International Journal of Advanced Mathematical Sciences, 11 (1) (2025) 7-14

International Journal of Advanced Mathematical Sciences

Website: www.sciencepubco.com/index.php/IJAMS

Research paper

Combinatorial results and green's relation of star-like $T\alpha\omega_n^*$ full transformation semigroups

M. Ahmadu[1*], M. M. Mogbonju[1] and A.O. Adeniji[1]

[1] *Department of Mathematics, University of Abuja*
[*] *Corresponding author E-mail:michaelahmadu1982@gmail.com*

Abstract

$X_n = \{1,2,...\}$ be a distinct non negative integer and star-like full transformation semigroup $T\alpha\omega_n^*$ be a semigroup of Full Transformation semigroup T_n of X_n. Let Height of α^* be $H^+(\alpha^*) = \mid Im\alpha^* \mid$, Fixed point of α^* be $F(\alpha^*) = \mid \{x \in X : x\alpha^* = x\} \mid$, Idempotent of α^* be $\mid F(\alpha^*) \mid = \mid Im\alpha^* \mid$, Collapse of (α^*) be $\mid \cup\{t\alpha^{-1} : t \in T\alpha\omega^*\} \mid$ and Relapse of (α^*) be $\mid n - C^+((\alpha^*)) \mid$, Green's relation of semigroup $T\alpha\omega_n^*$ were characterized using the general method and definitions. The methods employed in carrying out this research work were that the elements in each of the functions were listed and some tables were formed for $H^+(\alpha^*)$, $J^*(\alpha^*)$, $E \mid q^*(\alpha^*) \mid$, $C^*(\alpha^*)$, $C^-(\alpha^*)$, and $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$, $\mathscr{H}$ and $\mathscr{J}$ equivalence relations from these tables, triangular array and sequences were formed; the patterns of the arrangement were studied, formulae were deduced in different cases through the combinatorial principle. The star-like operator $\mid K_{n+1} - \lambda K_n \mid \leq \mid K_n - \lambda K_{n+1} \mid$ was used to generate some tables of results from the star-like element.

Keywords: *Green's relation, Full transformation, Semigroup, Star-like, Collapse, Relapse, and Idempotent*

1. Introductions and Preliminaries

The full transformation semigroup denoted as T_n defined on $X_n = \{1,2,3,...,n\}$ such that α:Dom $\alpha = X_n$, commonly known as full or total transformation semigroup. The general notation for a semigroup with respective operation is $(S,*)$. Transformation semigroups are associative then:(α,β,μ):$(\alpha*\beta)*\mu$=$\alpha*(\beta*\mu)$. It is also known as the full symmetric semigroup or monoid with composition of mappings as the semigroup operator. The star-like full transformation semigroup denoted as $T\alpha\omega_n^*$. A Star-like transformation semigroup is said to satisfy collapse function if $c^+(\alpha^*) = \mid \bigcup t\alpha^{-1} : t \in T\alpha\omega_n^* \mid$ while Relapse function is denoted as $r^+(\alpha) = \mid n - c^+(\alpha^*) \mid$ where $n \in N$. The Green's relations are useful for understanding the nature of divisibility in a semigroup, instead of working directly with a semigroup S, it is current to define Green's relation over the monoid S'. S' is S with an identity adjoined if necessary" if S is not already a monoid, a new element is adjoined and defined to be an identity. Let S' be a semigroup, $a,b \in S$. If a and b generate the same left principal ideal, that is, $S'a = S'b$, then we say that a and b are L equivalent and write a $\mathscr{L}$ b or $(a,b) \in \mathscr{L}$. If a and b generate the same right principal ideal, that is, $aS' = bS'$, then we say that a and b are $\mathscr{R}$ equivalent and write a$\mathscr{R}$ b or $(a,b) \in \mathscr{R}$. If a and b generate the same principal ideal, that is, $S'aS' = S'bS'$, then we say that a and b are $\mathscr{J}$ equivalent and write a $\mathscr{J}$ b or $(a,b) \in \mathscr{J}$. Let $\mathscr{H} = \mathscr{L} \bigcap \mathscr{R}$, $\mathscr{D} = \mathscr{L} \cup \mathscr{R}$, then $\mathscr{H}$ and $\mathscr{D}$ are equivalences on S, too. It is a well known fact that $\mathscr{J} = \mathscr{D}$ in any finite semigroup. These five equivalences are usually called Green equivalence relations on **S**.

