note that:

1) In the modified Alia method for studying the general convex multi-objective programming problem (AP1) [20], the distance constraint can be replaced by:

$$\sqrt{\sum_{i=1}^{k}(f_i - f_i^*)^2} - h \leq 0, \; i = 1, 2, 3, ..., k. \tag{16}$$

However, the proposed methods are more simple and easier than the modified Alia methods to obtain the accurate best point.
2) If the number or power degree of variables increases, it is necessary to apply the weighted sum method for objective functions.
3) The right-hand side of the distance constraint "B" in the first model (FDM) can equal zero value or/ and the negative of absolute optimum value of the total objective functions (without weights).
4) To facilitate the work of these proposed methods, the maximum objective functions problem can be multiplied by minus one to become the minimum objective functions problem.
5) To correct the error still found in the LINGO software when the resulting solution of a problem is negative, you can multiply the objective function(s) as well as the resulting solution by minus one.
6) There is no doubt that the distance of objectives from the ideal point for the conflicting objectives is greater than zero, but for the non-conflicting objectives is zero.

4. Numerical examples

To illustrate the formulation and solution procedure of the proposed methods, some numerical examples are presented in this section. They are solved using LINGO 14.0 software.

Example (1):
This example described in paper by P. & Bharti [21], Mustafa et al [22], and Alia [1], where the linear multi-objective problem has the following form:

Minimize: $f_1 = (x_1 - 2x_2)$,

$f_2 = (-2x_1 - x_2)$,

Subject to:

$-x_1 + 3x_2 \leq 21$,

$x_1 + 3x_2 \leq 27$,

$4x_1 + 3x_2 \leq 45$,

$3x_1 + x_2 \leq 30$,

$x_1, x_2 \geq 0$.

This problem consists of two minimized objectives, two decision variables, and four constraints with non-negative feasibility condition of the variables. Now, by solving each objective separately for obtaining the ideal objective values, they are found to be $f_1^*(x_1^* = 0, x_2^* = 7) = -14$ and $f_2^*(x_1^* = 9, x_2^* = 3) = -21$. Mathematically, the above problem can be formulated using the (FDM) as follows:

Minimize: $\sqrt{w_{11}(x_1 - 2x_2 + 14)^2 + w_{12}(-2x_1 - x_2 + 21)^2} + Ɗ$,

Subject to:

$\sqrt{w_{21}(x_1 - 2x_2 + 14)^2 + w_{22}(-2x_1 - x_2 + 21)^2} - Ɗ \leq 0$,

$-x_1 + 3x_2 \leq 21$,

$x_1 + 3x_2 \leq 27$,

$4x_1 + 3x_2 \leq 45$,

$3x_1 + x_2 \leq 30$,

$x_1, x_2 \geq 0$, $w_{11}, w_{12}, w_{21}, w_{22} > 0$, $w_{11} + w_{12} = w_{21} + w_{22} = w_{11} + w_{21} = w_{12} + w_{22} = 1$, $w_{11} = w_{22}$, $w_{12} = w_{21}$, or $w_{11} = w_{12}$, $w_{21} = w_{22}$.

The formulation of (SDM) is as follows:

Minimize: $\sqrt{w_1(x_1 - 2x_2 + 14)^2 + w_2(-2x_1 - x_2 + 21)^2}$,

Subject to:

$-x_1 + 3x_2 \leq 21$,

$x_1 + 3x_2 \leq 27$,

$4x_1 + 3x_2 \leq 45$,

$3x_1 + x_2 \leq 30$,

$x_1, x_2 \geq 0$, $w_1 = w_2$, $w_1, w_2 > 0$.

