On J₂ short-term orbit predictions in terms of KS elements

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Abstract

Sharma's singularity-free analytical theory for the short-term orbital motion of satellites in terms of KS elements in closed form in eccentricity with Earth's zonal harmonic term J₂, is improved by using King-Hele's expression for the radial distance 'r' which includes the effect of J₂, and is suitable for low eccentricity orbits. Numerical experimentation with four test cases with perigee altitude of 200 km and eccentricity varying from 0.01 to 0.3 for different inclinations is carried out. It is found that the orbital elements computed with the analytical expressions in a single step during half a revolution match very well with numerically integrated values and show significant improvement over the earlier theory. The solution can be effectively used for computation of mean elements for near-Earth orbits, where the short-term orbit perturbations due to J₂ play most important role. The theory will be very useful in computing the state vectors during the coast phase of rocket trajectories and flight algorithms for on-board implementation.

Keywords: *Analytical Integration; KS Element Equations; Earth's Oblateness; Short-Term Orbit Prediction; Radial Distance.*

1. Introduction

The development of analytical theories for the motion of an Earth satellite under the influence of Earth's oblateness, viz. the second zonal harmonic J₂, draws great attention due to the non-integrability of the main problem in the artificial satellite theory and it may also induce chaotic motion under certain conditions [4]. Brouwer [3] and Kozai [9] made significant contributions on the study of J₂ perturbed satellite motion, among several theories that emerged in the past. Lyddane [12] re-worked Brouwer's theory to improve the solution for small eccentricities and inclinations. Garfinkel [7] and Aksnes [1] employed the use of intermediary orbit to obtain a solution in terms of Delaunay's variables. Deprit's Radial Intermediary [5] is a particular canonical transformation theory of Lie type to remove the parallactic factors by the elimination of parallax, which was utilized to produce a viable alternative for the onboard orbit propagators [8].

The classical Newtonian equations, being unstable, are not very ideal choice for numerical integration as well as for development of analytical theories. Regularization of the classical Newtonian equations by Kustaanheimo and Stiefel[11] and presented in details in Stiefel and Scheifele [15], drew significant attention when J₂ perturbed Lambert-type solutions in terms of KS (Kustaanheimo-Stiefel) transformation was derived in closed-form by Andrus [2] and Engels & Junkins [6]. Sharma [13] obtained a singularity-free analytical solution for short-term orbit computation in terms KS elements with J₂, in series expanded up to fourth power in eccentricity. He revisited the problem and provided an accurate theory by analytical integration in closed form in eccentricity for short-term orbit prediction [14]. The equations are regular everywhere and the generalized eccentric anomaly is the independent variable. The elegant feature of KS element equations is the existence of symmetry which allows for solving only one out of the eight equations, which improves the efficiency in terms of memory usage and computation time.

In spite of the good accuracy of Sharma's closed form solution, some avenue for further improvement is available. The radial distance 'r', appearing in the denominator of the equations of motion, was assumed by Sharma to be unperturbed, i.e., r = a(1- e cos E), where a, e and E are semi-major, eccentricity and eccentric anomaly, respectively. However, it needs to be function of J₂. In the present study, we have modified the above theory by using King-Hele's [10] expression for the radial distance 'r' which includes the effect of J₂. It is derived for low eccentricity orbits. Numerical experimentation brings out that the orbital elements computed from the new analytical theory in a single step during half a revolution show significant improvement over the earlier theory.

2. Equations of motion

The KS element equations in terms of four-vectors α, β with Earth's zonal harmonic term J₂ are [14]:

$$\frac{d\alpha_i}{dE} = \frac{\mu J_2 R^2}{8w^2}\left[\frac{1}{r^3}\{q_{0_i} + q_{1_i}\cos E + q_{2_i}\sin E\} + \frac{3}{r^4}\{g_{0_k} + g_{1_k}\cos E + g_{2_k}\cos^2 E + g_{3_k}\sin E + g_{4_k}\cos E \sin E\} - \frac{6}{r^5}\{f_{0_i} + f_{1_i}\cos E + f_{2_i}\cos^2 E + f_{3_i}\cos^3 E + f_{4_i}\sin E + f_{5_i}\sin E \cos E + f_{6_i}\sin E \cos^2 E\}\right] \quad (1)$$