A transformation $\alpha \in T_n$ is said to be full contraction transformation semigroup if $\mid x\alpha - y\alpha \mid \leq \mid x - y \mid$ for all $x,y \in X_n$. The set of all order-preserving full contraction transformation semigroup is denoted by OCT_n and it is the subsemigroup of T_n. The concept of the *eggbox* structure, introduced by [1], describes each $\mathscr{D}$-class in a semigroup S as a union of $\mathscr{L}$-classes and $\mathscr{R}$-classes. The intersection of these classes is either empty or forms an $\mathscr{H}$-class. This structure may consist of a single row or a single column of cells. The equivalence relation $\mathscr{L}$ on S is defined such that two elements a and b are $\mathscr{L}$-related if and only if they generate the same principal left ideal, which is expressed as $S'a = S'b$. Similarly, $\mathscr{R}$-equivalence is characterized by $a\mathscr{R}b$ if $aS' = bS'$, and $\mathscr{J}$-equivalence holds when $S'aS' = S'bS'$, representing the principal ideal generated by $a \in S$ [2]. Green's relations, defined on algebras of type T, are applicable to any semigroup or monoid, which are algebras with an associative binary operation. Their behavior has been further analyzed within varieties V of semigroups [3]. Ibrahim, [7] explained some combinatorial results on Green's relation of partial injective transformation semigroup and characterized the Greens relation on CI_n and also solve the contraction mapping injective partial transformation semigroup on $n-objects$. He used two parameters $F(n,p)$ and found that the order of $\mathscr{L}$ -classes and $\mathscr{R}$ classes are the same but $\mathscr{D}$ class is different.

Results on the collapse of the full transformation semigroup, derived from studies on combinatorial properties, provide valuable insights into

this area of research [4]. On Idempotents on transformation semigroups (see [12, 13, 14, 15, 16]). The collapse of transformation semigroups was also studied by [11, 16]. On star-like $T\alpha w_n^{\star}$(see, [8], [9], [10]).
The study of [5] showed the combinatorial problems in the theory of symmetric inverse semigroup and some relevant results from his work are:

1. Proposition Umar, [5])
   Let $S = I_n$, then $F(n;\ p,\ k) = \begin{pmatrix} n \\ p \end{pmatrix} \begin{pmatrix} k-1 \\ p-1 \end{pmatrix} p!\ \forall\, n \geq k \geq p \geq 0$
2. Corollary Umar, [5])
   Let $S = I_n$, then $F(n,\ p) = \begin{pmatrix} n \\ p \end{pmatrix}^2 p!$ for all $n \geq p \geq 0$.

Some combinatorial results obtained by [6] on $ORCT_n$ and $ODCT_n$ are presented below:

1. Corollary [6]
   Let $S = ORCT_n$, then $|S| = |ORCT_n| = (n+1)2^{(n-1)} - n$, for $n \geq 1$
2. Corollary [6]
   Let $S = ODCT_n$, then
   $F(n,\ k) = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} for\ k \geq 1$
   $F(n,\ m) = 2^{(n-m-1)}, for\ n \geq m \geq 1$
   $F(n,\ p) = \begin{pmatrix} n-1 \\ p-1 \end{pmatrix}, for\ p \geq 1.$

