International Journal of Advanced Mathematical Sciences, 2 (1) (2014) 71-80
©*Science Publishing Corporation*
www.sciencepubco.com/index.php/IJAMS
doi: 10.14419/ijams.v2i1.1761
Research Paper

S_s-Open sets and S_s-Continuous Functions

Alias B. Khalaf [1]*, Abdulrahman H. Majeed [2], Jamil M. Jamil [2]

[1] *University of Duhok, Faculty of Science, Department of Mathematics*
[2] *University of Baghdad, College of Science, Department of Mathematics*
**Corresponding author E-mail: aliasbkhalaf@gmail.com*

Abstract

In this paper we introduce and study the concept of S_s-open sets .also, a study new class of functions called $S_S$continuous functions, the relationships between S_s-continuity and other types of continuity are investigated.

Keywords: *S_s-open set, S_s-continuous function, semi-open set, semi-continuous functions.*

1. Introduction

In 1963, Levine [16], introduced the concept of semi-open set and semi continuity and gave several properties about these functions. Njastad [18] introduced the concepts of α-sets and Abd-El-Monsef et al [1] defined β-open sets and β-continuous functions. Khalaf and Ameen in [14], defined the concept of S_c-open sets and in 2012, Khalaf and Ahmed [15], introduced another type of semi-open sets called S_β-open sets. Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represents non-empty topological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset A of X, $Cl(A)$ and $Int(A)$ represents the closure and Interior of A respectively. A subset A is said to be preopen [17] (resp., α-open [18], semi-open [16], regular open [21], β-open [1]) set if $A \subseteq IntCl(A)$ (resp. $A \subseteq IntClInt(A)$, $A \subseteq ClInt(A)$, $A = IntCl(A)$, $A \subseteq ClIntCl(A)$). The complement of a preopen (resp., α-open , semi-open , regular open, β-open) set is called pre-closed (resp., α-closed, semi-closed, regular closed, β-closed) set. The intersection of all semi-closed sets containing A is called the semi-closure [5] of A and it is denoted by $sClA$. The semi-interior of a set A is the union of all semi-open sets contained in A and is denoted by $sIntA$. A subset A of a topological space (X, τ) is said to be θ-open [23] (resp., θ-semi-open [13], semi-θ-open [6]) set if for each $x \in A$, there is an open (resp., semi-open, semi-open) set U such that $x \in U \subseteq Cl(U) \subseteq A$ (resp., $x \in U \subseteq Cl(U) \subseteq A$, $x \in U \subseteq sCl(U) \subseteq A$). For more properties of semi-θ-open sets (see [24]) also. A subset A of a topological space X is said to be regular-semi-open [4] if there exists a regular-open set U such that $U \subseteq A \subseteq ClU$ equivalently A is regular-semi-open [22] if and only if $A = sIntsClA$. A set A is called semi-regular [12], if it is both semi-open and semi-closed. The family of all regular-semi-open (resp., θ-open, θ-semi-open, semi-θ-open, semi-regular) sets of X is denoted by $RSO(X)$ (resp., $\theta O(X)$, $\theta SO(X)$, $S\theta O(X)$, $SR(X)$). The aim of the present paper is to define a new type of sets, we call it S_s-open set. Since the families $SO(X)$ and $PO(X)$ are incomparable [17], so the it is obvious that the concept of S_s-open sets incomparable with S_p-open sets but it is strictly weaker than S_c-open sets and stronger than S_β-open sets.

2. Preliminaries

In this section, we recall the following definitions and results:

Lemma 2.1 *Let* (Y,τ_Y) *be a subspace of a space* (X,τ)*.*

1. If $A \in SO(X,\tau)$ and $A \subseteq Y$, then $A \in SO(Y,\tau_Y)$.[16]
2. *If* $A \in SO(Y,\tau_Y)$ *and* $Y \in SO(X,\tau)$*, then* $A \in SO(X,\tau)$*.[8]*

Lemma 2.2 *Let* A *be a subset of a space* X*, then the following properties hold.*

1. *If* $A \in SO(X)$*, then* $sCl(A) \in RSO(X)$ *[22]*
2. *If* $A \in SO(X)$*, then* $sCl(A) = sCl_\theta(A)$*. [3]*
3. *If* A *is open subset of* X*, then* $sCl(A) = IntCl(A)$*.[12]*

Lemma 2.3 *[9] For any topological space* X*. If* $A \in \alpha O(X)$ *and* $B \in SO(X)$*, then* $A \cap B \in SO(X)$*.*

Definition 2.4 *A semi-open subset* A *of a space* X *is called* S_c*-open [14] (resp.,* S_β*-open [15],* S_p*-open [20]) set if for each* $x \in A$*, there exists a closed set (resp.,* β*-closed, pre-closed set)* F *such that* $x \in F \subseteq A$*.*

Definition 2.5 *A topological space* (X,τ) *is called:*

1. *semi-*T_1 *[2], if for every two distinct points* x, y *in* X*, there exist two semi-open sets, one containing* x *but not* y *and the other containing* y *not* x*.*
2. *semi-regular [11], if for each* $x \in X$ *and each* $H \in SO(X)$ *containing* x*, there exists* $G \in SO(X)$ *such that* $x \in G \subseteq sCl(G) \subseteq H$*.*

Lemma 2.6 *[2] A space* X *is semi-*T_1 *, if and only if, the singleton set* $\{x\}$ *is semi closed for any point* $x \in X$*.*

Lemma 2.7 *The following properties hold:*

1. *If a space* X *is semi-regular, then each* $SO(X) = S\theta O(X)$*.*
2. *If a space* X *is semi-regular, then* $sCl(A) = sCl_\theta(A)$ *for each subset* A *of* X*.*

Proof. It is clear that each semi-θ-open is semi-open. If X is semi-regular space and if G is a non-empty semi-open set in X, the by Definition 2.5, there exists a semi-open set U such that $x \in U \subseteq sCl(U) \subseteq G$, this implies that G is semi-θ-open. Therefore, $SO(X) = S\theta O(X)$.
Part (2). Follows from part (1).

