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Research Paper

A note on fuzzy PS-ideals in PS-algebra and its level subsets

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Abstract

In this paper, a new notion, named fuzzification of PS – Algebra, which is a generalization of BCK/BCI/TM/BH/Q/d/KU-algebras, is introduced, along with PS-ideal and we have discussed some of their properties in detail.

Keywords: *PS-Algebra, PS-ideal, Fuzzy PS-Ideal, Level Subsets, PS-Subalgebra and Fuzzy PS-Subalgebra.*

1. Introduction

The concept of fuzzy set was initiated by L.A.Zadeh in 1965 [15]. Since then these ideas have been applied to other algebraic structures such as groups, rings, modules, vector spaces and topologies. K.Iseki and S.Tanaka [2] introduced the concept of BCK-algebras in 1978 and K.Iseki [3] introduced the concept of BCI-algebras in 1980. It is known that the class of BCK –algebras is a proper subclass of the class of BCI algebras. J.Neggers and H.S.Kim introduced a notion called B-algebra in 2002. T.Priya and T.Ramachandran [8-13] introduced the new algebraic structure, PS-algebra, which is an another generalization of BCI / BCK/Q /d/ KU algebras and investigated its properties in detail. In this paper we introduce a new notion, called fuzzification of PS-algebra, which is a generalization of BCK / BCI / BH/Q /d /TM / KU algebras, and investigate some of its properties.

2. Preliminaries

In this section we site the fundamental definitions that will be used in the sequel.
Definition 2.1 [2]: *A BCK- algebra is an algebra (X,*, 0) of type (2, 0) satisfying the following conditions:*
i) $(x * y) * (x * z) \leq (z * y)$
ii) $x * (x * y) \leq y$
iii) $x \leq x$
iv) $x \leq y$ and $y \leq x \Rightarrow x=y$
v) $0 \leq x \Rightarrow x=0$, where $x \leq y$ is defined by $x * y = 0$,for all $x, y, z \in X$

Definition 2.2 [3]: *A BCI- algebra is an algebra (X,*, 0) of type (2, 0) satisfying the following conditions:*
i) $(x * y) *(x * z) \leq (z*y)$
ii) $x * (x * y) \leq y$
iii) $x \leq x$
iv) $x \leq y$ and $y \leq x \Rightarrow x = y$
v) $x \leq 0 \Rightarrow x = 0$, where $x \leq y$ is defined by $x * y = 0$, for all $x, y, z \in X$.

Definition 2.3 [5]: *A Q- algebra is an algebra (X,*, 0) of type (2, 0) satisfying the following conditions:*
i) $x * x = 0$
ii) $x * 0 = x$

iii) $(x * y)*z = (x * z) * y$, where $x \leq y$ is defined by $x * y = 0$, for all $x, y, z \in X$.

Definition 2.4 [6]: *A d- algebra is an algebra (X,*, 0) of type(2,0) satisfying the following conditions:*
i) $x * x = 0$
ii) $0 * x = 0$
iii) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$.

Definition 2.5 [7,14]: *A KU- algebra is an algebra (X,*,0) of type(2,0) satisfying the following conditions:*
i) $(x * y) * ((y * z) * (x * z)) =0$
ii) $x * 0 = 0$
iii) $0 * x = x$
iv) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y, z \in X$.

Remark:
- Every BCK-algebra is a TM-algebra but not the converse.
- Every BCK-algebra is a BCI-algebra but not the converse.
- Every BCI-algebra is a BCH-algebra but not the converse.
- Every BCH-algebra is a Q-algebra but not the converse.
- Every TM-algebra is a BH-algebra but not the converse.
- Every BCK-algebra is a d-algebra but not the converse.

Definition 2.6 [8]: *Let S be a non-empty subset of an algebra X , then S is called a subalgebra of X if* $x * y \in S$, *for all* $x, y \in S$.

Definition 2.7 [15]: *Let X is a non-empty set. A fuzzy subset* μ *of the set X is a mapping* $\mu : X \rightarrow [0, 1]$.

Definition 2.8 [9]: *Let* μ *be a fuzzy set of X. For a fixed* $t \in [0, 1]$, *the set* $\mu^t = \{x \in X / \mu(x) \geq t\}$ *is called the upper level subset of* μ. *Clearly* $\mu^t \cup \mu_t = X$ *for* $t \in [0,1]$ *if* $t_1 < t_2$, *then* $\mu_{t1} \subseteq \mu_{t2}$.

3. Fuzzy PS-ideal and Fuzzy PS-Sub algebra

Definition 3.1 (PS-algebra): *A nonempty set X with a constant 0 and a binary operation ' * ' is called PS – Algebra if it satisfies the following axioms.*
1. $x * x = 0$
2. $x * 0 = 0$
3. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$, $\forall\ x, y \in X$.

