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Research paper

Diagonalization of Hamiltonian in the photogravitational restricted three body problem with P-R drag

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Abstract

In this paper, restricted, three-body problem (RTBP) is generalised to study the non-linear stability of equilibrium points in the photogravitational RTBP with P-R drag. In the present study, both primaries are considered as a source of radiation and effect of P-R drag. Hence the problem will contain four parameters q_1, q_2, W_1 and W_2. At first, the Lagrangian and the Hamiltonian of the problem were computed, then the Lagrangian function is expanded in power series of the coordinates of the triangular equilibrium points x and y. Lastly, diagonalized the quadratic term of the Hamiltonian of the problem, which is obtained by expanding original Lagrangian or Hamiltonian by Taylor's series about triangular equilibrium point. Finally, the study concluded that the diagonalizable Hamiltonian is $H_2=\omega_1 I_1-\omega_2 I_2$.

Keywords: *Hamiltonian, Normalization, Photogravitational, P-R drag, RTBP.*

1. Introduction

Poynting [8] investigated the effect of radiation pressure on the moving particle in Interplanetary space and stated that the particle such as meteors or cosmic dust are comparably affected by gravitational and light radiation pressure force as they approach uminous Celestial bodies. Robertson [10] modified Poynting theory in keeping with the principle of relativity. He considered only terms of first order in the ratio of velocity of the particle to that of light. The radiation force is given by,

$$\vec{F} = F_p\left[\frac{\vec{R}}{R} - \frac{\vec{V}.\vec{R}}{c,R} - \frac{\vec{V}}{c}\right]$$

where, F_p denotes the measure of the radiation pressure force, $\vec{R}$ is the position vector of the particle with respect to the source, $\vec{V}$ is the velocity vector and c is the velocity of the light. The first term in the above expression expresses the radiation pressure, the second term expresses Doppler shift due to the motion of the particle and the third term is due to the absorption and subsequent re-emission part of induced radiation. The last two terms constitute Poynting-Robertson (P-R) effect. Chernikov [2] described the photogravitational RTBP and dealt with Sun-planet-particle model and concludes that due to P-R drag triangular equilibrium points are unstable. Schuerman [11] studied classical RTBP by including radiation pressure and P-R effect. Murray [7] discussed the dynamical effect of drag in the planar circular RTBP. Ragos and Zafiropoulos [9] studied the existence and stability of equilibrium points for particle moving in the vicinity of two massive bodies, which exerts light radiation pressure with P-R drag numerically and concludes that none of the equilibrium points is stable. Ishwar [3] analyzed non linear stability in the generalized RTBP. Singh and Ishwar [12] examined the stability of triangular points in the generalised photogravitational RTBP by considering both primaries as oblate spheroid and shown that triangular points are stable. Kushvah *et al*. [6] investigated non linear stability in the generalised photogravitational RTBP with P-R drag. Vivek Kumar Mishra *et al*. [13] examined the stability of triangular equilibrium points in photogravitational elliptic RTBP with P-R drag. Jaiyeola Sefinat *et al*. [4] studied the stability of the photogravitational RTBP, when the primaries are considered to be oblate spheroid as well as sources of radiation. Vivek Kumar Mishra and Ishwar [14] examined the non-linear stability of triangular equilibrium points in the photogravitational elliptic RTBP with P-R drag.

In this paper, the effect of P-R drag in the photogravitational RTBP is studied, when both primaries are intense emitter of radiation. Hence the problem will contain four parameters q_1, q_2, W_1 and W_2. Using the method described in Jorba [5], diagonalization of the quadratic part of the Hamiltonian of the photogravitational RTBP with P-R drag is carried out.

2. Equations of motion and location of triangular equilibrium points

Equations of motion of infinitesimal mass are given by

$$\ddot{x} - 2\dot{y} = x - \frac{q_{1(1-\mu)(x+\mu)}}{r_1^3} - \frac{q_{2\mu(x+\mu-1)}}{r_2^3}$$
$$-\frac{W_1}{r_1^2}\left[\frac{(x+\mu)\{(x+\mu)\dot{x}+y\dot{y}\}}{r_1^2} + \dot{x} - y\right]$$
$$-\frac{W_2}{r_2^2}\left[\frac{(x+\mu-1)\{(x+\mu-1)\dot{x}+y\dot{y}\}}{r_1^2} + \dot{x} - y\right],$$

$$\ddot{y} - 2\dot{x} = y - \frac{q_{1(1-\mu)y}}{r_1^3} - \frac{q_{2\mu y}}{r_2^3}$$