Definition 2.8 *A space* X *is locally indiscrete [9], if every open set is closed.*

Lemma 2.9 *[9] A space* X *is locally indiscrete if and only if every semi open set in* X *is closed.*

Definition 2.10 *[19] A function* $f : X \to Y$ *is said to be strongly* θ*-semi-continuous at a point* $x \in X$*, if for each open set* V *containing* $f(x)$*, there exists a semi-open set* U *containing* x *such that* $f(sCl(U)) \subseteq V$*.*
The function f *is said to be strongly* θ*-semi-continuous on* X *if it is strongly* θ*-semi-continuous at every point of* X *, we shall denote by* f *is st.sc on* X*.*

Definition 2.11 *[10] A function* $f : X \to Y$ *is said to be semi-continuous (resp., contra-semi-continuous) if the inverse image of every open set in* Y *is semi-open (resp., semi-closed) in* X*.*

Theorem 2.12 *[2] For any spaces* X *and* Y*. If* $A \subseteq X$ *and* $B \subseteq Y$ *then,*

1. $sInt_{X\times Y}(A \times B) = sInt_X(A) \times sInt_Y(B)$.
2. $sCl_{X\times Y}(A \times B) = sCl_X(A) \times sCl_Y(B)$.

3. S_s-Open Sets

In this section, we Introduce the concept of S_s-open sets in topological spaces.

Definition 3.1 *A semi-open subset A of a space X is called S_s-open if for each $x \in A$, there exists a semi-closed set F such that $x \in F \subseteq A$.*

The family of all S_s-open subsets of a topological space (X, τ) is denoted by $S_sO(X, \tau)$ or $S_sO(X)$.

Proposition 3.2 *A subset A of a space X is S_s-open if and only if $A = \cup F_\gamma$ where A is semi-open set and F_γ semi-closed set for each γ.*

Proof. Obvious.

Remark 3.3 *It is clear from the definition that every S_s-open subset of a space X is semi-open, but the converse is not true in general as it is shown in Example 3.11.*

Proposition 3.4 *If a space X is semi-T_1, then $S_sO(X) = SO(X)$.*

Proof. Follows from the fact that in a semi-T_1 space, every singleton set is semi-closed (Lemma 2.6).

Remark 3.5 *Since any union of semi-open sets is semi-open [16], so any union of S_s-open sets in a topological space (X, τ) is also S_s-open. The intersection of two S_s-open sets need not be S_s-open in general as it is shown by the following example:*

Example 3.6 *Consider the intervals $[0, 1]$ and $[1, 2]$ in R with the usual topology. Since R is T_1 space and hence it is semi-T_1, so by Proposition 3.4, both the intervals are S_s-open sets and we have $[0, 1] \cap [1, 2] = \{1\}$ which is not S_s-open.*

Proposition 3.7 *Every semi-θ-open subset of a space X is S_s-open.*

Proof. Suppose that the subset A of X is semi-θ-open, then clearly it is semi-open and by definition, for each $x \in A$, there exists a semi-open set U such that $x \in U \subseteq sClU \subseteq A$. Hence, $sClU$ is the semi-closed set containing x contained in A, so A is S_s-open.

The relation of S_s-open sets to some other types of sets is illustrated in the following remark:

Remark 3.8 *If X is any topological space, then the following properties hold:*

1. *Since every θ-semi-open subset of X is semi-θ-open, so, from Proposition 3.7, we obtain that every θ-semi-open set is S_s-open.*
2. *It is obvious that every semi-regular subset of X is S_s-open.*
3. *It is obvious that every s_c-open set is s_s-open.*
4. *Every s_s-open set is s_β-open, because every semi open set is β-open.*

Although not every open set is an S_s-open set as we can see in Example 3.11 but we have the following results:

Proposition 3.9 *Let (X, τ) be a semi regular space, then $\tau \subseteq S_sO(X)$.*

Proof. Let A be any non-empty open subset of X, then for each $x \in A$, there is a semi-open set G such that $x \in G \subseteq sCl(G) \subseteq A$ implies that $x \in sCl(G) \subseteq A$. Hence A is S_s-open.

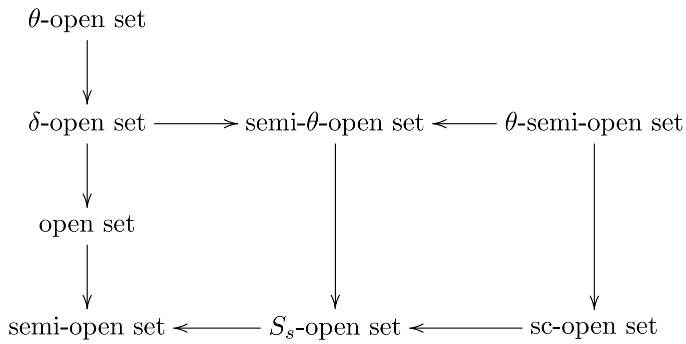
Proposition 3.10 *The following properties hold.*

1. *If A is a semi-open subset of a space X, then $sClA$ is S_s-open.*
2. *If A is a semi-closed subset of a space X, then $sIntA$ is S_s-open.*
3. *$sClsIntA$ is S_s-open subset, for every subset A of X.*
4. *$sIntsClA$ is S_s-open subset, for every subset A of X.*

5. *Every regular semi-open subset of X is S_s-open.*

Proof. (1) For any subset A of X, we have $sClA = A \cup IntClA$ [2]. Hence $sClA$ is both semi-open and semi-closed, so it is S_s-open.
The proof of parts (2), (3), (4) and (5) are similar.

