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Research paper

Stability and bifurcation analysis of a prey-predator model with Smith growth rate in presence of environmental toxins

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Abstract

The aim of the present article is to formulate a prey-predator model by assuming that the preys inhabit in contaminated environments and undergo Smith growth rate instead of logistic growth rate. Stability and bifurcation phenomenon of the model have been analyzed with utmost care.

Keywords: *Prey-predator equilibrium, bifurcation, Smith growth rate, environmental toxins.*

1. Introduction

Presently, the study of prey-predator interactions with different constraints has become a rapid growing branch in the field of research in Mathematical Biology. Prey-predator dynamics depends on several parameters like birth rate, death rate, food availability, environmental effects, intrinsic behavior of species etc. In the perspective of prey population growth, generally researchers use the logistic growth function to describe growth rate of prey population. This growth rate is based on the assumption that the average growth rate is a non-linear function of the prey population But, this consideration may not be practical for populations living in contaminated environment and environment with restricted food availability. To remove this limitation of the logistic growth model, another model, namely Smith growth [6] model appeared in the literature. In the present article, we are involved with such type of growth rate.

Survival of any living organisms are not independent of environmental impacts. Environmental toxins are substances that can adversely impact living organisms, and can be present in air, soil, water, food and consumer products. Nowadays our environment is being polluted for several reagents like pesticide, leakage of oil tanks, garbage and wasts of towns and factories. Recently a tremendious danger of environment is being caused by plastic pollution. Plastic pollution is a rapid growing and alarming crisis everywhere. A recent study sponsored by Science and Engineering Board, India has pointed out presence of huge levels of microplastics in the costal water and sediments of Digha and Puri two important coastal towns of Bay of Bengal [7]. It warns serious danger about the impact of microplastics on marine life and people associated with this region. The potentiality of the grave situation is an alarming threat to the delicate marine ecosystem of this region. Researchers are contemplating that these microplastics have every possibility to enter into the food chain of the ecosystem of this region as the people of this region mostly consume fish and other sea foods which are primarily harvested from Bay of Bengal. In present days, the implications of environmental toxins are being studied in mathematical modeling by several authors. For instance we refer [3] and references therein. But most of the models are formulated on the basis of logistic growth of preys. In contrary, in the present article, we would like to consider Smith growth rate of preys which is more practical in the perspective of environmental impacts. The aim of the present article is to formulate a hypothetical prey-predator model by considering the people dependent on sea foods as predator and the eatable sea creatures of the coastal region as prey facing environmental contamination. Stability and bifurcation phenomenon of the model have been analyzed with utmost care.

The present article is organized as follows: After the introduction in Section 1, we formulate the model in Section 2. In the same section, we find the equilibrium points of the system and verify the validity i.e., positivity and boundedness of the solutions of the model. In Section 3, we analyze the stability of the equilibrium points. The aspects of bifurcations are treated in Section 4. Finally in Section 5, we strengthen or findings by supporting numerical and graphical representations. We conclude the article in Section 6.

2. Model formulation and validity of the solutions

Consider the eatable sea creatures as preys and human being consuming such foods as predators. Consider, at certain time the density of prey and predator populations respectively be $x(t)$ and $y(t)$. It is known that, any population undergoes through logistic growth rate whenever it faces no interruption regarding availability of foods and free and fair environment. But in the presence of environmental toxicants a population cannot grow logistically. In such cases the growth rate is assumed as Smith growth rate [6] which is expressed by

$$\frac{dx(t)}{dt} = rx(t)\left(\frac{K-x(t)}{K+cx(t)}\right),$$

where r and c denote the parameters representing the intrinsic growth rate and the mass in the prey population at the carrying capacity K, respectively. The consumption of the sea creatures by human being is expressed by Holling type II functional responses [1] that is given by

$$\frac{\lambda x(t)}{1+\lambda x(t)},$$

λ being catching rate of the sea creatures and h denotes the duration after which the foods are consumed by human being after harvesting. Let l be the death rate of sea creatures due to environmental toxins and m be the death rate of human being for consumption of contaminated sea foods. Let s be the natural death rate of predators. In such a situation the dynamics is governed by the following system.