where

$$q_{0_i} = \beta_i \quad q_{1_i} = -\beta_i \quad q_{2_i} = \alpha_i$$

where

$$k = i + 2, g_{p_{j+4}} = g_{p_j} (j = 1,2 ; p = 0,1,2,3,4)$$

$$g_{0_k} = a_0 q_{0_k} + a_2 q_{2_k}$$

$$g_{1_k} = a_1 q_{0_k} + a_0 q_{1_k}$$
$$g_{2_k} = a_1 q_{1_k} - a_2 q_{2_k} \tag{2}$$

$$g_{3_k} = a_2 q_{0_k} + a_0 q_{2_k}$$

$$g_{4_k} = a_1 q_{2_k} + a_2 q_{1_k}$$

$$f_{0_i} = b_0 q_{0_i} + b_3 q_{2_i}$$

$$f_{1_i} = b_1 q_{0_i} + b_0 q_{1_i} + b_4 q_{2_i}$$

$$f_{2_i} = b_2 q_{0_i} + b_1 q_{1_i} - b_3 q_{2_i}$$

$$f_{3_i} = b_2 q_{1_i} - b_4 q_{2_i} \tag{3}$$

$$f_{4_i} = b_3 q_{0_i} + b_0 q_{2_i}$$

$$f_{5_i} = b_4 q_{0_i} + b_3 q_{1_i} + b_1 q_{2_i}$$

$$f_{6_i} = b_4 q_{1_i} + b_2 q_{2_i}$$

$$a_0 = \alpha_1 \alpha_3 + \alpha_2 \alpha_4 + \beta_1 \beta_3 + \beta_2 \beta_4$$

$$a_1 = \alpha_1 \alpha_3 + \alpha_2 \alpha_4 - \beta_1 \beta_3 - \beta_2 \beta_4$$

$$a_2 = \alpha_1 \beta_3 + \alpha_2 \beta_4 + \beta_1 \alpha_3 + \beta_2 \alpha_4$$

$$b_0 = a_0^2 + a_2^2 \quad b_1 = 2a_0 a_1$$

$$b_2 = a_1^2 - a_2^2 \quad b_3 = 2a_0 a_2 \quad b_4 = 2a_1 a_2$$

$$x_3 = a_0 + a_1 \cos E + a_2 \sin E \tag{4}$$

$$x_3^2 = b_0 + b_1 \cos E + b_2 \cos^2 E + b_3 \sin E + b_4 \sin E \cos E. \tag{5}$$

If α_i is changed to β_i in equation (1), we substitute

$$q_{0_i} = \alpha_i \quad q_{1_i} = \alpha_i \quad q_{2_i} = \beta_i$$

for initial conditions.

3. Expression for radial distance 'r' with J₂

Restricting up to J_2 term, the radial distance r is given by [10]:

$$\frac{1}{r} = L \left[1 + e \cos \theta + \frac{3}{2} J_2 V^* \right] \tag{6}$$

where

$$V^* = L^2 R^2 \left[\frac{2 - 5 \sin^2 i}{2} - \frac{\sin^2 i}{6} \cos(2\theta + 2\omega) \right] \tag{7}$$

$$L = \frac{1}{a(1-e^2)}$$

$$(1 + e \cos \theta) = \frac{1 - e^2}{1 - e \cos E},$$

where θ, i and ω are true anomaly, inclination and argument of perigee, respectively.
The initial value of V^* is equal to V^* at θ minus V^* at $\theta = 0°$, i.e.