We get the following diagram of implications:



The following examples show that the above implications are not reversible.

Example 3.11 *Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then we have:*
$SO(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, and hence $SC(X) = \{\phi, \{c\}, \{b\}, \{b, c\}, X\}$.
So, $S_sO(X) = \{\phi, X\}$ implies that the set $\{a\} \in SO(X)$, but $\{a\} \notin S_sO(X)$.

Example 3.12 *Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then :*
$SO(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\{a, b, c\}, \{a, b, d\}, \{b, c, d\}\{a, c, d\}, X\}$.
Hence, the set $\{a, d\}$ is an S_s-open set which is not S_c-open.

Example 3.13 *Consider $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, X\}$. Then we can easily find the following families of sets:*
$SO(X) = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{b, c, d\}\{a, c, d\}, X\}$, also $SC(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, X\}$.
Hence, the set $\{a, c, d\}$ is an S_s-open set which is not θ-semi-open set also it is not semi-regular set.

Proposition 3.14 *For any space X, $sCl(sInt(\{x\})) = \{x\}$ if and only if $\{x\}$ is S_s-open.*

proof. Let $sCl(sInt(\{x\})) = \{x\}$, this implies that $sInt(\{x\} = \{x\}$ and so, $\{x\}$ is both semi-open and semi-closed, then $\{x\}$ is semi regular open. Hence, $\{x\} \in S_sO(X)$.
Conversely. Let $\{x\}$ be an S_s-open set in X, then there exists a semi-closed F such that $x \in F \subseteq \{x\}$, this implies that $x \in \{x\} \subseteq \{x\}$, so, $\{x\}$ is semi-open and semi-closed. Therefore, $sCl(sInt(\{x\})) = \{x\}$.

Proposition 3.15 *Let (x, τ) be a topological space. Then $\{x\} \in S_so(X)$ if and only if it is semi-regular.*

proof. If $\{x\}$ is semi-regular, then, by Remark 3.8, $\{x\} \in S_sO(X)$.
Conversely. Suppose $\{x\}$ is $S_sO(X)$, then $\{x\}$ is semi-open and by definition it is semi-closed. Hence, $\{x\}$ is semi-regular.

Proposition 3.16 *A subset A of a space (X, τ) is S_s-open if and only if for each $x \in A$, there exists an S_s-open set B such that $x \in B \subseteq A$.*

proof. If A is an S_s-open subset in the space (X, τ), then for each $x \in A$, putting $A = B$, which is S_s-open containing x such that $x \in B \subseteq A$.
Conversely. Suppose that for each $x \in A$, there exists a S_s-open set B such that $x \in B \subseteq A$. So, $A = \cup B_\gamma$ where $B_\gamma \in S_sO(X)$ for each γ. Therefore, by Remark 3.5, A is S_s-open.

Proposition 3.17 *Let X be a topological space, and $A, B \subseteq X$. If $A \in S_sO(X)$ and B is both α-open and semi-closed, then $A \cap B \in S_sO(X)$.*

proof. Let $A \in S_sO(X)$ and B be α-open, then A is semi open set, so by Lemma 2.3, we have $A \cap B \in SO(X)$. Now let $x \in A \cap B$, then $x \in A$ and therefore, there exists a semi-closed set F such that $x \in F \subseteq A$. Since B is semi-closed, so $F \cap B$ is semi-closed set. Hence, $x \in F \cap B \subseteq A \cap B$. Thus $A \cap B$ is S_s-open set in X.

Proposition 3.18 *Let (Y,τ_Y) be an α-open subspace of a space (X,τ). If $A \in S_sO(X,\tau)$ and $A \subseteq Y$, then $A \in S_sO(Y,\tau_Y)$.*

proof. Let $A \in S_sO(X,\tau)$, then $A \in SO(X,\tau)$ and for each $x \in A$, there exists a semi-closed set F in X such that $x \in F \subseteq A$. Since $A \in SO(X,\tau)$ and $A \subseteq Y$, so, by Lemma 2.1, $A \in SO(Y,\tau_Y)$. Since F semi-closed set in X, then $X \setminus F$ is semi-open and hence, by Lemma 2.3, $Y \cap (X \setminus F)$ is semi-open in X. So, by Lemma 2.1, $Y \cap (X \setminus F)$ is semi-open in Y. Therefore, $F = Y \setminus (Y \cap X \setminus F)$ is semi-closed set in Y. Hence $A \in S_sO(Y,\tau_Y)$.

Proposition 3.19 *Let Y be a semi-regular set in a space (X,τ). If $A \in S_sO(Y,\tau_Y)$, then $A \in S_sO(X,\tau)$.*

proof. Let $A \in S_sO(Y,\tau_Y)$, then $A \in SO(Y,\tau_Y)$ and for each $x \in A$, there exists a semi-closed set F in Y such that $x \in F \subseteq A$. Since $A \in SO(Y,\tau_Y)$ and Y is semi-regular. So, by Lemma 2.1, $A \in SO(X,\tau)$. Since F semi-closed set in Y, then $Y \setminus F$ is semi-open in Y and also, by Lemma 2.1, $Y \setminus F$ is semi-open in X. Again Y is semi-regular in X implies that $X \setminus Y$ is semi-open. Hence, $Y \setminus F \cup X \setminus Y = X \setminus F$ is semi-open in X. So, F is semi-closed in X. Therefore, $A \in S_sO(X,\tau)$.