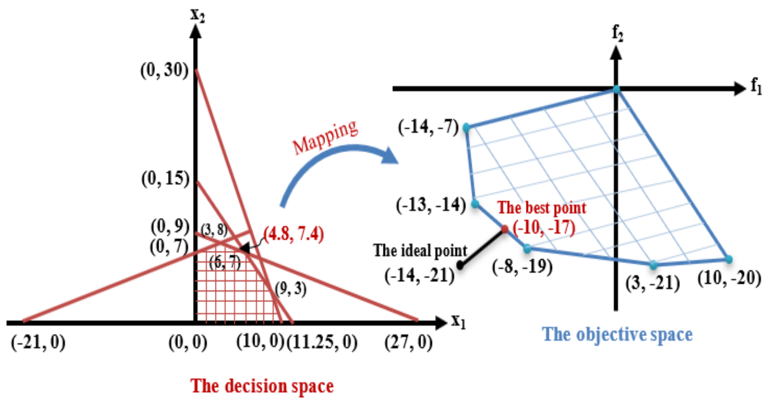

Fig. 7: The Best Point of Example (1).

Since the optimum value of total objectives (without weights) is –27, the formulation of (TDM) is as follows:
Minimize: Ɗ,
Subject to:

$\sqrt{w_1(x_1 - 2x_2 + 14)^2 + w_2(-2x_1 - x_2 + 21)^2} - Ɗ \leq -27$ or 0,

$-x_1 + 3x_2 \leq 21$,

$x_1 + 3x_2 \leq 27$,

$4x_1 + 3x_2 \leq 45$,

$3x_1 + x_2 \leq 30$,

$x_1, x_2 \geq 0$, $w_1 = w_2$, $w_1, w_2 > 0$.

As shown in figure 7, the best efficient point is: $x_1^* = 4.8$, $x_2^* = 7.4$, $f_1^* = -10$, $f_2^* = -17$. The first constraint = C_1 is 17.4, the second constraint = C_2 is 27, the third constraint = C_3 is 41.4 and the fourth constraint = C_4 is 21.8. The standard Euclidean distance from the utopian point is 5.65685. That is achieved when the value of B= 0 or –27 without weights or with equal weights, $w_i > 0$ in the first and third methods. In the first method, this solution is given using the total weights of every objective in both the objective function and distance constraint equal to one. In addition, the second method provides the best point only when the weights are equal ($w_1 = w_2 =$ any value of $w_i > 0$, i=1, 2) or without using weights.
Overall, the results of this study indicate that the value of D is equal to the distance when the weights are equal to one or without using weights in both the objective function and the constraint of Euclidean distance with the value of B equals to zero. If |B| = | the total of objectives| = 27, then the Euclidean distance = ((The resulted D= 32.65685) – 27) without weighting of objectives or with equal weights=1.
It seems that the results of these models are the same as Alia's result with less time. On the other side, they are more accurate than the solution by P. & Bharti, and Mustafa et al.

Example (2):
Consider the following two-dimensional nonlinear multi-objective problem:

Minimize: ($f_1 = x$, $f_2 = y$),

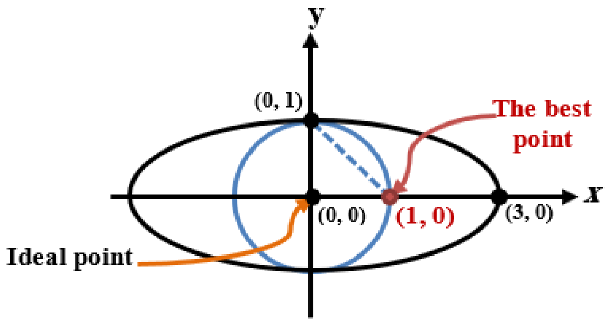

Fig. 8: The Best Point of Example (2).

Subject to:

$x^2 + y^2 \geq 1$,

$\frac{1}{9}x^2 + y^2 \leq 1$,

$x \geq 0$, $y \geq 0$.