Garba et al. (2022) On some combinatorial result on star-like tansformation semigroup. $T\alpha\omega_n^*$ and one of the result that is obtained from there work is presented in the lemma below: Lemma 2.1: For any transfomation $\alpha^* \in T\omega_n^*$ there are finitely many star-like $\alpha^* \in T\omega_n^*$ such that

$$F(n, r^+n(\alpha^*)) = \binom{2(n-2)+(n-m(\alpha^*))}{n-m(\alpha)}$$

for all $r^+n(\alpha^*) \geq m \geq 1$ and $n \in N$

2. Main Results

Lemma 3.1
Let $\mathscr{R} \in T\alpha\omega_n^*$ then a$\mathscr{R}$b if and only if ker(a) = ker(b)and also $\mathscr{H} \in T\alpha\omega_n^*$ such that $\mathscr{H} = im(\alpha) = im(\beta), ker(a) = ker(a)$ then

i

$$F(n,\mathscr{R}) = \frac{\binom{14n^4}{14p^4-k} - \binom{418n^2}{418p^2-k}}{4!} - \frac{\binom{31n^3}{31p^3-k} - \binom{143n}{143p-k}}{3!} + 12$$

ii

$$F(n,\mathscr{H}) = \frac{\binom{115n^4}{115p^4-k} + \binom{3629n^2}{3629p^2-k}}{4!} - 2\frac{\binom{537n^3}{537p^3-k} + \binom{2535n}{2535p-k}}{4!} + 101$$

Proof:
Since $a\mathscr{R}b$ is ker(a) = ker(b) then the sequence generated are 1,2,5,14,47 and System of equation for $\mathscr{R}$ is given below.
$a_4 + a_3 + a_2 + a_1 + a_0 = 1$

$16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0 = 2$

$81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0 = 5$

$256a_4 + 64a_3 + 16a_2 + 4a_1 + a_0 = 14$

$625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0 = 47$

Using maple 18, The following were obtained

$$a_n = \frac{7}{12}n^4 - \frac{31}{6}n^3 + \frac{209}{12}n^2 - \frac{143}{6}n + 12$$

with recursive formular

$$F(n,\mathscr{R}) = \frac{\binom{14n^4}{14p^4-k} - \binom{418n^2}{418p^2-k}}{4!} - \frac{\binom{31n^3}{31p^3-k} - \binom{143n}{143p-k}}{3!} + 12$$

while System of equation for $\mathscr{H}$

$a_4 + a_3 + a_2 + a_1 + a_0 = 1$

$16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0 = 2$

$81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0 = 8$

$256a_4 + 64a_3 + 16a_2 + 4a_1 + a_0 = 38$

$625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0 = 226$

Using maple 18, The following were obtained
$a_n = \frac{115}{24}n^4 - \frac{537}{12}n^3 + \frac{3629}{24}n^2 - \frac{2535}{12}n + 101$
with recursive formular

$$F(n,\mathscr{H}) = \frac{\binom{115n^4}{115p^4-k} + \binom{3629n^2}{3629p^2-k}}{4!} - 2\frac{\binom{537n^3}{537p^3-k} + \binom{2535n}{2535p-k}}{4!} + 101$$

with recursive formular For $\mathscr{D}$
$\mathscr{D}$ = n where n = $\{1,2,3,4....\}$. The results follows in Tables 11, 12 and 13

Table 1: $\mathscr{R}$ - Classes of $T\alpha\omega_n^*$

n/r	1	2	3	4	5	$\sum\mathscr{R}$
1	1					1
2	1	1				2
3	1	3	1			5
4	1	7	5	1		14
5	1	15	22	8	1	47

Table 2: $\mathscr{D}$ - Classes of $T\alpha\omega_n^*$

n/r	1	2	3	4	5	$\sum\mathscr{D}$
1	1					1
2	1	1				2
3	1	1	1			3
4	1	1	1	1		4
5	1	1	1	1	1	5

Table 3: $\mathscr{H}$ - Classes of $T\alpha\omega_n^*$

n/r	1	2	3	4	5	$\sum\mathscr{H}$
1	1					1
2	1	1				2
3	1	6	1			8
4	1	21	15	1		38
5	1	60	132	32	1	226