Definition 3.20 *Let A be a subset of a topological space (X,τ).*

1. *The union of all S_s-open sets which are contained in A is called the S_s-interior of A and is denoted by $S_sInt(A)$.*
2. *The intersection of all S_s-closed sets containing A is called the S_s-closure of A and we denote it by $S_sCl(A)$.*
3. *The S_s-boundary of A is $S_sCl(A) \setminus S_sInt(A)$ and is denoted by $S_sBd(A)$.*

Proposition 3.21 *Let A be any subset of a space X. If a point x is in the S_s-interior of A, then there exists a semi closed set F of X containing x such that $F \subseteq A$.*

proof. Suppose that $x \in S_sInt(A)$, then there exists a S_s-open set U of X containing x such that $U \subseteq A$. Since U is S_s-open set, so there exists a semi closed set F containing x such that $F \subseteq U \subseteq A$. Hence $x \in F \subseteq A$.

Proposition 3.22 *For any subset A of a topological space X. The following statements are true:*

1. *$S_sInt(A)$ is the largest S_s-open set contained in A.*
2. *A is S_s-open if and only if $A = S_sInt(A)$.*
3. *$S_sCl(A)$ is the smallest S_s-Closed set in X containing A.*
4. *A is S_s-closed set if and only if $A = S_sCl(A)$.*

Some other properties of S_s-interior of a set A are in the following result:

Theorem 3.23 *If A and B are any subsets of a topological space (X,τ), then the following properties hold:*

1. *if $A \subseteq B$, then $S_sInt(A) \subseteq S_sInt(B)$ and $S_sCl(A) \subseteq S_sCl(B)$.*
2. *$S_sInt(A) \cup S_sInt(B) \subseteq S_sInt(A \cup B)$.*
3. *$S_sInt(A) \cap S_sInt(B) \subseteq S_sInt(A \cap B)$.*
4. *$S_sCl(A) \cup S_sCl(B) \subseteq S_sCl(A \cup B)$.*
5. *$S_sCl(A \cap B) \subseteq S_sCl(A) \cap S_sCl(B)$.*

proof. Obvious.

In general, $S_sInt(A) \cup S_sInt(B) \neq S_sInt(A \cup B)$. and $S_sInt(A) \cap S_sInt(B) \neq S_sInt(A \cap B)$. Also, the equalities in (4) and (5) does not hold as shown in the following example:

Example 3.24 *Consider the space (X, τ) defined in Example 3.13, then we have the following cases:*

1. *if $A = \{a, c\}$ and $B = \{a, d\}$, then $S_sInt(A) = \{a\}$, $S_sInt(B) = \{a\}$. Hence, $S_sInt(A) \cup S_sInt(B) = \{a\}$ and $S_sInt(A \cup B) = S_sInt(\{a, b, c\}) = \{a, b, c\}$. It follows that $S_sInt(A) \cup S_sInt(B) \neq S_sInt(A \cup B)$.*

2. *If $A = \{a, b\}$ and $B = \{b, c, d\}$, then $S_sInt(A) = \{a, b\}$, $S_sInt(B) = \{b, c, d\}$, so $S_sInt(A) \cap S_sInt(B) = \{b\}$ and $S_sInt(A \cap B) = S_sInt(\{b\}) = \phi$. It follows that $S_sInt(A) \cap S_sInt(B) \neq S_sInt(A \cap B)$.*

3. *If $A = \{a\}$ and $B = \{c, d\}$, then $S_sCl(A) = F$, $S_sCl(B) = B$. Hence, $S_sCl(A) \cup S_sCl(B) = \{a, c, d\}$, and $S_sCl(A \cup B) = S_sCl(\{a, c, d\}) = X$. It follows that $S_sCl(A) \cup S_sCl(B) \neq S_sCl(A \cup B)$.*

4. *If $A = \{a, c, d\}$ and $B = \{b, c, d\}$, then $S_sCl(A) = X$ and $S_sCl(B) = B$, so $S_sCl(A) \cap S_sCl(B) = B$, and $S_sCl(A \cap B) = S_sCl(\{c, d\}) = \{c, d\}$. It follows that $S_sCl(A \cap B) \neq S_sCl(A) \cap S_sCl(B)$.*

Proposition 3.25 *Let A be a subset of a topological space X. Then $x \in S_sCl(A)$ if and only if for any S_s-open set U containing x, $U \cap A \neq \phi$.*

proof. Let $x \in S_sCl(A)$ and suppose that $U \cap A = \phi$ for some S_s-open set U which contains x. Then $(X \setminus U)$ is S_s-closed set and $A \subseteq (X \setminus U)$, thus $S_sCl(A) \subseteq (X \setminus U)$. But this implies that $x \in (X \setminus U)$, which is contradiction. Therefore $U \cap A \neq \phi$.
Conversely. Suppose that there exists an S_s-open set containing x with $A \cap U = \phi$, then $A \subseteq X \setminus U$ and $X \setminus U$ is an S_s-closed with $x \notin X \setminus U$. Hence, $x \notin S_sCl(A)$.

Proposition 3.26 *For any subset A of a topological space X. The following statements are true.*

1. $X \setminus S_sCl(A) = S_sInt(X \setminus A)$.

2. $S_sCl(A) = X \setminus S_sInt(X \setminus A)$.