$$\begin{aligned} \frac{dx(t)}{dt} &= rx(t)\left(\frac{K-x(t)}{K+cx(t)}\right) - \frac{\lambda x(t)y(t)}{1+h\lambda x(t)} - lx(t), \\ \frac{dy(t)}{dt} &= \frac{\varepsilon\lambda x(t)y(t)}{1+h\lambda x(t)} - m\lambda x(t)y(t) - sy(t), \end{aligned} \tag{1}$$

ε being the conversion efficiency of predators from preys.

2.1. Positivity and boundedness

By some routine calculations, one can establish that

$$x(t) = x(0)exp\left(\int_0^t \left(\frac{r(K-x(s))}{K+cx(s)} - \frac{\lambda y(s)}{1+h\lambda x(s)} - l\right)ds\right),$$

$$y(t) = y(0)\left(\int_0^t \left(\frac{\varepsilon\lambda x(s)}{1+h\lambda x(s)} - m\lambda x(s) - s\right)ds\right).$$

The above expressions convey that if $x(t)$ and $y(t)$ originates from the region $R_+^2\{(x(t),y(t)) : x(t) \geq 0, y(t) \geq 0\}$, they will continue as non-negative. Obviously, R_+^2 is a positively invariant set for the model. This establishes the positivity and boundedness of the model.

2.2. Equilibrium points

For the system (1) we get the following equilibrium points:
(i) The trivial rquilibrium point $(0,0)$
(ii) The axial equilibrium point $\left(\frac{K(r-l)}{r+cl}, 0\right)$
(iii) The interior equilibrium point $(x(t),y(t))$, where $x(t)$ is solution of

$$rx(t)\left(\frac{K-x(t)}{K+cx(t)}\right) - \frac{\lambda x(t)y(t)}{1+h\lambda x(t)} - lx(t) = 0,$$

and $y(t)$ is solution of

$$\frac{\varepsilon\lambda x(t)y(t)}{1+h\lambda x(t)} - m\lambda x(t)y(t) - sy(t) = 0.$$

The existence of interior equilibrium is shown in the numerical section.

3. Stability analysis

Now our goal is to search whether the equilibrium points are stable. To this end, we first compute the Jacobian of the system as follows:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = \frac{r(K-2x)}{xc+K} - \frac{rx(K-x)c}{(xc+K)^2} + \frac{\lambda^2 xyh - \lambda y(xh\lambda+1)}{(xh\lambda+1)^2} - l$, $a_{12} = -\frac{\lambda x}{xh\lambda+1}$, $a_{21} = \frac{\varepsilon\lambda y}{xh\lambda+1} - \frac{\varepsilon\lambda^2 xyh}{(xh\lambda+1)^2} - m\lambda$, $a_{22} = \frac{\varepsilon\lambda x}{xh\lambda+1} - s - m\lambda x$.
Now from the nature of the Jacobian we have the following:
Theorem 3.1. The trivial equilibrium is stable when $r < l$, it is unstable and exhibit saddle behaviour whenever $r > l$; if $r = s$ the point is degenerate, so no precise conclusion can be drawn about stability.

Proof. The Jacobian matrix at the origin takes the form

$$\begin{bmatrix} r-l & 0 \\ 0 & -s \end{bmatrix}.$$

Hence the eigenvalues are $r-l$ and $-s$. Thus the theorem is easily concluded.

Theorem 3.2. The axial equilibrium point $\left(\frac{K(r-l)}{r+cl}, 0\right)$ is degenerate if $l = r$. If $l < r$, then the axial equilibrium is stable whenever $\frac{\varepsilon\lambda Ka^2}{(cr-ca^2+r)+Ka^2h\lambda} < s + \frac{m\lambda Ka^2}{cr-ca^2+r}$, where $a^2 = r-l$. If $l > r$, then the equilibrium point is stable whenever $\frac{m\lambda Kb^2}{cr+cb^2+r} < s + \frac{\varepsilon\lambda Kb^2}{(cr+cb^2+r)-Kb^2h\lambda}$, where $b^2 = l-r$.