Lemma 3.2
Let $\delta^* \in$ T $\alpha\omega_n^*$ be a star-like transformation, then

$$3 \mid T\alpha\omega_n^* \mid = \frac{\binom{13n^4}{13p^4-b} - 23\binom{5n^3}{5p^3-b}}{2} + 5\binom{38n^2}{38p^2-b} - 7\binom{37n}{37p-b} + 123$$

such that $n = p \geq 1$ where b is a star-like algebraic constant

Proof:

Suppose $N_i = \{i, i+1, i+2, i+3, \cdots, n\}$, $i = \{0, 1, 2, ...\}$ is non-negative with $N_0 = 0, 1, 2, ...$ if $\delta^* \in$ T $\alpha\omega_n^*$ is a star- like transformation with $P^* \leq 1$ and $P^* \geq 1$ there exist some star-like sequences U_n with (ς_n^* is vertical order)
$\varsigma_n^* = \mid \delta^* \in$ T $\alpha\omega_n^* \mid$, then

$$U_n = \varsigma_n^{*1}\binom{n}{1} + \varsigma_n^{*2}\binom{n}{2} + \varsigma_n^{*3}\binom{n}{3} + \cdots + \varsigma_n^{*k+1}\binom{n}{k}$$

posses a unique integer difference of 52 at ς_n^{*4}, for all $i \geq n \geq 1$. We generate a system of equation.
$U_0 + U_1 + U_2 + U_3 + U_4 = 1$

$16U_0 + 8U_1 + 4U_2 + 2U_3 + U_4 = 3$

$81U_0 + 27U_1 + 9U_2 + 3U_3 + U_4 = 10$

$256U_0 + 64U_1 + 16U_2 + 4U_3 + U_4 = 37$

$625U_0 + 125U_1 + 25U_2 + 5U_3 + U_4 = 151$

Since δ is a bijective mapping,the system of equation may be re-written as **A U = X** such that

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & :1 \\ 16 & 8 & 4 & 2 & :1 \\ 81 & 27 & 9 & 3 & :1 \\ 256 & 64 & 16 & 4 & :1 \\ 625 & 125 & 25 & 5 & :1 \end{pmatrix}, U = \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}, X = \begin{pmatrix} 1 \\ 3 \\ 10 \\ 37 \\ 151 \end{pmatrix}$$

using Maple 18, we obtained

$$U_0 = \frac{13}{6}, U_1 = \frac{-115}{6}, U_2 = \frac{190}{6}, U_3 = \frac{-518}{6}, U_4 = 41$$

We see that ς_n^* gives the required star-like recursive relation of $T\alpha\omega_n^*$ for $P^* \leq 1$, and $P^* \geq 1$.

Lemma 3.3:
Let $T\alpha\omega_n^* \subseteq P\alpha\omega_n^* \subset \alpha\omega_n^*$ be a star-like full finite semigroup, given any $\rho^* \in T\alpha\omega_n^*$

$$F(n, m^*) = \binom{2_n - 1}{n - m^*}; m^* \geq n \geq 1$$

Proof:

Suppose $\rho^* \in T\alpha\omega_n^*$ such that $n \in N_i = \{i, i+1, i+2, ...\}$ (i = $\{0, 1, 2, ...\}$), then $m^*(\alpha)$ has an identity element **e** in which m^* is bijective under composition of mapping with $m^* \in Dom(\alpha)$. Each element of $Dom(\alpha)$ in $T\alpha\omega_n^*$ can be chosen from N_i in

$$F(n, m^*) = \binom{2_n - 1}{n - m^*}; ways$$

Theorem 3.4
Let $\delta^* \in T\alpha\omega_n^*$ be a star-like transformation, if $| b_n |$ denote the cardinality of maximum fixed element in $Dom\delta^*$, there exist a non- negative integer k such that ρ_n^{k+1} = 0, then