$$-\frac{W_1}{r_1^2}\left[\frac{y\{(x+\mu)\dot{x} + y\dot{y}\}}{r_1^2} + \dot{y} + x + \mu\right]$$

$$-\frac{W_2}{r_2^2}\left[\frac{y\{(x+\mu-1)\dot{x} + y\dot{y}\}}{r_1^2} + \dot{y} + x + \mu - 1\right],$$

where,

$$W_1 = \frac{(1-q_1)(1-\mu)}{C_d}, \quad W_2 = \frac{(1-q_1)\mu}{C_d},$$

$$r_1^2 = (x+\mu)^2 + y^2 \quad and \quad r_2^2 = (x+\mu-1)^2 + y^2$$

with C_d as the dimensionless velocity of light.

The coordinates of the triangular equilibrium points, when the terms up to first order in q_1, q_2, W_1 and W_2 are considered,

$$x = \frac{\gamma}{2} - \frac{2}{\sqrt{3}}W_1\left(1+\frac{\gamma}{2}\right) - \frac{2}{\sqrt{3}}W_2\left(1-\frac{\gamma}{3}\right) - \frac{\epsilon_1}{3} + \frac{\epsilon_2}{3},$$

$$y = \pm\left\{\frac{\sqrt{3}}{2} + \frac{2}{3}W_1\left(\gamma+\frac{1}{3}\right) + \frac{2}{3}W_2\left(\gamma-\frac{1}{3}\right) - \frac{\epsilon_1}{3\sqrt{3}} + \frac{\epsilon_2}{3\sqrt{3}}\right\},$$

where, $\gamma = 1\text{-}2\mu$ and $\varepsilon_i = 1\text{-}q_i$, for $i = 1, 2$, as in Avdhesh Kumar *et al*. [1].

3. Diagonalization of Hamiltonian

The Lagrangian function of the problem is

$$L = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{x^2+y^2}{2} + x\dot{y} - \dot{x}y + q_1\frac{(1-\mu)}{r_1} + q_2\frac{\mu}{r_2}$$

$$+W_1\left[\frac{(x+\mu)\,\dot{x} + y\dot{y}}{2r_1^2} - tan^{-1}\frac{y}{x+\mu}\right]$$

$$+W_1\left[\frac{(x+\mu-1)\,\dot{x} + y\dot{y}}{2r_2^2} - tan^{-1}\frac{y}{x+\mu-1}\right].$$

Now, shift the origin at the triangular equilibrium point L_4. For that, change $x \to x+a-\mu$ and $y \to y+b$, so that

$$a = \frac{1}{2} - \frac{2}{\sqrt{3}}W_1\left(1+\frac{\gamma}{2}\right) - \frac{2}{\sqrt{3}}W_2\left(1-\frac{\gamma}{3}\right) - \frac{\epsilon_1}{3} + \frac{\epsilon_2}{3},$$

$$b = \frac{\sqrt{3}}{2} + \frac{2}{3}W_1\left(\gamma+\frac{1}{3}\right) + \frac{2}{3}W_2\left(\gamma-\frac{1}{3}\right) - \frac{\epsilon_1}{3\sqrt{3}} + \frac{\epsilon_2}{3\sqrt{3}}.$$

Expanding L in power series of x and y, we get

$L=L_0+L_1+L_2+$----------

Therefore, Hamiltonian

$H=H_0+H_1+H_2+$----------

as $H = -L + p_x\dot{x} + p_y\dot{y}$,

where L_0, L_1, L_2 are constant, first order, second order term respectively and p_x, p_y are momenta coordinates given by

$$p_x = \frac{\partial L}{\partial \dot{x}} \quad and \quad p = \frac{\partial L}{\partial \dot{y}}.$$

Second order Hamiltonian H_2 is written as

$$H_2 = \frac{p_x^2+p_y^2}{2} + yp_x - xp_y + Ex^2 + Gxy + Fy^2, \qquad (1)$$

where,

$$E = \frac{127}{2} + \frac{76}{\sqrt{3}}W_1(3\gamma+1) + \frac{76}{\sqrt{3}}W_2(3\gamma-1)$$

$$-8\,\epsilon_1\,(8+25\gamma) - 8\,\epsilon_2\,(8-25\gamma),$$

$$F = \frac{7}{2} + 16\sqrt{3}(W_1+W_2) - 2\,\epsilon_1\,(1-3\gamma) - 2\epsilon_2(1+3\gamma),$$

$$G = -24\sqrt{3}\gamma - \frac{4}{3}W_1(291+104\gamma) - \frac{4}{3}W_2(291-104\gamma)$$

$$-\frac{52}{\sqrt{3}}\,\epsilon_1\,(3-\gamma) + \frac{52}{\sqrt{3}}\,\epsilon_2\,(3+\gamma).$$