Proof. The Jacobian matrix at the point $\left(\frac{K(r-l)}{r+cl}, 0\right)$ is of the form

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix},$$

where $a_{11} = \frac{r(K+\frac{k(l-r)}{cl+r})}{-\frac{K(l-r)c}{cl+r}+K} + \frac{rK(l-r)}{(cl+r)(-\frac{K(l-r)c}{cl+r})+K} + \frac{rK(l-r)(K+\frac{K(l-r)}{cl+r})c}{(cl+r)(-\frac{K(l-r)c}{cl+r}+K)^2} - l$,
$a_{12} = \frac{\lambda K(l-r)}{(cl+r)(-\frac{K(l-r)h\lambda}{cl+r})+1}$,
$a_{22} = -\frac{\varepsilon\lambda K(l-r)}{(cl+r)(-\frac{K(l-r)h\lambda}{cl+r}+1)} - s + \frac{m\lambda K(l-r)}{cl+r}$. The eigenvalues are $\frac{(cl+r)(l-r)}{r(c+1)}$ and $-\frac{\varepsilon\lambda K(l-r)}{(cl+r)-K(l-r)h\lambda} - s + \frac{m\lambda K(l-r)}{cl+r}$. If $r = l$, one of the eigen values is zero. Hence the case is degenerate. If $l < r$, and $\frac{\varepsilon\lambda Ka^2}{(cr-ca^2+r)+Ka^2h\lambda} < s + \frac{m\lambda Ka^2}{cr-ca^2+r}$, where $a^2 = r-l$, then both the eigenvalues are negative.

If $l > r$, and $\frac{m\lambda Kb^2}{cr+cb^2+r} < s + \frac{\varepsilon\lambda Kb^2}{(cr+cb^2+r)-Kb^2h\lambda}$, where $b^2 = l-r$ then also both the eigenvalues are negative. Hence, the result follows.

Theorem 3.3. The interior equilibrium point is asymptotically stable if $a_{11} + a_{22} < 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$, where where $a_{11} = \frac{r(K-2x^*)}{x^*c+K} - \frac{rx^*(K-x^*)c}{(x^*c+K)^2} - \frac{\lambda y^*}{x^*h\lambda+1} + \frac{\lambda^2 x^* y^* h}{(x^*h\lambda+1)^2} - l$, $a_{12} = -\frac{\lambda x^*}{x^*h\lambda+1}$, $a_{21} = \frac{\varepsilon\lambda y^*}{x^*h\lambda+1} - \frac{\varepsilon\lambda^2 x^* y^* h}{(x^*h\lambda+1)^2} - m\lambda y^*$, $a_{22} = \frac{\varepsilon\lambda x^*}{x^*h\lambda+1} - s - m\lambda x^*$.

Proof. Let (x^*, y^*) be the interior equilibrium point. The Jacobian at the interior equilibrium point is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = \frac{r(K-2x^*)}{x^*c+K} - \frac{rx^*(K-x^*)c}{(x^*c+K)^2} - \frac{\lambda y^*}{x^*h\lambda+1} + \frac{\lambda^2 x^* y^* h}{(x^*h\lambda+1)^2} - l$, $a_{12} = -\frac{\lambda x^*}{x^*h\lambda+1}$, $a_{21} = \frac{\varepsilon\lambda y^*}{x^*h\lambda+1} - \frac{\varepsilon\lambda^2 x^* y^* h}{(x^*h\lambda+1)^2} - m\lambda y^*$, $a_{22} = \frac{\varepsilon\lambda x^*}{x^*h\lambda+1} - s - m\lambda x^*$.
At the interior equilibrium point the characteristic equation of the Jacobian matrix is given by $\mu^2 - \tau\mu + \Delta = 0$, where $\tau = \text{Trace}J = a_{11} + a_{22}$ and $\Delta =$ Determinant of $J = a_{11}a_{22} - a_{12}a_{21}$. The interior equilibrium point becomes stable if $\tau < 0$ and $\Delta > 0$. In this case the eigenvalues are real, distinct and negative. Otherwise, it is unstable.