$$F(n, J^*, e^*) = \binom{2J^* - e^0}{2n - 2^2};$$

Proof:
Suppose $\rho_n^* b_n = 0$ is the diagonal star-like operate of the maximal element of $\delta^* \in T\alpha\omega_n^*$ with e^0 as a star-like exponential constant. If $u_1, u_2, ...u_n$ are star-like sequences of number in which there exist a non - negative integer k for all $n \geq 2$ in $n \in N_i = \{i, i+1, i+2, ...\}$ such that $\rho_n^{k+1} = 0$ by Theorem 2, $\delta^* \in T\alpha\omega_n^*$ can be expressed as a reducible star-like polynomial and since δ^* is bijective under composition of mapping, we see that $f(J^*)$ can be chosen in $\binom{J}{n-1}$; ways for all $J \geq 1$, such that

$$F(n, J^* e^*) = \binom{2J^* - e^0}{2n - 2^2};$$

Proposition 3.5:
By (Howie, 1995), the Green's relation on the Star-like semigroup T_n shows that

i. $\alpha\mathscr{L}\beta$ if and only if $im(\alpha) = im(\beta)$
ii. $\alpha\mathscr{R}\beta$ if and only if $ker(\alpha) = ker(\beta)$
iii. $\alpha\mathscr{D}\beta$ if and only if $| im(\alpha) | = | im(\beta) |$
iv. $\mathscr{D} = \mathscr{J}$

Proof:
With critical investigation of the star-like full semi-group on the green relations we deduced the following:

If $n = 1$

Number of $\mathscr{L}$ - Classes of $T\alpha\omega_n^*$ is 1
Number of $\mathscr{R}$ - Classes of $T\alpha\omega_n^*$ is 1
Number of $\mathscr{D}$ - Classes of $T\alpha\omega_n^*$ is 1
Number of $\mathscr{H}$ - Classes of $T\alpha\omega_n^*$ is 1

when n = 2
Number of $\mathscr{L}$ - Classes of $T\alpha\omega_n^*$ is 2
Number of $\mathscr{R}$ - Classes of $T\alpha\omega_n^*$ is 2
Number of $\mathscr{D}$ - Classes of $T\alpha\omega_n^*$ is 2
Number of $\mathscr{H}$ - Classes of $T\alpha\omega_n^*$ is 2

when n = 3
Number of $\mathscr{L}$ - Classes of $T\alpha\omega_n^*$ is 4
Number of $\mathscr{R}$ - Classes of $T\alpha\omega_n^*$ is 5
Number of $\mathscr{D}$ - Classes of $T\alpha\omega_n^*$ is 3
Number of $\mathscr{H}$ - Classes of $T\alpha\omega_n^*$ is 8

when n = 4
Number of $\mathscr{L}$ - Classes of $T\alpha\omega_n^*$ is 8
Number of $\mathscr{R}$ - Classes of $T\alpha\omega_n^*$ is 14
Number of $\mathscr{D}$ - Classes of $T\alpha\omega_n^*$ is 4
Number of $\mathscr{H}$ - Classes of $T\alpha\omega_n^*$ is 38

when n = 5
Number of $\mathscr{L}$ - Classes of $T\alpha\omega_n^*$ is 16
Number of $\mathscr{R}$ - Classes of $T\alpha\omega_n^*$ is 47
Number of $\mathscr{D}$ - Classes of $T\alpha\omega_n^*$ is 5
Number of $\mathscr{H}$ - Classes of $T\alpha\omega_n^*$ is 226

Theorem 3.6
Suppose $\delta^* \in$ T $\alpha\omega_n^*$ be a star-like transformation, then

$$F(n,\mathscr{L}) = \binom{x^n}{x^p - k}$$

such that $n = p \geq 0$ where k is a star-like algebraic constant.