$$V^* = -\frac{L^2 R^2}{3} \left[\sin^2 i \sin^2 \omega - \left(\frac{z}{r} \right)^2 \right], \tag{8}$$

$$\left(\frac{z}{r} \right) = \sin i \sin(\theta + \omega). \tag{9}$$

Substituting the value of V* from Equation (8) and (9) into Equation (6) and using binomial theorem and retaining terms up to J_2, we get

$$\frac{1}{r^3} = L^3 (1 + e \cos \theta)^3 \left[1 - \frac{3 J_2 L^2 R^2}{2(1 - e^2)} (1 - e \cos E) \left\{ \sin^2 i \sin^2 \omega - \left(\frac{z}{r} \right)^2 \right\} \right] \tag{10}$$

$$\frac{1}{r^4} = L^4 (1 + e \cos \theta)^4 \left[1 - \frac{2 J_2 L^2 R^2}{(1 - e^2)} (1 - e \cos E) \left\{ \sin^2 i \sin^2 \omega - \left(\frac{z}{r} \right)^2 \right\} \right] \tag{11}$$

$$\frac{1}{r^5} = L^5 (1 + e \cos \theta)^5 \left[1 - \frac{5 J_2 L^2 R^2}{2(1 - e^2)} (1 - e \cos E) \left\{ \sin^2 i \sin^2 \omega - \left(\frac{z}{r} \right)^2 \right\} \right] \tag{12}$$

Substituting Equations (10), (11) and (12) in Equation (1) and simplifying considerably, we get

$$\Delta \alpha_i = \frac{\mu J_2 R^2}{8 w^2 a^3} \left[q_{0_i} \Lambda_3^{00} + q_{1_i} \Lambda_3^{10} + q_{2_i} \Lambda_3^{01} - \right.$$
$$\frac{3 J_2 R^2}{2(1-e^2)^3 a^2} \left\{ A (q_{0_i} \Lambda_2^{00} + q_{1_i} \Lambda_2^{10} + q_{2_i} \Lambda_2^{01}) - \right.$$
$$\left. \frac{(f_{0_i} \Lambda_4^{00} + f_{1_i} \Lambda_4^{10} + f_{2_i} \Lambda_4^{20} + f_{3_i} \Lambda_4^{30} + f_{4_i} \Lambda_4^{01} + f_{5_i} \Lambda_4^{11} + f_{6_i} \Lambda_4^{21})}{a^2} \right\} + \frac{3}{a} \left\{ g_{0_k} \Lambda_4^{00} + \right.$$
$$g_{1_k} \Lambda_4^{10} + g_{2_k} \Lambda_4^{20} + g_{3_k} \Lambda_4^{01} + g_{4_k} \Lambda_4^{11} \right\} + \frac{2 J_2 R^2}{(1-e^2)^3 a^2} \left(A \{ g_{0_k} \Lambda_3^{00} + \right.$$
$$g_{1_k} \Lambda_3^{10} + g_{2_k} \Lambda_3^{20} + g_{3_k} \Lambda_3^{01} + g_{4_k} \Lambda_3^{11} \} -$$
$$\left. \left. \frac{(h_{0_i} \Lambda_5^{00} + h_{1_i} \Lambda_5^{10} + h_{2_i} \Lambda_5^{20} + h_{3_i} \Lambda_5^{30} + h_4 \Lambda_5^{40} + h_{5_i} \Lambda_5^{01} + h_{6_i} \Lambda_5^{11} + h_{7_i} \Lambda_5^{21} + h_{8_i} \Lambda_5^{31})}{a^2} \right) \right\} -$$
$$\frac{6}{a^2} \left\{ \{ f_{0_i} \Lambda_5^{00} + f_{1_i} \Lambda_5^{10} + f_{2_i} \Lambda_5^{20} + f_{3_i} \Lambda_5^{30} + f_{4_i} \Lambda_5^{01} + f_{5_i} \Lambda_5^{11} + \right.$$
$$f_{6_i} \Lambda_5^{21} \} + \frac{5 J_2 R^2}{2(1-e^2)^3 a^2} \left(A \{ f_{0_i} \Lambda_4^{00} + f_{1_i} \Lambda_4^{10} + f_{2_i} \Lambda_4^{20} + f_{3_i} \Lambda_4^{30} + \right.$$
$$f_{4_i} \Lambda_4^{01} + f_{5_i} \Lambda_4^{11} + f_{6_i} \Lambda_4^{21} \} -$$
$$\frac{(m_{0_i} \Lambda_6^{00} + m_{1_i} \Lambda_6^{10} + m_{2_i} \Lambda_6^{20} + m_{3_i} \Lambda_6^{30} + m_{4_i} \Lambda_6^{40} + m_{5_i} \Lambda_6^{50})}{a^2} -$$
$$\left. \left. \left. \frac{(m_{6_i} \Lambda_6^{01} + m_{7_i} \Lambda_6^{11} + m_{8_i} \Lambda_6^{21} + m_{9_i} \Lambda_6^{31} + m_{10_i} \Lambda_6^{41})}{3 a^2} \right) \right\} \right] \tag{13}$$