In X, we define a binary relation $\leq$ by $x \leq y$ if and only if $y * x = 0$.
In any PS-algebra (X, *, 0), the following holds good for all $x, y \in X$.
1. $x *(y*x)= y * (x *x)$
2. $y * (x * (y* x)) = 0$
3. $x * (x * (x*y)) = x * y$
4. $y * (x * (x * y)) = 0$

Example 3.1: Let X = {0, a, b, c} be the set with the following Cayley table.

*	0	a	b	c
0	0	b	a	c
a	0	0	0	b
b	0	0	0	b
c	0	b	b	0

Then (X, *, 0) is a PS – algebra.
Remark: Every KU algebra is a PS-algebra but not the converse, since $(a*0)*((0*c)*(a*c)) = a \neq 0$.

Definition 3.2: *Let X be a PS-algebra and I be a subset of X, then I is called a PS-ideal of X if it satisfies the following conditions:*
1. $0 \in I$
2. $y * x \in I$ and $y \in I \Rightarrow x \in I$

Definition 3.3: *Let X be a PS-algebra. A fuzzy set μ in X is called a fuzzy PS-ideal of X if it satisfies the following **conditions**.*

i) $\mu(0) \geq \mu(x)$

ii) $\mu(x) \geq \min\{\mu(y * x), \mu(y)\}$, for all x, y $\in$ X

Definition 3.4: *A fuzzy set μ in a PS-algebra X is called a fuzzy PS- sub algebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all x, y$\in$ X.*

Theorem 3.1: *Every fuzzy PS-ideal of a PS-algebra X is order reversing.*

Proof: *Let μ be a fuzzy PS-ideal of a PS-algebra X and let x, y $\in$ X be such that* $x \leq y$, then $y * x = 0$

Now $\mu(x) \geq \min\{\mu(y * x), \mu(y)\}$

$= \min\{\mu(0), \mu(y)\}$

$= \{\mu(y)\}$

$\Rightarrow \mu(x) \geq \mu(y)$

Theorem 3.2: *If μ is a fuzzy PS-ideal then it satisfies the condition $\mu(x * (y * x)) \geq \mu(y)$.*

Proof : *Let μ be a fuzzy PS-ideal. Then*

$\mu(x * (y * x)) \geq \text{Min}\{\mu(y * (x * (y * x))), \mu(y)\}$

$= \text{Min}\{\mu(0), \mu(y)\}$

$= \mu(y)$.

Theorem 3.3: *Let X be a PS-algebra. μ is a fuzzy PS-ideal of X iff μ is a fuzzy PS-subalgebra of X.*

Proof: *By definition, every fuzzy PS-ideal of a PS-algebra X is a fuzzy PS-subalgebra of X.*

Let μ be a fuzzy PS-ideal.

To prove**:** μ is a fuzzy PS- subalgebra of X.

By definition of PS-ideal, $\mu(x) \geq \min\{\mu(y * x), \mu(y)\}$, for all x, y $\in$ X

Now $\mu(x * y) \geq \min\{\mu(y * (x * y), \mu(y)\}$,

$= \min\{\mu(0), \mu(y)\}$

$\geq \min\{\mu(x), \mu(y)\}$

$\Rightarrow$μ is a fuzzy PS- subalgebra of X.

Conversely, let μ be a fuzzy PS-subalgebra of X.

To prove: μ is a fuzzy PS-ideal of X

Now $\mu(0) = \mu(x * x)$

$\geq \min\{\mu(x), \mu(x)\}$

$= \mu(x)$

$\Rightarrow \mu(0) \geq \mu(x)$

And $\mu(x) \geq \mu(y)$

$= \min\{\mu(0), \mu(y)\}$

$= \min\{\mu(y * x), \mu(y)\}$

$\Rightarrow \mu(x) \geq \min\{\mu(y * x), \mu(y)\}$

Hence μ is a fuzzy PS-ideal of X.

Theorem 3.4: *The intersection of any set of fuzzy PS-ideal in PS-algebra X is also a fuzzy PS-ideal.*

Proof: *Let $\{\mu_i\}$ be a family of fuzzy PS-ideals of PS-algebras X.*

Then for any x, y $\in$ X.