Proof:
Let $N_i = \{i, i+1, i+2, i+3, \cdots, n\}$, $i = \{0,1,2,...\}$ is non-negative with $N_0 = 0,1,2,...$ since $\delta^* \in$ T $\alpha\omega_n^*$ is a star- like transformation with $n \geq 0$ and $P \geq 1$ there exist some star-like sequences N_n with ς_n^* is vertical order which posses a unique integer difference of 1 at ς_4^* for all $i \geq n \geq 0$
ς_n^* **=** $|\ \delta^* \in$ **T** $\alpha\omega_n^*\ |$**,** then
System of equation for $\mathscr{L}$

$a_4 + a_3 + a_2 + a_1 + a_0$ **= 1**

$16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0$ **= 2**

$81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$ **= 4**

$256a_4 + 64a_3 + 16a_2 + 4a_1 + a_0$ **= 8**

$625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0$ **= 16**

Using maple 18, The following were obtained
$a_n = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{18}{24}n + 1$

Recursive Formular

$$F(n,\mathscr{L}) = \binom{x^n}{x^p - k}$$

such that x = 2 and n,p = $\{0,1,2,3,...\}$ k= Algebraic constant

Theorem 3.7
Let $\mathscr{R} \in T\alpha\omega_n^*$ then a$\mathscr{R}$b if and only if ker(a) = ker(b)and also $\mathscr{H} \in T\alpha\omega_n^*$ such that $\mathscr{H}$ = $im(\alpha) = im(\beta), ker(a) = ker(a)$ then the following results were obtained

i

$$F(n,\mathscr{R}) = \frac{\binom{14n^4}{14p^4-k} - \binom{418n^2}{418p^2-k}}{4!} - \frac{\binom{31n^3}{31p^3-k} - \binom{143n}{143p-k}}{3!} + 12$$

ii

$$F(n,\mathscr{H}) = \frac{\binom{115n^4}{115p^4-k} + \binom{3629n^2}{3629p^2-k}}{4!} - 2\frac{\binom{537n^3}{537p^3-k} + \binom{2535n}{2535p-k}}{4!} + 101$$

proof:

Since $a\mathscr{R}b$ is ker(a) = ker(b) then the sequence generated are 1,2,5,14,47 and System of equation for $\mathscr{R}$ is given below.

$a_4 + a_3 + a_2 + a_1 + a_0$ = **1**

$16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0$ = **2**

$81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$ = **5**

$256a_4 + 64a_3 + 16a_2 + 4a_1 + a_0$ = **14**

$625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0$ = **47**

Using maple 18, The following were obtained

$$a_n = \frac{7}{12}n^4 - \frac{31}{6}n^3 + \frac{209}{12}n^2 - \frac{143}{6}n + 12$$

Recursive Formular

$$F(n,\mathscr{R}) = \frac{\binom{14n^4}{14p^4-k} - \binom{418n^2}{418p^2-k}}{4!} - \frac{\binom{31n^3}{31p^3-k} - \binom{143n}{143p-k}}{3!} + 12$$

while System of equation for $\mathscr{H}$

$a_4 + a_3 + a_2 + a_1 + a_0$ = **1**

$16a_4 + 8a_3 + 4a_2 + 2a_1 + a_0$ = **2**

$81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$ = **8**

$256a_4 + 64a_3 + 16a_2 + 4a_1 + a_0$ = **38**

$625a_4 + 125a_3 + 25a_2 + 5a_1 + a_0$ = **226**

Using maple 18, The following were obtained

$$a_n = \frac{115}{24}n^4 - \frac{537}{12}n^3 + \frac{3629}{24}n^2 - \frac{2535}{12}n + 101$$

Recursive Formular

$$F(n,\mathscr{H}) = \frac{\binom{115n^4}{115p^4-k} + \binom{3629n^2}{3629p^2-k}}{4!} - 2\frac{\binom{537n^3}{537p^3-k} + \binom{2535n}{2535p-k}}{4!} + 101$$