where

$$\Lambda_q^{ps} = \int \frac{\cos^p E \sin^s E}{(1 - e \cos E)^q} dE \quad (14) \qquad \Lambda_q^{ps} = \int \frac{\cos^p E \sin^s E}{(1 - e \cos E)^q} dE \tag{14}$$

$$A = 2 - \frac{16 \sin^2 i}{3}$$

$$\Lambda_0^{00} = E, \Lambda_1^{00} = \frac{2}{\eta^{0.5}} \tan^{-1} \left[\left(\frac{1+e}{1-e} \right)^{0.5} \tan \frac{E}{2} \right]$$

$$\Lambda_n^{00} = \frac{1}{(n-1)\eta} \left[\frac{e \sin E}{\varphi^{n-1}} + (2n - 3)\Lambda_{n-1}^{00} - (n - 2)\Lambda_{n-2}^{00} \right] \quad n > 1$$

$$\Lambda_n^{01} = -\frac{1}{(n-1)e \Lambda^{n-1}} \quad n > 1$$

$$\Lambda_n^{11} = \frac{1}{e} \left(\Lambda_n^{01} - \Lambda_{n-1}^{01} \right) \quad n > 2$$

$$\Lambda_5^{21} = \frac{1}{e^2} \left[\Lambda_5^{01} - 2\Lambda_4^{01} + \Lambda_3^{01} \right]$$

$$\Lambda_n^{m0} = 1/(-e)^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \Lambda_{n-k}^{00}$$

$$\Lambda_5^{31} = \frac{\frac{1}{1-e \cos E} - \frac{1.5}{(1-e \cos E)^2} + \frac{1}{(1-e \cos E)^3} - \frac{0.25}{(1-e \cos E)^4}}{e^4}$$

$$\Lambda_6^{21} = \frac{-\frac{0.2}{(1-e\cos E)^5} + \frac{0.5}{(1-e\cos E)^4} - \frac{1}{3(1-e\cos E)^3}}{e^3}$$

$$\Lambda_6^{31} = \frac{\frac{-0.2}{(1-e\cos E)^5} + \frac{3}{4(1-e\cos E)^4} \frac{1}{(1-e\cos E)^3} + \frac{0.5}{(1-e\cos E)^2}}{e^4}$$

$$\Lambda_6^{41} = -\frac{\cos^5 E}{5} - e \cos^6 E - 3e^2 \cos^7 E - 7e^3 \cos^8 E$$

with

$$\varphi = \frac{r}{a} \quad \eta = 1 - e^2$$

$$L = \frac{1}{a(1-e^2)}$$

$$(1 + e\cos\theta) = \frac{1-e^2}{1-e\cos E}$$

$$h_{0_k} = b_3 g_{3_k} + b_0 g_{0_k}$$

$$h_{1_k} = b_3 g_{4_k} + b_4 g_{3_k} + b_0 g_{1_k} + b_1 g_{0_k}$$

$$h_{2_k} = b_4 g_{4_k} - b_3 g_{3_k} + b_0 g_{2_k} + b_1 g_{1_k} + b_2 g_{0_k}$$