Now, follow the method described in Jorba [5] to find the real symplectic change of real linear form of Hamiltonian (1). For that purpose, find the characteristic vector of matrix M corresponding to

$$\dot{X} = MX \qquad (2)$$

where,

$$\dot{X} = \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{p}_x\\ \dot{p}_y\end{bmatrix}, \quad M = \begin{bmatrix}0 & 1 & 1 & 0\\ -1 & 0 & 0 & 1\\ -2E & -G & 0 & 1\\ -G & -2F & -1 & 0\end{bmatrix} \quad and \quad X = \begin{bmatrix}x\\ y\\ p_x\\ p_y\end{bmatrix}.$$

Suppose, $A = M - \lambda I_4'$, where I_4' is the identity matrix of order 4. Hence,

$$A = \begin{bmatrix}-\lambda & 1 & 1 & 0\\ -1 & -\lambda & 0 & 1\\ -2E & -G & -\lambda & 1\\ -G & -2F & -1 & -\lambda\end{bmatrix} \qquad (3)$$

Clearly $|A| = 0$ implies that the characteristic equation corresponding to the Hamiltonian is given by

$$\lambda^4 + 2(E+F+1)\lambda^2 + 4EF - G^2 - 2(E+F) + 1 = 0.$$

Discriminant is given by

$$D = 4(E+F+1)^2 - 4(4EF - G^2 - 2E - 2F + 1)$$

Stability is assured, when $D>0$ only. When $D>0$, the roots of the characteristic equation $\pm i\omega_1$ and $\pm i\omega_2$ are related to each other as

$$\omega_1^2 + \omega_2^2 = 2E + 2F + 2, \qquad (4)$$

$$\omega_1^2\omega_2^2 = 4EF - G^2 - 2E - 2F + 1 \qquad (5)$$

Substituting the values of E, F and G in equations (4) and (5) we get,

$$\omega_1^2 + \omega_2^2 = 136 + \frac{8}{\sqrt{3}}W_1(69\gamma+19) + \frac{8}{\sqrt{3}}W_2(69\gamma-19)$$

$$-4\epsilon_1(33+97\gamma) - 4\epsilon_2(33-97\gamma),$$

$$\omega_1^2\omega_2^2 = 756 - 304\sqrt{3}W_1(45-\gamma) - 304\sqrt{3}W_2(45+\gamma)$$

$$-24\,\epsilon_1\,(53+349\gamma) - 24\,\epsilon_2\,(53-349\gamma).$$

From the above two equations the values of ω_1 and ω_2 can be found out.
To obtain the real linear form of the Hamiltonian, find out a real symplectic change of variable. For that purpose, obtain the characteristic vectors of the matrix A as in Jorba [5].

Let $A = \begin{bmatrix} A'_\lambda & I'_2 \\ A' & A'_\lambda \end{bmatrix}$,

where, I'_2 is the identity matrix of order 2.

Using (3), we get

$$A'_\lambda = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \text{ and } A' = \begin{bmatrix} -2E & -G \\ -1 & -2F \end{bmatrix}.$$

Since λ is the root of the matrix A, the Kernel of A is obtained by solving the matrix equation

$$\begin{bmatrix} A'_\lambda & I'_2 \\ A' & A'_\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

with $X_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ and $X_2 = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$.

This gives,

$$A'_\lambda X_1 + I'_2 X_2 = 0 \tag{6}$$

$$A' X_1 + A'_\lambda X_2 = 0 \tag{7}$$

From the above equations, we get

$$\begin{bmatrix} 2E + \lambda^2 - 1 & G - 2\lambda \\ 2\lambda + G & 2F + \lambda^2 - 1 \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which gives either

$$x=2\lambda-G, \quad y=\lambda^2+2E-1 \tag{8}$$

or

$$x=1-\lambda^2-2F, \quad y=2\lambda+G \tag{9}$$

Now, use any one set of x and y in (8) or (9). Suppose, if we use the second set of x and y given in equation (9) into (7), we get

$$p_x=-\lambda^3-2\lambda F-\lambda-G \tag{10}$$

$$p_y=\lambda^2+\lambda G-2F+1 \tag{11}$$

From the equations (9) to (11), the characteristic vector of the matrix A is

$$\begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix} = \begin{bmatrix} 1 - \lambda^2 - 2F \\ 2\lambda + G \\ -\lambda^3 - 2\lambda F - \lambda - G \\ \lambda^2 + \lambda G - 2F + 1 \end{bmatrix} \tag{12}$$