These two objectives are conflicting as shown in figure 8. Their individual optimal solutions are (0, 1), and (1, 0). The ideal point is (0, 0), and the total of objectives "B" is 1.0. The best solution obtained by the proposed models is highly dependent on the used weights. Using (FDM), the best solution is: $f_1^* = x^* = 1.0$, $f_2^* = y^* = 0.0$, $w_{11} = w_{21} = w_{22} =$ [0.11- 0.489], $w_{12} = w_{22} = w_{21} =$ [0.89- 0.511], $w_{11}, w_{12}, w_{21}, w_{22} > 0$, $w_{11} + w_{12} = w_{21} + w_{22} = w_{11} + w_{21} = w_{12} + w_{22} = 1$ and B= –1 or 0. Therefore, $C_1 = 1.0$, $C_2 = \frac{1}{9}$ and the distance of objectives is 1.0. This solution is obtained by (SDM) when $w_1 =$ [0.000001- 0.4999], and $w_2 =$ [0.999999- 0.5001]. But, (TDM) uses weights $0 < w_1 \leq 0.4999$, and $1 > w_2 \geq 0.5001$ to get the best point.

Example (3):
Consider the following non-linear multi-objective problem taken from Alia [19]:

Min: F = ($f_1 = x$, $f_2 = y$),

Subject to: $x^2 + y^2 \leq 1$.

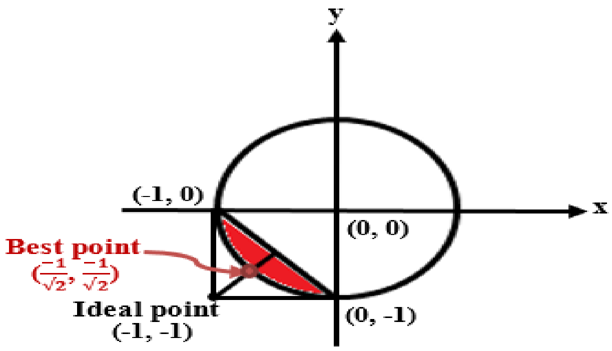

Fig. 9: The Best Point of Example (3).

The example consists of two linear objectives and one nonlinear constraint. To solve this example by the proposed methods using Lingo 14 software, the objective functions is multiplied by negative one. This problem is reformulated as follows:

Min: $F = (f_1 = -x,\ f_2 = -y)$,

Subject to: $x^2 + y^2 \leq 1$.

Therefore, the results, also, will multiply by negative one. As seen in figure 9, the best point is $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$ that is as Alia's solution with the distance of objectives is 0.414. All proposed methods give this solution with equal weights of objectives. In addition, the first method produces the best point when the total weights of every objective are equal to one.

Example (4):

Consider the linear multi-objective problem taken from Mustafa et al [22]:

Maximize: $f_1 = 50x_1 + 100x_2 + 17.5x_3$,

Maximize: $f_2 = 50x_1 + 50x_2 + 100x_3$,

Maximize: $f_3 = 20x_1 + 50x_2 + 100x_3$,

Maximize: $f_4 = 25x_1 + 75x_2 + 12x_3$,

Subject to:

$12x_1 + 17x_2 \leq 1400$,

$3x_1 + 9x_2 + 8x_3 \leq 1000$,

$10x_1 + 13x_2 + 15x_3 \leq 1750$,

$6x_1 + 16x_3 \leq 1325$,

$12x_2 + 7x_3 \leq 900$,

$9.5x_1 + 9.5x_2 + 4x_3 \leq 107$,

$x_1, x_2, x_3 \geq 0$.

All maximization objectives are transformed to minimization to facilitate the calculation process. The individual optimum of each objective is: $f_1^*(x_1^*= 0, x_2^*= 11.26316, x_3^*= 0) = -1126.316$, $f_2^*= f_3^*(x_1^*= 0, x_2^*= 0, x_3^*= 26.75) = -2675$, and $f_4^*(x_1^*= 0, x_2^*= 11.26316, x_3^*= 0) = -844.7368$. The total of objectives without weights is –6139.125. Thus, the best point by all proposed methods with equal weights in both the objective function and distance constraint ($w_1= w_2 = w_3 = w_4$ = any value of $w_i > 0$, i=1, 2, 3, 4) or without using weights is: $x_1^*= 0$, $x_2^*= 0.8277382$, $x_3^*= 24.78412$, $f_1^*= -516.4959$, $f_2^*= f_3^*= -2519.7989$, and $f_4^*= -359.4898$. The Euclidean distance from the utopian point is 809.64175. In this case, the value of D is the distance of objectives with B=0. But if |B| = 6139.125, the Euclidean distance is equal to ((The resulted D= 6948.76675) – 6139.125) without weighting of objectives or when weights =1.