$$h_{3_k} = -b_3 g_{4_k} - b_4 g_{3_k} + b_1 g_{2_k} + b_2 g_{1_k}$$

$$h_{4_k} = b_2 g_{2_k} - b_4 g_{4_k}$$

$$h_{5_k} = b_0 g_{3_k} + b_3 g_{0_k}$$

$$h_{6_k} = b_0 g_{4_k} + b_1 g_{3_k} + b_3 g_{1_k} + b_4 g_{0_k}$$

$$h_{7_k} = b_1 g_{4_k} + b_2 g_{3_k} + b_3 g_{2_k} + b_4 g_{1_k}$$

$$h_{8_k} = b_2 g_{4_k} + b_4 g_{2_k}$$

$$m_{0_i} = b_0 f_{0_i} + b_3 f_{4_i}$$

$$m_{1_i} = b_1 f_{0_i} + b_3 f_{5_i} + b_4 f_{4_i} + b_0 f_{1_i}$$

$$m_{2_i} = b_2 f_{0_i} + b_3 f_{6_i} + b_4 f_{5_i} - b_3 f_{4_i} + b_2 f_{0_i} + b_1 f_{1_i}$$

$$m_{2_i} = b_2 f_{0_i} + b_3 f_{6_i} + b_4 f_{5_i} - b_3 f_{4_i} + b_2 f_{0_i} + b_1 f_{1_i}$$

$$m_{3_i} = b_4 f_{6_i} - b_3 f_{5_i} - b_4 f_{4_i} + b_0 f_{3_i} + b_1 f_{2_i} + b_2 f_{1_i}$$

$$m_{4_i} = -b_3 f_{6_i} - b_4 f_{5_i} + b_1 f_{3_i} + b_2 f_{2_i}$$

$$m_{5_i} = b_2 f_{3_i} - b_4 f_{6_i}$$

$$m_{6_i} = b_3 f_{0_i} - b_0 f_{4_i}$$

$$m_{7_i} = b_4 f_{0_i} + b_0 f_{5_i} + b_1 f_{4_i} + b_3 f_{1_i}$$

$$m_{8_i} = b_0 f_{6_i} + b_1 f_{5_i} + b_2 f_{4_i} + b_3 f_{2_i} + b_4 f_{1_i}$$

$$m_{9_i} = b_1 f_{6_i} + b_2 f_{5_i} + b_3 f_{3_i} + b_4 f_{2_i}$$

$$m_{10_i} = b_2 f_{6_i} + b_4 f_{3_i}.$$

4. Numerical results

To compute $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we have programed Equation (13) in double precision arithmetic on a NV57H26u Gateway computer. Four test cases A, B, C and D with eccentricities 0.01, 0.1, 0.2 and 0.3 at high inclination of 85° having a perigee height of 200 km are chosen for detailed numerical study. The initial values chosen for right ascension of ascending node (Ω) = 60º and argument of perigee (ω) = 0º. Numerical integration of the KS element equations of motion with J_2 [14] is carried out with fixed step-size fourth-order Runge-Kutta-Gill method with a step-size of one degree in eccentric anomaly. The difference between the numerically integrated and analytically computed values with the modified theory (ANAL1) and existing theory (ANAL2) during half a revolution with a single analytical step size are given in Table 1. It may be noted that the modified theory provides more accurate values of the semi-major axis during most of the part of the orbit in half a revolution. The improvement in the semi-major axis with the new theory at 90º step size is found to be 19.2 m, 11.4 m, 5 m and 3.7 m, respectively for the four test cases, against the error of 20.1 m, 23.7 m, 21.4 m and 8.6 m, respectively. The difference between the new theory and the old theory for i = 85 degrees in semi-major axis, eccentricity and inclination are given in Figures 1, 2 and 3, respectively.