3. $X \setminus S_sInt(A) = S_sCl(X \setminus A)$.

4. $S_sInt(A) = X \setminus S_sCl(X \setminus A)$.

proof. We only prove (1), and the other parts can be proved similarly. (1) For any point $x \in X$, if $x \in X \setminus S_sCl(A) \Leftrightarrow x \notin S_sCl(A) \Leftrightarrow$ for each $B \in S_sO(X)$ containing x, we have $A \cap B = \phi \Leftrightarrow x \in B \subseteq X \setminus A \Leftrightarrow x \in S_sInt(X \setminus A)$.

Theorem 3.27 *If A is a subset of a topological space X. Then $Int_\theta(A) \subseteq sInt_\theta(A) \subseteq S_sInt(A) \subseteq sInt(A) \subseteq A \subseteq sCl(A) \subseteq S_sCl(A) \subseteq sCl_\theta(A) \subseteq Cl_\theta(A)$.*

proof. Obvious.

Proposition 3.28 *Let A be any subset of a space X. If $A \in S_sO(X)$, then $sCl_\theta(A) \subseteq S_sCl(A)$.*

proof. Assume that $x \notin S_sCl(A)$, then there exists an S_s-open set U containing x such that $A \cap U = \phi$ and $A \cap S_sCl(U) = \phi$ since $A \in S_sO(X)$, but $sCl(U) \subseteq S_sCl(U)$ implies that $A \cap sCl(U) = \phi$ and hence $x \notin sCl_\theta(A)$.

Proposition 3.29 *Let (X, τ) be a semi regular space and A be any subset of X. Then, $sCl_\theta(A) = S_sCl(A) = sCl(A)$.*

proof. From Theorem 2.6, we have $sCl(A) = sCl_\theta(A)$, so we get that $sCl_\theta(A) = S_sCl(A) = sCl(A)$.

4. S_s-Continuous Functions

Definition 4.1 *A function $f : X \to Y$ is called S_s-continuous at a point $x \in X$, if for each open set V of Y containing $f(x)$, there exists an S_s-open set U of X containing x such that $f(U) \subseteq V$.*
If f is S_s-continuous at every point x of X, then it is called S_s-continuous.

Proposition 4.2 A function $f : X \to Y$ is S_s-continuous if and only if the inverse image of every open set in Y is an S_s-open in X.

proof. Let f be S_s-continuous, and V be any open set in Y. If $f^{-1}(V) \neq \phi$, then there exists $x \in f^{-1}(V)$ which implies $f(x) \in V$. Since, f is S_s-continuous, there exists an S_s-open set U in X containing x such that $f(U) \subseteq V$. This implies that $x \in U \subseteq f^{-1}(V)$. This shows that $f^{-1}(V)$ is S_s-open.
Conversely, let V be any open set in Y, $f(x) \in V$, then $x \in f^{-1}(V)$. By hypothesis, $f^{-1}(V)$ is an S_s-open set in X containing x, thus $f(f^{-1}(V)) \subseteq V$. Therefore, f is S_s-continuous.

Proposition 4.3 If a function $f : X \to Y$ is strongly θ-semi-continuous, then f is S_s-continuous.

proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. Since, f is strongly θ-semi-continuous, then, there exists a semi-open set G in X containing x such that $f(sCl(G)) \subseteq V$. Hence, by Proposition 3.10(1), $sCl(G)$ is an S_s-open set. Therefore, f is S_s-continuous.

Corollary 4.4 If a function $f : X \to Y$ is strongly θ-continuous, then f is S_s-continuous.

proof. Follows from Remark 3.4 of [19] and Proposition 4.3.

The following example shows that the converse of Corollary 4.4 is not true in general.

Example 4.5 *Let* $X = \{a, b, c, d\}$ *equipped with the two topologies* $\tau = \sigma = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, X\}$.
If $f : (X, \tau) \to (X, \sigma)$ *is the identity function, then* f *is* S_s*-continuous, but it is not strongly* θ*-continuous because* $f^{-1}(\{a, c, d\}) = \{a, c, d\}$ *which is not* θ*-open.*

The proof of the following result follows directly from their definitions.

Corollary 4.6 Every S_s-continuous function is semi-continuous.

proof. Obvious.

Example 4.7 *Let* $X = \{a, b, c\}$ *with the topology* $\tau = \sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. *Let* $f : (X, \tau) \to (X, \sigma)$ *be the identity function. Then,* f *is semi-continuous, but it is not* S_s*-continuous, because* $\{a\}$ *is an open set in* (X, σ) *and* $f^{-1}(\{a\})$ *is not* S_s*-open.*

Corollary 4.8 If a function $f : X \to Y$ is both semi-continuous and contra-semi-continuous, then it is S_s-continuous.

proof. Follows from Definition 2.11 and Proposition 4.2.

Remark 4.9 *The function* f *in Example 4.5 is not contra-semi-continuous.*

Proposition 4.10 A function $f : X \to Y$ is S_s-continuous if and only if f is semi-continuous and for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a semi-closed set G of X containing x such that $f(G) \subseteq V$.

proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. By hypothesis, there exists an S_s-open set U of X containing x such that $f(U) \subseteq V$. Since U is S_s-open, then for each $x \in U$, there exists a semi-closed set G of X such that $x \in G \subseteq U$. Therefore, we have $f(G) \subseteq V$.
Conversely, let V be any open set of Y. It should be shown that $f^{-1}(V)$ is S_s-open set in X. Since, f is semi-continuous, then $f^{-1}(V)$ is semi-open set in X. Let $x \in f^{-1}(V)$, then $f(x) \in V$. By hypothesis, there exists a semi-closed set G of X containing x such that $f(G) \subseteq V$, which implies that $x \in G \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is S_s-open in X. Hence, by Proposition 4.2, f is S_s-continuous.