If we substitute $\lambda=i\omega$ into the above characteristic vector (12), and finding out the real and imaginary parts as u and v respectively, we get,

$$u = \begin{bmatrix} 1 + \omega^2 - 2F \\ G \\ -G \\ -\omega^2 - 2F + 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 0 \\ 2\omega \\ \omega^2 - 2\omega F - \omega \\ \omega G \end{bmatrix}$$

Now, consider the matrix $C=(v_1, v_2, u_1, u_2)$, where v_i, u_i for i=1, 2 corresponds to v, u for frequencies ω_i for i=1, 2. Hence, it is obvious that the symplectic change satisfy $C^T J_4 C = J_4$.

Substituting the expressions (2) and (3) in the above matrix equation $C^T J_4 C = J_4$ and simplifying, we get

$$C^T J_4 C = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix},$$

with $D = \begin{bmatrix} d(\omega_1) & 0 \\ 0 & d(\omega_1) \end{bmatrix}$,

where,

$$d(\omega) = \omega[\omega^4 + (2 - 4F)\omega^2 + G^2 + 4F^2 + 4F - 3] \tag{13}$$

to satisfy the symplectic property $d(\omega) = 1$.

If $d(\omega) \neq 1$, then scale the column of C matrix by $\sqrt{d(\omega_k)}$ for k=1, 2 to obtain the symplectic matrix C. Hence,

$C = \left[\frac{v_1}{\sqrt{d(\omega_1)}}, \frac{v_2}{\sqrt{d(\omega_2)}}, \frac{u_1}{\sqrt{d(\omega_1)}}, \frac{u_2}{\sqrt{d(\omega_2)}}\right]$, ie.

$$C = \begin{bmatrix} 0 & 0 & \frac{1 - 2F + \omega_1^2}{\sqrt{d(\omega_1)}} & \frac{1 - 2F + \omega_2^2}{\sqrt{d(\omega_1)}} \\ \frac{2\omega_1}{\sqrt{d(\omega_1)}} & \frac{2\omega_2}{\sqrt{d(\omega_2)}} & \frac{G}{\sqrt{d(\omega_1)}} & \frac{G}{\sqrt{d(\omega_2)}} \\ \frac{\omega_1^3 - 2F\omega_1 - \omega_1}{\sqrt{d(\omega_1)}} & \frac{\omega_2^3 - 2F\omega_2 - \omega_2}{\sqrt{d(\omega_2)}} & \frac{-G}{\sqrt{d(\omega_1)}} & \frac{-G}{\sqrt{d(\omega_2)}} \\ \frac{G\omega_1}{\sqrt{d(\omega_1)}} & \frac{G\omega_2}{\sqrt{d(\omega_2)}} & \frac{1 - 2F - \omega_1^2}{\sqrt{d(\omega_1)}} & \frac{1 - 2F - \omega_2^2}{\sqrt{d(\omega_2)}} \end{bmatrix}$$

where, $d(\omega_i)$ for i=1, 2 can be obtained from equation (13).
Now, the matrix C is symplectic, but C to be real also.
ie $d(\omega_i)>0$, for i=1, 2. This will determine the sign, to be chosen for the frequencies ω_i, for i=1, 2.

Since, $\omega_1^2 < \frac{1}{2}$, and $d(\omega_1)>0$, it is necessary to take $\omega_1>0$ and conversely, as $\omega_2^2 < \frac{1}{2}$, implies that $\omega_2<0$ in order to have $d(\omega_2)>0$. Hence the change obtained is real, symplectic and it brings the Hamiltonian (1) in to the normal form as in Jorba [5],

$$H_2 = \omega_1 \left(\frac{x^2+p_x^2}{2}\right) + \omega_2 \left(\frac{y^2+p_y^2}{2}\right) \tag{14}$$

Since,

$I_1 = \frac{x^2+p_x^2}{2}$ and $I_2 = \frac{y^2+p_y^2}{2}$ are action variables, then Hamiltonian (14) becomes,

$$H_2=\omega_1 I_1-\omega_2 I_2 \tag{15}$$

The Hamiltonian obtained in (15) is diagonalized form of the Hamiltonian (1).

4. Conclusion

The detailed analysis on non linear stability of triangular equilibrium point of the perturbations radiation pressure and P-R drag is performed. Digitalization of second order Hamiltonian is carried out and obtained $H_2=\omega_1 I_1-\omega_2 I_2$, which is the required, diagonalizable Hamiltonian.

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