Table 1: Variation in Semi-Major Axis during Half a Revolution with J_2

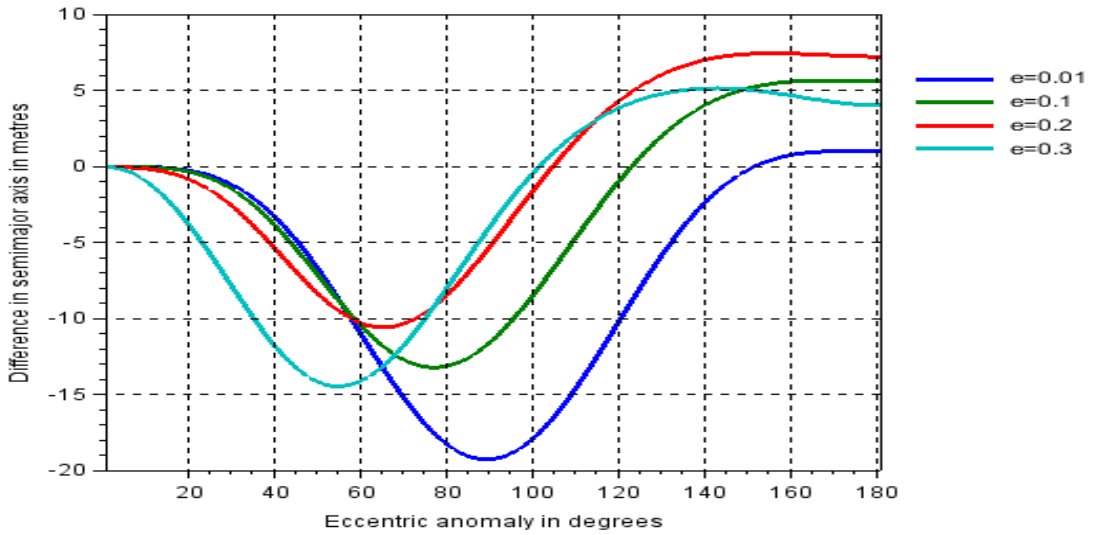| Parameter	Case	Method	ANAL steps (deg)					
			30	60	90	150	165	180
a(m)	A	NUM-ANAL1	-0.05	-3.6	-20.1	-3.8	21.9	43.8
		NUM-ANAL2	-1.33	-14.9	-39.3	-3.4	22.9	44.8
	B	NUM-ANAL1	-0.6	-8.6	-23.7	10.2	27.5	40.5
		NUM-ANAL2	-2.2	-19.2	-35.1	15.5	33.2	46.3
	C	NUM-ANAL1	-1.2	-14.0	-21.4	22.4	33.4	40.9
		NUM-ANAL2	-3.9	-24.4	-26.4	29.9	40.8	48.2
	D	NUM-ANAL1	0.8	-15.2	-8.6	36.7	43.5	47.9
		NUM-ANAL2	-7.3	-29.2	-12.3	41.7	47.9	51.9

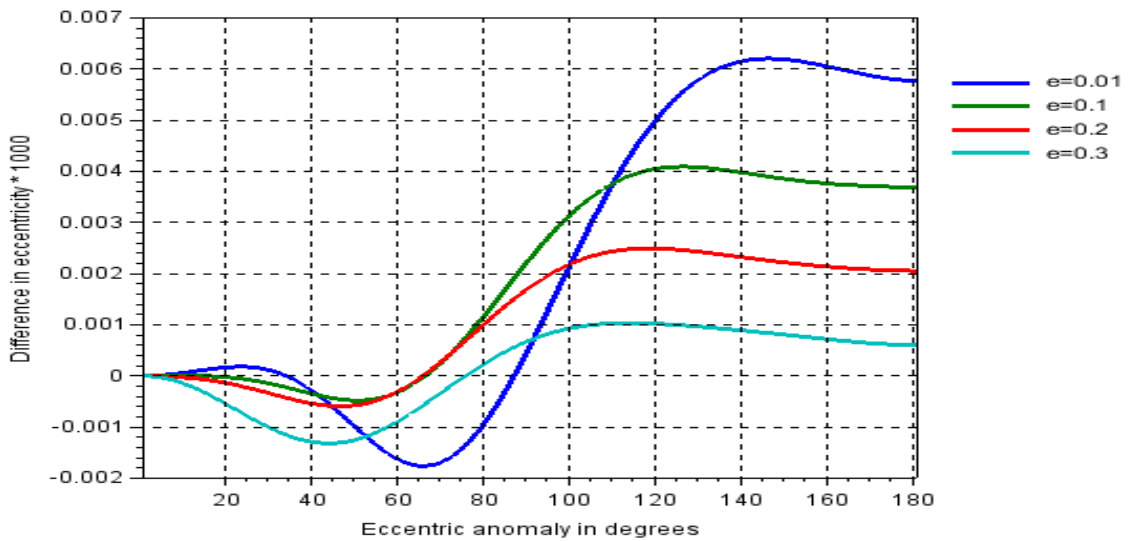**Fig. 1:** Difference in Semi-Major Axis between the Modified and Existing Theory.