4. Bifurcation analysis

If for change of a parameter the qualitative nature of solutions change, then it is called bifurcation and the transitional value of the parameter is called bifurcation value. In this section, we show the existence of transcritical and Hopf bifurcation.

4.1. Transcritical bifurcation

If for change of a parameter two equilibria interchange their stability, then transcritical bifurcation occurs. Now, we prove the following:

Theorem 4.1. The system experiences trans-critical bifurcation at the trivial equilibrium for $r = l$.

Proof. The Jacobian matrix at the trivial equilibrium is

$$\begin{bmatrix} r-l & 0 \\ 0 & -s \end{bmatrix}.$$

If $r < l$, both the eigenvalues are negative and the system is stable. If r crosses l, the trivial equilibrium point becomes unstable and the axial equilibrium point arises. For $r > l$, the axial equilibrium is stable for certain choices of parameters stated in Theorem 3.2. Hence, at the trivial equilibrium, the system experiences transcritical bifurcation at the trivial equilibrium point.

Theorem 4.2. The system experiences transcritical bifurcation at the axial equilibrium for $r = l$.

Proof. The Jacobian matrix at the point $\left(\frac{K(r-l)}{r+cl}, 0\right)$ is of the form

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix},$$

where $a_{11} = \frac{r(K+\frac{k(l-r)}{cl+r})}{-\frac{K(l-r)c}{cl+r}+K} + \frac{rK(l-r)}{(cl+r)(-\frac{K(l-r)c}{cl+r})+K} + \frac{rK(l-r)(K+\frac{K(l-r)}{cl+r})c}{(cl+r)(-\frac{K(l-r)c}{cl+r}+K)^2} - l$,
$a_{12} = \frac{\lambda K(l-r)}{(cl+r)(-\frac{K(l-r)h\lambda}{cl+r})+1}$,
$a_{22} = -\frac{\varepsilon\lambda K(l-r)}{(cl+r)(-\frac{K(l-r)h\lambda}{cl+r}+1)} - s + \frac{m\lambda K(l-r)}{cl+r}$. For $l = r$, the Jacobian is of the form

$$\begin{bmatrix} 0 & 0 \\ 0 & -s \end{bmatrix},$$

Clearly, the Jacobian has a zero eigenvalue at the axial equilibrium for $l = r$. It is easy to check that the conditions for transcritical bifurcation [4] are satisfied. Hence, the proof follows.