5. Characterizations and Properties

In this section, we give some characterizations and properties of S_s-continuous functions and we start with the following result.

Proposition 5.1 For a function $f : X \to Y$, the following statements are equivalent:

1. f is S_s-continuous.
2. $f^{-1}(V)$ is an S_s-open set in X, for each open set V of Y.
3. $f^{-1}(F)$ is an S_s-closed set in X, for each closed set F of Y.
4. $f(S_sCl(A)) \subseteq Cl(f(A))$, for each subset A of X.
5. $S_sCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for each subset B of Y.
6. $f^{-1}(Int(B)) \subseteq S_sInt(f^{-1}(B))$, for each subset B of Y.
7. $Int(f(A)) \subseteq f(S_sInt(A))$, for each subset A of X.

proof. (1) $\Rightarrow$ (2): Follows from Proposition 4.2.
(2) $\Rightarrow$ (3): Let F be any closed set of Y. Then, $Y \setminus F$ is an open set of Y. By (2), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is an S_s-open set in X and hence $f^{-1}(F)$ is S_s-closed in X.
(3) $\Rightarrow$ (4): Let A be any subset of X. Then, $f(A) \subseteq Cl(f(A))$ and $Cl(f(A))$ is a closed set in Y. By (3), we have $f^{-1}(Cl(f(A)))$ is S_s-closed in X. Therefore, $S_sCl(A) \subseteq f^{-1}(Cl(f(A)))$. Hence, $f(S_sCl(A)) \subseteq Cl(f(A))$.
(4) $\Rightarrow$ (5): Let B be any subset of Y, so $f^{-1}(B)$ is a subset of X. By (4), we have $f(S_sCl(f^{-1}(B))) \subseteq Cl(f(f^{-1}(B))) \subseteq Cl(B)$. Hence $S_sCl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$.
(5) $\Leftrightarrow$ (6): Let B be any subset of Y. Then apply (5) to $Y \setminus B$, we obtain $S_sCl(f^{-1}(Y \setminus B)) \subseteq f^{-1}(Cl(Y \setminus B)) \Leftrightarrow S_sCl(X \setminus f^{-1}(B)) \subseteq f^{-1}(Y \setminus Int(B)) \Leftrightarrow X \setminus S_sInt(f^{-1}(B)) \subseteq X \setminus f^{-1}(Int(B)) \Leftrightarrow f^{-1}(Int(B)) \subseteq S_sInt(f^{-1}(B))$. Therefore, $f^{-1}(Int(B)) \subseteq S_sInt(f^{-1}(B))$.
(6) $\Rightarrow$ (7): Let A be any subset of X. Then, $f(A)$ is a subset of Y. By (6), we have $f^{-1}(Int(f(A))) \subseteq S_sInt(f^{-1}(f(A))) \subseteq S_sInt(A)$. Therefore, $Int(f(A)) \subseteq f(S_sInt(A))$.
(7) $\Rightarrow$ (1): Let $x \in X$ and let V be any open set of Y containing $f(x)$. Then, $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. By (7), we have $Int(f(f^{-1}(V))) \subseteq f(S_sInt(f^{-1}(V)))$. Hence, $Int(V) \subseteq f(S_sInt(f^{-1}(V)))$. Since, V is an open set, so $V \subseteq f(S_sInt(f^{-1}(V)))$ implies that $f^{-1}(V) \subseteq S_sInt(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an S_s-open set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence, f is S_s-continuous.

Proposition 5.2 For a function $f : X \to Y$, the following statements are equivalent:

1. f is S_s-continuous.
2. $S_sCl(f^{-1}(V)) \subseteq f^{-1}(Cl_\theta(V))$, for each open set V of Y.
3. $f^{-1}(Int_\theta(V)) \subseteq S_sInt(f^{-1}(V))$, for each closed V of Y.

proof. (1) $\Rightarrow$ (2). Let V be any open set in Y. Suppose that $x \notin f^{-1}(Cl_\theta(V))$, then $f(x) \notin Cl_\theta(V)$ and there exists an open set G containing $f(x)$, such that $Cl(G) \cap V = \phi$ implies $G \cap V = \phi$. Since, f is S_s-continuous, there exists a S_s-open set U containing x such that $f(U) \subseteq G$. Therefore, we have $f(U) \cap V = \phi$ and $U \cap f^{-1}(V) = \phi$. This shows that $x \notin S_sCl(f^{-1}(V))$. Thus, we obtain $S_sCl(f^{-1}(V)) \subseteq f^{-1}(Cl_\theta(V))$.
(2) $\Rightarrow$ (3). It is quite similar to part (5) $\Rightarrow$ (6) in Proposition 5.1.
(3) $\Rightarrow$ (1). From the Proposition 5.1 (6) and the fact that $Int(V) = Int_\theta(V)$ for each closed set V.

Proposition 5.3 A $f : X \to Y$ is S_s-continuous if and only if $S_sBd(f^{-1}(B)) \subseteq f^{-1}(Bd(B))$, for each subset B in Y.

proof. Let B be any subset of Y, then we have $f^{-1}(Bd(B)) = f^{-1}(Cl(B) \setminus Int(B)) = f^{-1}(Cl(B)) \setminus f^{-1}(Int(B))$. Hence, by Proposition 5.1 (5) and (6), we have $f^{-1}(Cl(B)) \setminus f^{-1}(Int(B)) \supseteq f^{-1}(S_sCl(B) \setminus S_sInt(B))$. Hence, $S_sBd(f^{-1}(B)) \subseteq f^{-1}(Bd(B))$.