$(\cap \mu_{i)}(0) = \text{Inf}(\mu_i(0))$

$\geq \text{Inf}(\mu_i(x))$

$= (\cap \mu_{i)}(x)$

And $(\cap \mu_{i)}(x) = \text{Inf}(\mu_i(x))$

$\geq \text{Inf}\{\min\{\mu_i(y * x), \mu_i(y)\}\}$

$= \min\{\text{Inf}(\mu_i(y * x)), \text{Inf}(\mu_i(y))\}$

$= \min\{(\cap \mu_{i)}(y * x), (\cap \mu_{i)}(y)\}$

This completes the proof.

Theorem 3.5: *A fuzzy set μ of a PS-algebra X is a fuzzy PS- subalgebra iff for every t$\in$[0,1], μ^t is either empty or a subalgebra of X.*

Proof: *Assume that μ is a fuzzy PS- sub algebra of X and $\mu^t \neq \phi$*

Then for any x, y $\in$ μ^t, we have

$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} = t$
There fore $x * y \in \mu^t$
Hence μ^t is a sub algebra of X.
Conversely, assume that μ^t is subalgebra of X.
Let $x, y \in X$. Take $t = \min\{\mu(x), \mu(y)\}$
Then by assumption μ^t is a sub algebra of X, $x * y \in \mu^t$
$\mu(x * y) \geq t = \min\{\mu(x), \mu(y)\}$
Hence μ is a fuzzy PS- sub algebra of X.

Theorem 3.6: *Any sub algebra of a PS – algebra X can be realized as level sub algebra of some fuzzy PS-sub algebra of X.*
Proof: *Let μ be a sub algebra of the given PS– algebra X and let μ be a fuzzy set in X defined by*
$\mu(x) = t$, if $x \in A$
0, if $x \notin A$.
where $t \in [0, 1]$ is fixed. It is clear that $\mu^t = A$.
Now we prove such defined μ is a fuzzy PS- sub algebra of X.
Let $x, y \in X$. If $x, y \in A$, then $x*y \in A$.
Hence $\mu(x) = \mu(y) = \mu(x*y) = t$ and $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$
If $x, y \notin A$, then $\mu(x) = \mu(y) = 0$ and $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} = 0$.
If at most one of $x, y \in A$, then at least one of $\mu(x)$ and $\mu(y)$ is equal to 0.
Therefore, $\min\{\mu(x), \mu(y)\} = 0$ so that $\mu(x * y) \geq 0$, which completes the proof.
As a generalisation of theorem 3.6, we prove the following theorem.

Theorem 3.7 : *Let X be à PS- algebra. Then given any chain of subalgebra $S_0 \subset S_1 \subset S_2 \subset \ldots\ldots\ldots\ldots \subset S_r = X$,* there exists a fuzzy PS-subalgebra μ of X whose level subalgebras are exactly the sualgebras of this chain.
Proof : *Consider a set of numbers $t_0 > t_1 > t_2 > \ldots\ldots\ldots > t_r$, where each $t_i \in [0,1]$.*
Let $\mu : X \to [0,1]$ be a fuzzy set defined by $\mu(s_0) = t_0$ and $\mu(s_i - s_{i-1}) = t_i$, $0 < i \leq r$.
We claim that μ is a fuzzy PS-subalgebra of X.Let $x,y \in X$. Then we classify it into two cases as follows :
Case (1)
Let $x, y \in s_i - s_{i-1}$. Then by the definition of μ,
$\mu(x) = t_i = \mu(y)$.
Since S_i is a subalgebra,it follows that $x * y \in S_i$, and so either $x * y \in S_i - S_{i-1}$ (or) $x * y \in S_{i-1}$
In any case, we conclude that
$\mu(x * y) \geq t_i = \min\{\mu(x), \mu(y)\}$.
Case (2)
For $i > j$,
Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$.
Then $\mu(x) = t_i$; $\mu(y) = t_j$ and $x * y \in S_i$, since S_i is a subalgebra of X and $S_j \subset S_i$.
Hence $\mu(x * y) \geq t_j = \min\{\mu(x), \mu(y)\}$
Thus μ is a fuzzy PS-subalgebra of X.
From the definition of μ,it follows that $Im(\mu) = \{t_0, t_1, t_2, \ldots\ldots\ldots t_r\}$.
Hence the level subalgebras of μ are given by the chain of subalgebras.
$\mu_{t0} \subset \mu_{t1} \subset \mu_{t2} \subset \ldots\ldots\ldots\ldots \subset \mu_{tr} = X$.
Now $\mu_{t0} = \{x \in X / \mu(x) \geq t_0\} = S_0$.
Finally,we prove that $\mu_{ti} = S_i$ for $0 < i \leq r$.
Clearly $S_i \subseteq \mu_{ti}$.
If $x \in \mu_{ti}$, then $\mu(x) \geq t_i$ which implies that $x \notin S_j$ for $j > i$.
Hence $\mu(x) \in \{t_1, t_2, \ldots\ldots\ldots t_i\}$ and so $x \in S_k$ for some $k \leq i$.
As $S_k \subseteq S_i$, it follows that $x \in S_i$.
$\Rightarrow \mu_{ti} = S_i$ for $0 < i \leq r$.
This completes the proof.