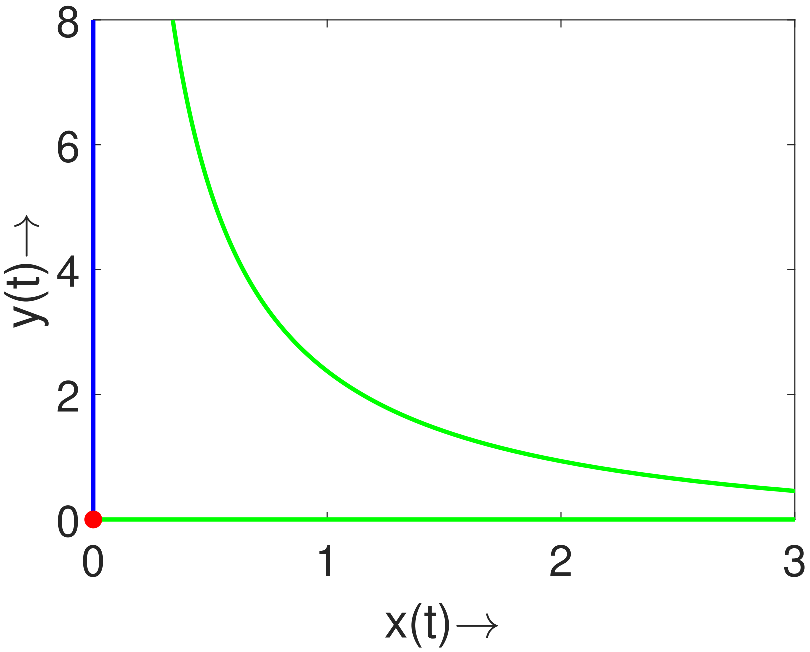

Fig. 1. Trivial equilibrium for $K = 0.02$

4.2. Hopf bifurcation

If for change of a certain parameter value, a stable equilibrium undergoes prolonged oscillation or the reverse, then Hopf bifurcation occurs. In such a case limit cycle appears or disappears. The existence of such bifurcation is shown in the following Numerical Section for change of the parameter r.

5. Numericals

In this section we verify the analytically obtained results by graphical representations. The graphs are drawn by MATLAB. Let us first exhibit existence of axial and interior equilibrium points by changing the parameter K. Other values are $r = 10.5, c = 0.15, \lambda = 0.15, h = 0.2, \varepsilon = 0.25, m = 0.1, l = 0.1, s = 0.1$.

In the following we plot the phase portrait for different values of r. Here the other parameters are $c = 0.1, \lambda = .15, h = .1, m = .9, K = 0.5, l = 0.2, s = 0.2, \varepsilon = 10$.

It is known that if for any change of parameter the limit cycle appears or diappears, then Hopf bifurcation occurs. The figures 4, 5 and 6 show disappearence of limit cycle for change in r. This ensures the existence of Hopf bifurcation in the system.