Conversely, let V be any open set in Y and $F = Y \setminus V$. Then, by hypothesis, we have $S_sBd(f^{-1}(F)) \subseteq f^{-1}(Bd(F)) \subseteq f^{-1}(Cl(F)) = f^{-1}(F)$ and hence $S_sCl(f^{-1}(F)) = S_sInt(f^{-1}(F)) \cup S_sBd(f^{-1}(F)) \subseteq f^{-1}(F)$. Thus, $f^{-1}(F)$ is S_s-closed and hence $f^{-1}(V)$ is S_s-open in X.

Theorem 5.4 Let $f : X \to Y$ be a function. Let $\mathbf{B}$ be any basis for τ in Y. Then, f is S_s-continuous if and only if for each $B \in \mathbf{B}$, $f^{-1}(B)$ is a S_s-open subset of X

proof. Necessity. Suppose that f is S_s-continuous. Then, since each $B \in \mathbf{B}$ is an open subset of Y. Therefore, by Theorem 5.1, $f^{-1}(B)$ is a S_s-open subset of X.
Sufficiency. Let V be any open subset of Y. Then, $V = \cup\{B_i : i \in I\}$ where every B_i is a member of $\mathbf{B}$ and I is a suitable index set. It follows that $f^{-1}(V) = f^{-1}(\cup\{B_i : i \in I\}) = \cup f^{-1}(\{B_i : i \in I\})$. Since, $f^{-1}(B_i)$ is a S_s-open subset of X for each $i \in I$. Hence, $f^{-1}(V)$ is the union of a family of S_s-open sets of X and hence is S_s-open set of X. Therefore, by Proposition 5.1, f is S_s-continuous.

Proposition 5.5 Let $f : X \to Y$ be a S_s-continuous function. If Y is any subset of a topological space Z, then $f : X \to Z$ is S_s-continuous.

proof. Let $x \in X$ and V be any open set of Z containing $f(x)$, then $V \cap Y$ is open in Y. But, $f(x) \in Y$ for each $x \in X$, then $f(x) \in V \cap Y$. Since, $f : X \to Y$ is S_s-continuous, then there exists a S_s-open set U containing x such that $f(U) \subseteq V \cap Y \subseteq V$. Therefore, $f : X \to Z$ is S_s-continuous.

Proposition 5.6 Let $f : X \to Y$ be a function and X is locally indiscrete space. Then, f is S_s-continuous if and only if f is semi-continuous .

proof. Follows from Lemma 2.9.

Proposition 5.7 Let $f : X \to Y$ be a function and X is semi-T_1 space. Then, f is S_s-continuous if and only if f is semi-continuous .

proof. Follows from Proposition 3.4.

Proposition 5.8 Let $f : X \to Y$ be an S_s-continuous function. If A is α-open and semi-closed subset of X, then $f|A : A \to Y$ is S_s-continuous in the subspace A.

proof. Let V be any open set of Y. Since, f is S_s-continuous. Then, by Proposition 4.2, $f^{-1}(V)$ is S_s-open set in X. Since, A is α-open and semi-closed subset of X. By Proposition 3.17, $(f|A)^{-1}(V) = f^{-1}(V) \cap A$ is an S_s-open subset of A. This shows that $f|A : A \to Y$ is S_s-continuous.

Proposition 5.9 A function $f : X \to Y$ is S_s-continuous, if for each $x \in X$, there exists a semi-regular set A of X containing x such that $f|A : A \to Y$ is S_s-continuous.

proof. Let $x \in X$, then by hypothesis, there exists a semi-regular set A containing x such that $f|A : A \to Y$ is S_s-continuous. Let V be any open set of Y containing $f(x)$, there exists an S_s-open set U in A containing x such that $(f|A)(U) \subseteq V$. Since, A is semi-regular set, by Remark 3.8, U is S_s-open set in X and hence $f(U) \subseteq V$. This shows that f is S_s-continuous.

Proposition 5.10 Let $f : X_1 \to Y$ and $g : X_2 \to Y$ be two S_s-continuous functions. If Y is Hausdorff, then the set $E = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1) = g(x_2)\}$ is S_s-closed in the product space $X_1 \times X_2$.

proof. Let $(x_1, x_2) \notin E$. Then, $f(x_1) \neq g(x_2)$. Since, Y is Hausdorff, there exist open sets V_1 and V_2 of Y such that $f(x_1) \subseteq V_1$, $g(x_2) \subseteq V_2$ and $V_1 \cap V_2 = \phi$. Since, f and g are S_s-continuous, then there exist S_s-open sets U_1 and U_2 of X_1 and X_2 containing x_1 and x_2 such that $f(U_1) \subseteq (V_1)$ and $g(U_2) \subseteq (V_2)$, respectively. Put $U = U_1 \times U_2$, then $(x_1, x_2) \in U$ and by Proposition 2.12, U is an S_s-open set in $X_1 \times X_2$ and $U \cap E = \phi$. Therefore, we obtain $(x_1, x_2) \notin S_sCl(E)$. Hence, E is S_s-closed in the product space $X_1 \times X_2$.

Proposition 5.11 *Let $f : X \to Y$ and $g : Y \to Z$ be two functions. If f is S_s-continuous and g is continuous. Then, the composition function $g \circ f : X \to Z$ is S_s-continuous.*

proof. Let V be any open subset of Z. Since, g is continuous, $g^{-1}(V)$ is open subset of Y. Since, f is S_s-continuous, then by Proposition 4.2, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is S_s-open subset in X. Therefore, $g \circ f$ is S_s-continuous.