Theorem 3.8: *Two level sub algebras μ^s, μ^t $(s < t)$ of a fuzzy PS- sub algebras are equal iff there is no $x \in X$ such that $s \leq \mu(x) < t$.*
Proof: *Let $\mu^s = \mu^t$ for some $s < t$. If there exist $x \in X$ such that $s \leq \mu(x) < t$, then μ^t is a proper subset of μ^s, which is a contradiction.*
Conversely, assume that there is no $x \in X$ such that $s \leq \mu(x) < t$.
Since $s < t$, $\mu^t \subseteq \mu^s$.
If $x \in \mu^s$, then $\mu(x) \geq s$ and so $\mu(x) \geq t$, because $\mu(x)$ does not lie between s and t.

Hence $x \in \mu^t$, which gives $\mu^s \subseteq \mu^t$.
This completes the proof.

Theorem 3.9: *Let μ be a fuzzy set in a PS-algebra X and let $t \in Im(\mu)$. Then μ is a fuzzy PS-ideal of X if and only if the level subset μ^t is a PS-ideal of X, which is called a level PS-ideal of X.*
Proof: *Assume that μ is a fuzzy PS-ideal of X. Clearly $0 \in \mu^t$.*
Let $y * x \in \mu^t$ and $y \in \mu^t$. Then $\mu(y * x) \geq t$ and $\mu(y) \geq t$
Now $\mu(x) \geq \min \{\mu(y * x), \mu(y)\}$
$\geq \min \{t, t\}$
$= t$
Hence the level subset μ^t is a PS-ideal of X.
Conversely assume that, the level subset μ^t is a PS-ideal of X, for any $t \in [0,1]$.
Suppose assume that there exist some $x_0 \in X$ such that $\mu(0) < \mu(x_0)$
Take $s = \frac{1}{2} [\mu(0) + \mu(x_0)]$
$\Rightarrow \mu(0) < s < \mu(x_0)$
$\Rightarrow x_0 \in \mu^s$ and $0 \notin \mu^s$, a contradiction, since μ^s is a PS-ideal of X.
Therefore, $\mu(0) \geq \mu(x)$ for all $x \in X$.
If possible, assume that $x_0, y_0 \in X$ such that $\mu(x_0) < \min \{ \mu(y_0 * x_0), \mu(y_0)\}$.
Take $s = \frac{1}{2} [\mu(x_0) + \min \{ \mu(y_0 * x_0), \mu(y_0)\}]$
$\Rightarrow s > \mu(x_0)$ and $s < \min \{\mu(y_0 * x_0), \mu(y_0)\}$.
$\Rightarrow s > \mu(x_0)$, $s < \mu(y_0 * x_0)$ and $s < \mu(y_0)$.
$\Rightarrow x_0 \notin \mu^s$, a contradiction, since μ^s is a PS-ideal of X.
Therefore $\mu(x) \geq \min \{\mu(y * x), \mu(y)\}$, for any $x, y \in X$.

Theorem 3.10 : *Let X be a PS-algebra & μ be a fuzzy PS-subalgebra of X. If Im(μ) is finite, say $\{t_1, t_2, \ldots, t_r\}$, then for any $t_i, t_j \in Im(\mu)$, $\mu_{t_i} = \mu_{t_j}$, implies $t_i = t_j$.*
Proof : *Assume that $t_i \neq t_j$ say $t_i < t_j$.*
If $x \in \mu_{t_j}$ then $\mu(x) \geq t_j > t_i$, which implies that $x \in \mu_{t_i}$.
Let $x \in X$ be such that $t_i < \mu(x) < t_j$
Then $x \in \mu_{t_i}$, but $x \notin \mu_{t_j}$.
Hence $\mu_{t_j} \not\subset \mu_{t_i}$ and $\mu_{t_j} = \mu_{t_i}$, a contradiction.

4. Conclusion

In this article authors have been discussed fuzzy PS-ideal in fuzzy PS-algebra. The relationship between level subsets and subalgebra also established. It has been observed that PS-algebra as a generalization of BCK/BCI/Q/d/TM/KU-algebras. Interestingly, the chain concept adds another dimension to the defined PS-algebra. This concept can further be generalized to Intuitionistic fuzzy set, interval valued fuzzy sets, Lie algebra for new results in our future work.

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