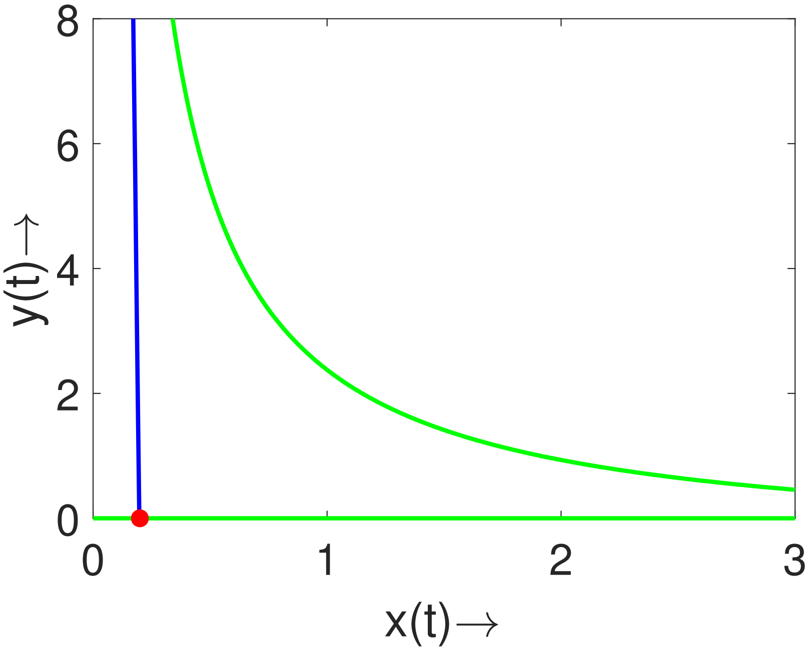

Fig. 2. Axial equilibrium for $K = 0.2$

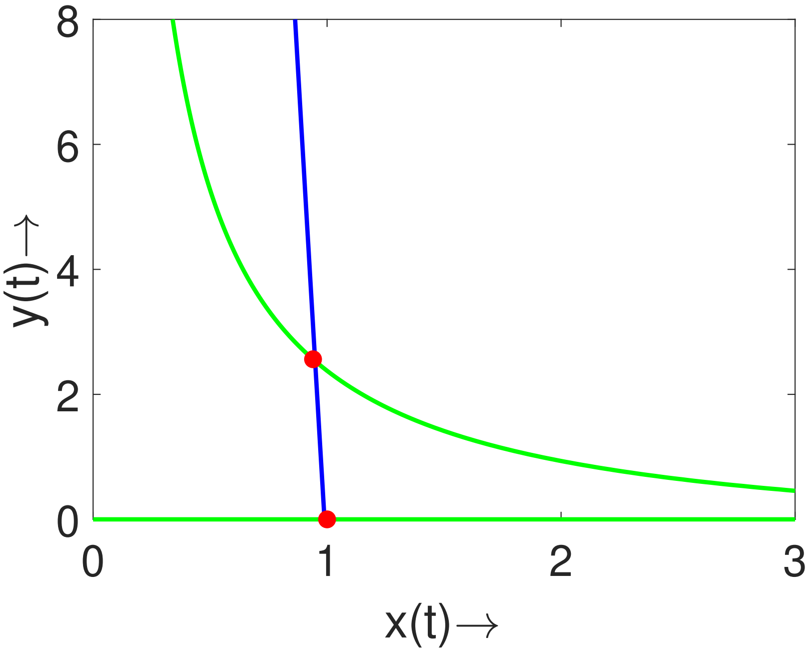

Fig. 3. Axial and interior equilibrium for $K = 1$

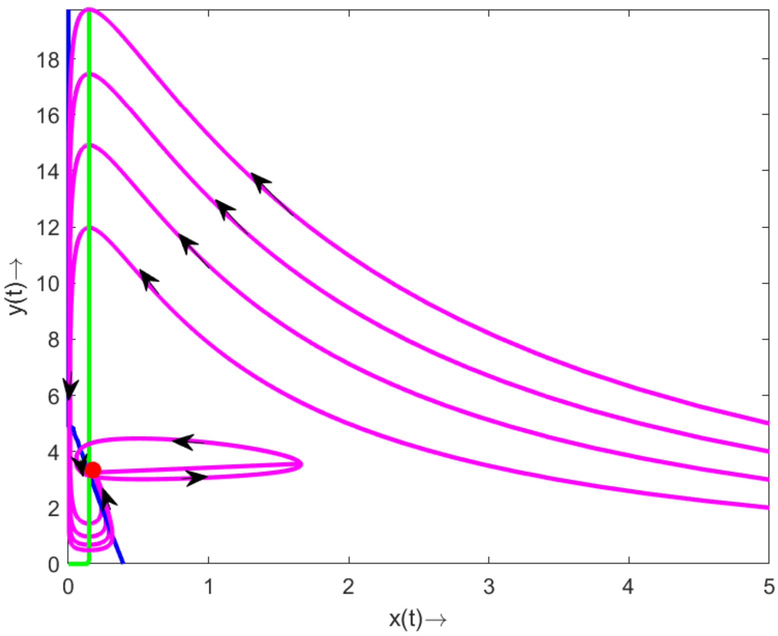

Fig. 4. Phase portrait for $r = 1$

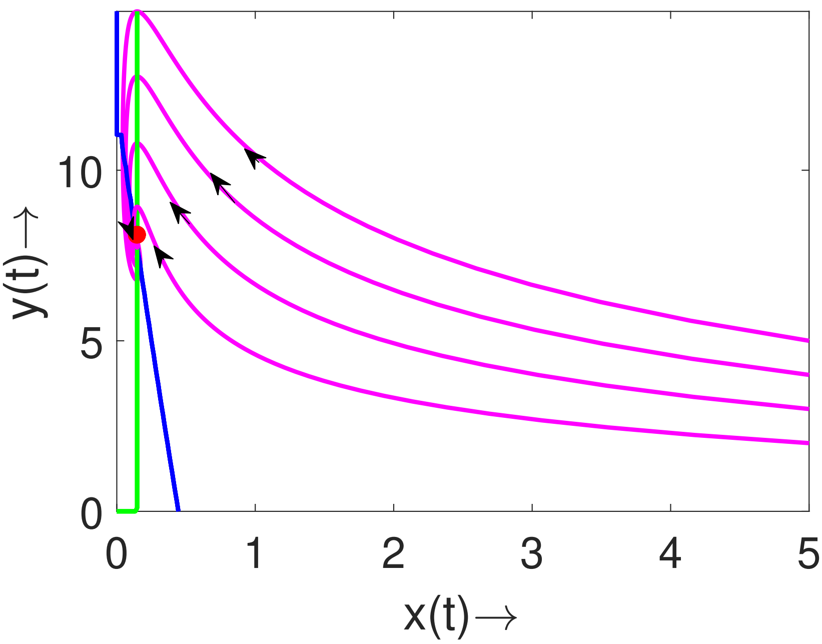

Fig. 5. Phase portrait for $r = 2$

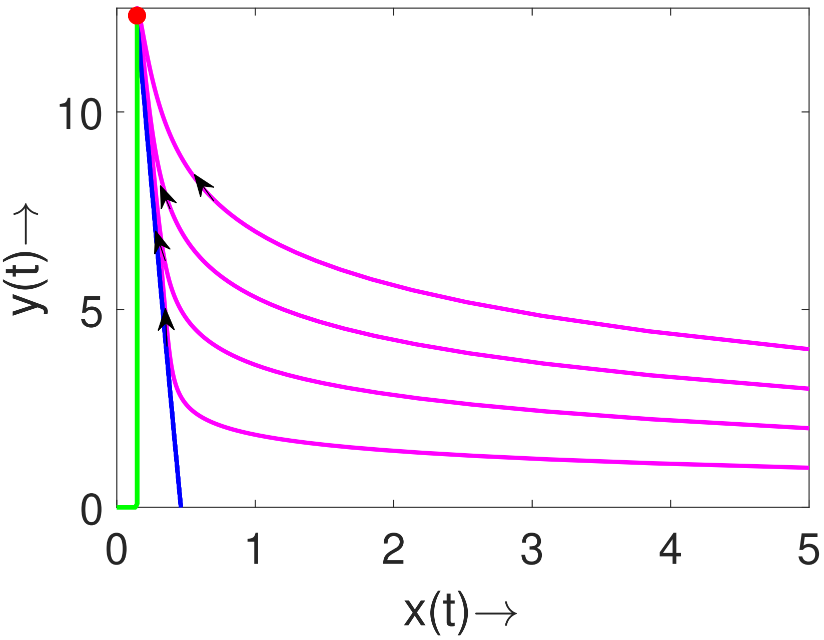

Fig. 6. Phase portrait for $r = 3$

6. Conclusions and Future Work

Stability of equilibrium points for a model with Smith growth rate in presence of environmental toxins has been analyzed and existence of transcritical and Hopf Bifurcation has been established with graphical representation. In future this model can be generalized to multi prey and multi predator model with more than one contamination factors which have different impacts on different species.

References

[1] Holling, C. S., The functional response of predators to prey density and its role in mimicry and population regulation, Mem. Entomol. Soc.,Can., 97(1965), 05-60.
[2] Maiti, T., Mondal, B. and Sarkar, A., Dynamic Interplay: Cooperative hunting strategies in a predator-prey model with Smith growth dynamics, Brazilian J. Physics, 54(2024):82, 11pp.
[3] Mandal, A., Tiwari, P. K., and Pal, S., Impact of waarness on environmental toxins affecting plankton dynamics: A mathematical implication, J. Appl. Math. Comput., 66(2021), 369-395.
[4] Perko, L., Differential Equations and Dynamical Systems, Springer Verlag, 2013.
[5] Sivakumara, M., Sambath, M. and Balachandran, K., Stability and Hopf bifurcation analysis of a diffusive predator-prey model with Smith growth, Int. J. Biomath., 8(2015), 1550013.
[6] Smith, F. E., Population dynamics in Daphina Magna and a new model for population growth, Ecology, 44(1963), 651-663.
[7] Times of India, August 4, 2024.