But, Mustafa et al presented a game theory-based approach to generate three compromise solutions for this problem as follows: ($x_1^*= 45.22$, $x_2^*= 49.61$, $x_3^*= 43.52$), ($x_1^*= 44.94$, $x_2^*= 50.63$, $x_3^*= 41.77$), and ($x_1^*= 22.28$, $x_2^*= 31.57$, $x_3^*= 74.46$). Their Euclidean distances from the utopian point are 11591.0277, 11512.4289, and 11423.723, respectively. The proposed methods can offer the best solution directly more than another related method by Mustafa et al.

Example (5):

Consider the nonlinear multi-objective problem taken from Majid et al [18] has the following form:

Maximize: $f_1 = x_1^2 + x_2^2 + x_3^2$,

Maximize: $f_2 = (x_1 - 1)^2 + x_2^2 + (x_3 - 2)^2$,

Minimize: $f_3 = 2x_1 + x_2^2 + x_3$,

Subject to:

$-x_1 + 3x_2 - 4x_3 + 6 \geq 0$,

$-2x_1^2 - 3x_2 - x_3 + 10 \geq 0$,

$x \in R^3$, $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 4$, $0 \leq x_3 \leq 2$.

After transforming the first two maximization objectives to minimization objectives, the ideal point is (–11.1111, –16.1111, 0.0). Afterward, this problem is solved by the proposed methods with using equal weights of objectives or without. The best point of the given problem by three proposed methods is: $x_1^*= 0.0$, $x_2^*= 2.721654$, $x_3^*= 0.0$, $f_1^*= -7.4074$, $f_2^*= -12.4074$, $f_3^*= 7.4074$, $C_1= 14.164962$, $C_2= 1.835038$, the resulting D= 25.183 when B= –16.1111 and its distance from the utopian point is 9.072. This solution is the same result as Alia's approach [2]. On the other hand, one of the Pareto-optimal solutions in the paper by Majid [18] is: x = (0.00, 0.05, 0.99) which has a distance from the utopian point = 17.3798. Therefore, the proposed methods achieve better results than the previous work.

5. Conclusion

No one can deny the importance of finding the best solution for a multi-objective optimization problem, where solving such a problem leads to a large number of solutions that are not all optimal. This paper succeeded in its tasks to introduce three new methods for finding the best solution to multi-objective programming problems. They depend on minimizing the weighting Euclidean distance of objectives from the ideal point. This work consists of two tasks. The first task is to reformulate a multi-objective programming problem by any one of the proposed methods. The second task is to solve this formulated problem to obtain its best solution with any nonlinear solver. The weighted sum of objectives is used with these proposed approaches as a criterion to ensure that the best solution is accurately found. There is a guarantee that the proposed methods will find the best Pareto-optimal solution in a finite number of steps for an arbitrary multi-objective programming problem.

In addition, some illustrative examples of the multi-objective optimization problems are presented to clarify the idea of the proposed methods. These examples include both convex and non-convex spaces. The experimental results indicate that these methods can obtain explicitly the best efficient point compared to the other classical and artificial multi-objective approaches. In some of these examples, the proposed methods give explicitly better results compared to that of the other experiments and also show the same best point in other references with less time. Finally, these new methods have flexibility in modeling, simplicity in concept, and robust performance with high speed. Undoubtedly, the third method produces the best point immediately than the others.

In future work, these proposed approaches will be applied to different practical applications of multi-objective optimization, because they are easy to use and understand.

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