Recursive Formular For $\mathscr{D}$

$\mathscr{D}$ = **n where n** = $\{1,2,3,4....\}$ The results follows in Tables 11 and 13

References

[1] **Clifford, A. H., & Preston, G. B. (1961).** ***The Algebraic Theory of Semigroups, Volume 1*****. American Mathematical Society.**
[2] **Howie, J. M. (1995).** ***Fundamentals of Semigroup Theory*****. Clarendon Press.**
[3] **Denecke, K., Lück, D., & Reichel, H. (2007).** ***General Algebra and Applications in Theoretical Computer Science*****. Springer.**
[4] **Adeniji, A. O. and Makanjuola, S. O. (2008), On some combinatorial results of collapse and properties of height in full transformation semigroups.** ***African Journal of Computer and ICT. Vol. 1, No. 2, pp. 61 - 63.***
[5] Umar A., (2010) Some combinatorial in the theory of partial transformation semigroups. **5th NBSAN meeting University of Saint Andrews** ***134.***
[6] Adeshola, D. A. (2013), Some semigroups of full contraction mapping of a finite chain. **Ph.D Thesis,** ***University of Ilorin. Nigeria.***
[7] Ibrahim, G.R., (2015). Some Combinatorial Results On Green's Relation Of Partial Injective Transformation Semigroup. **Journal Semigroup Theory Appl.*****2015,2015:4 ISSN: 2051-2937***
[8] Akinwunmi, S. A. Mogbonju, M. M. and Adeniji, A. O. (2021(a)), Multiplicative invertibility characterizations on star-like cyclicpoid $C_yP\omega_n^*$ finite partial transformation semigroups. **Hikari Journal of Pure Mathematical Science, 10(** ***1), pp. 45 - 55.***

[9] *Ibrahim G.R., Lawal O.J., Akinwunmi S.A., Bakare G.N., and Dauda A.A., (2022).On Some Combinatorial Results on star-like Transformatiom Semigroup $T\alpha\omega_n^*$* **Nigerian Journal of Mathematics and Application***32, 56-67*
[10] *Akinwunmi, S.A., Mogbonju, M.M., Adeniji, A.O., Ibrahim, G. R., and Fati, M.O. (2021(b))Combinatorial magic right angle triangle characterization on partial $t^{\triangle}$- symmetric contration semigroups.* **Adamawa State University journal of scientific research (SDSUJSR)***. Vol. 9 issue 2, pp 18-23*
[11] *Adeniji, A. O., and Makanjuola, S. O., (2008). On some combinatorial results of collapse and properties of height in full transformation semigroups. African Journal of Computer and ICT, 1(2):61–63.*
[12] *Garba, G.U.,(1990). Idempotents in partial transformation semigroups* **fric Roy.Soc, Edinburgh***, 116 pp. 359-366.*
[13] *Zubairu, M. M., (2023). On the subsemigroup generated by idempotents of the semigroup of order-preserving and decreasing contraction mapping of finite chain. Journal of the Nigerian Mathematical Society, 42:140–152.*
[14] *Mogbonju, M. M., and Azeez, R. A., (2018). On some signed semigroup of order-preserving transformation. International Journal of Mathematics and Statistics Studies, 6(2):38–45.*
[15] *Ibrahim, A., Akinwunmi, S. A., Mogbonju, M. M., and Onyeozili, I. A., (2024). Combinatorial model of 3-dimensional nildempotency star-like classes N_c^{ω} partial one-one semigroups. International Journal of Mathematical Science and Optimization: Theory and Application, 10(1):25–35.*
[16] *Ibrahim, A., Nwaibeh, E. A., Francis, M. O., and Okigbo, C. J., (2024). On Collapse of Order-Preserving and Idempotent of Order-Preserving Full Contraction Transformation Semigroup. International Journal of Mathematical Sciences and Optimization: Theory and Applications, 10(2).* https://doi.org/10.5281/zenodo.10936738