Proposition 5.12 Let $f : (X, \tau) \to (Y, \rho)$ be a surjection function such that $f(U)$ is S_s-open in Y, for any S_s-open set U in X and let $g : (Y, \rho) \to (Z, \sigma)$ be any function. If $g \circ f$ is S_s-continuous then g is S_s-continuous.

proof. Let $y \in Y$. Since, f is surjection, there exists $x \in X$ such that $f(x) = y$. Let $V \in \sigma$ with $g(y) \in V$, then $(g \circ f)(x) \in V$. Since, $g \circ f$ is S_s-continuous, there exists an S_s-open set U in X containing x such that $(g \circ f)(U) \subseteq V$. By assumption $H = f(U)$ is an S_s-open set in Y and contains $f(x) = y$. Thus, $g(H) \subseteq V$. Hence, g is S_s-continuous.

Proposition 5.13 If $f_i : X_i \to Y_i$ is S_s-continuous functions for $i = 1, 2$. Let $f : X_1 \times X_2 \to Y_1 \times Y_2$ be a function defined as follows: $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then, f is S_s-continuous.

proof. Let $R_1 \times R_2 \subseteq Y_1 \times Y_2$, where R_i is open set in Y_i for $i = 1, 2$. Then, $f^{-1}(R_1 \times R_2) = f_1^{-1}(R_1) \times f_2^{-1}(R_2)$. Since, f_i is S_s-continuous for $i = 1, 2$. By Proposition 4.2, $f^{-1}(R_1 \times R_2)$ is S_s-open set in $X_1 \times X_2$.

Proposition 5.14 Let $f : X \to Y$ be any function. If the function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$, is an S_s-continuous function, then f is S_s-continuous.

proof. Let H be an open subset of Y, then $X \times H$ is an open subset of $X \times Y$. Since g is S_s-continuous, then $g^{-1}(X \times H) = f^{-1}(H)$ is an S_s-open subset of X. Hence f is S_s-continuous.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci. Assuit. Univ.*, 12 (1) (1983), 1-18.

[2] N. K. Ahmed, On Some Types of Separation Axioms, *M. Sc. Thesis*, College of Science, Salahaddin Univ., (1990).

[3] M. Caldas, S. Jafari, T. Noiri and R. Saraf, Functions with strongly semi-θ-closed graphs, *J. of Advanced Research in Applied Mathematics* 2(2) (2010), 38-47.

[4] D. E. Cameron, Properties of S-closed spaces, *Proc. Amer. Math. Soc.* 72 (1978), 581-586.

[5] S. G. Crossley and S. K. Hildebrand, Semi-closure, *Texas J. Sci.*, 22 (1971), 99-112.

[6] G. Di Maio and T. Noiri, On s-closed spaces, *Indian J. Pure Appl. Math.*, 18 (3) (1987), 226-233.

[7] J. Dontchev, on submaximal spaces, *Tamkang Math. J.*, 26(1995), 243-250.

[8] J. Dontchev and M. Ganster , On covering space with semi regular sets, *Ricerche di Matematica*, 45(1), (1996), 229-245.

[9] J. Dontchev, Contra-continuous functions and strongly S-closed spaces, *Internat. J. Math. and Math. Sci.*, 19 (2) (1996), 303-310.

[10] J. Dontchev and T. Noiri, Contra-Semi-Continuous Functions, *Mathematica Pannonica*, 10(2)(1999), 159-168.

[11] C. Dorsett, Semi-regular spaces, *Soochow J. Math*, 8(1982), 45-53.

[12] K. Dlaska, N. Ergun and M. Ganster, On the topology generated by semi-regular sets, *Indian J. Pure Appl. Math.*, 25(11) (1994), 1163-1170.

[13] J. E. Joseph and M. H. Kwack, On S-closed spaces, *Proc. Amer. Math. Soc.*, 80 (2) (1980), 341-348.

[14] A. B. Khalaf and Z. H. Ameen, S_c-open sets and S_c-continuous functions, *J. of Advanced Research in Pure Mathematics*, 2 (3) (2010), 87-101.

[15] A. B. Khalaf and N. K. Ahmed, S_β-open sets and S_β-continuous functions, *Thai J. Math.*, 11(2) (2013), 319-335.

[16] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70 (1) (1963), 36-41.

[17] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and week precontinuous mappings, *Proc. Math. Phys. Soc.*, [Egypt], 53 (1982), 47-53.

[18] O. Njastad, On some classes of nearly open sets, *Pacific J. Math.*, 15 (3) (1965), 961-970.

[19] N. Puturong, On strongly θ-semi-continuous functions, *Thai Journal of Mathematics*, Special Issue Annual Meeting in Mathematics, (2007), 11-23.

[20] H. A. Shareef, S_p-Open Sets, S_p-Continuity and S_p-Compactness in Topological Spaces , *M. Sc. Thesis*, University of Sulaimani (2007).

[21] J. N. Sharma, Topology (General and Algebraic), *Krishna Parkashan Mandir*, 4^{th} Edition (2011).

[22] S. F. Tadros and A. B. Khalaf, On regular semi-open sets and s*-closed spaces, *Tamkang J. of Math.*, 23 (4) (1992), 337-348.

[23] N. V. Velicko, H-closed topological spaces, *Amer. Math. Soc. Transl.*, 78 (2) (1968), 103-118.

[24] R. H. Yunis, Properties of θ-semi-open sets, *Zanco J. of Pure and Applied Sciences*, 19 (1) (2007), 116-122.