Fig. 2: Difference in Eccentricity between the Modified and Existing Theory.
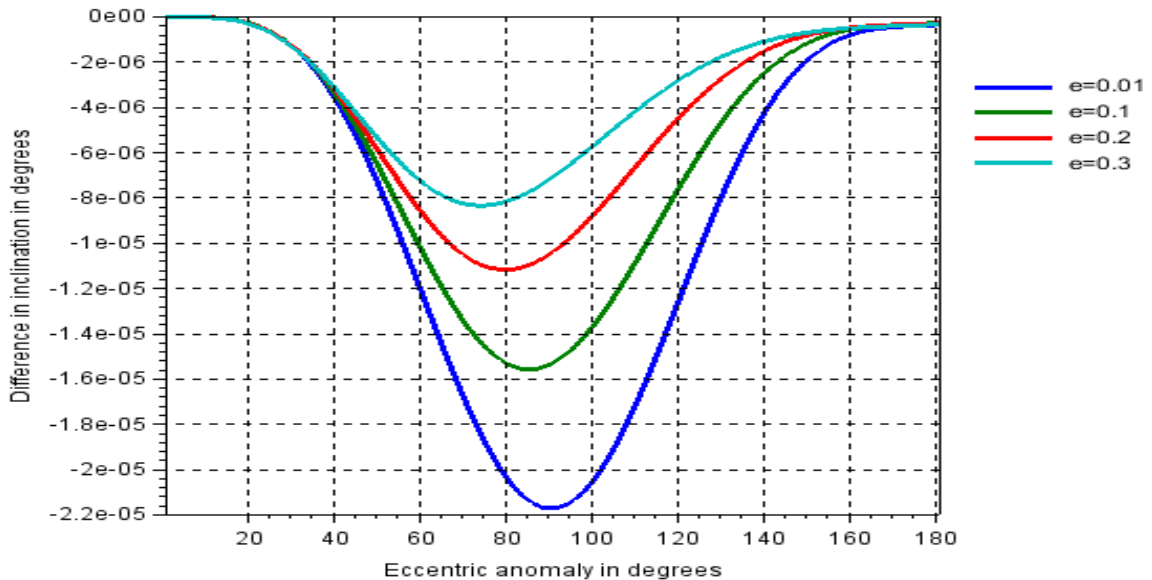

Fig. 3: Difference in Inclination between the Modified and Existing Theory.

Figures 1-3 provide the differences in semi-major axis, eccentricity and orbital inclination, respectively, between the modified and the old theory. It may be noted that the maximum improvement with the new theory is for the small eccentricity (0.01) case A and the magnitude of improvement in semi-major axis, eccentricity and inclination are ~19 m, 6.2 x 10⁻⁶ and 2.19° x 10⁻⁵, respectively. The maximum improvement happened with a single step-size in eccentric anomaly of 90° for semi-major axis and inclination, and a single step-size of ~148° for eccentricity.

5. Conclusion

A modified non-singular theory for analytical solution of short-term motion of satellites with J_2 is evolved using KS regular elements. The study showed that with the inclusion of the perturbed 'r' from King-Hele [10], the accuracy of the solution increases significantly. The results are documented up to half a revolution. The theory provides accurate solution for small as well as high inclination orbits.

Acknowledgement

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