International Journal of Advanced Astronomy, 4 (1) (2016) 49-56

International Journal of Advanced Astronomy

Website: www.sciencepubco.com/index.php/IJAA
doi: 10.14419/ijaa.v4i1.5893
Research paper

Existence and linear stability of triangular points in the perturbed relativistic R3BP when the bigger primary is an oblate spheroid

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Abstract

We study the effects of oblateness and small perturbations in the Coriolis and centrifugal forces on the locations and stability of the triangular points in the relativistic R3BP. It is observed that the positions are affected by the oblateness, relativistic, and a small perturbation in the centrifugal force, but are unaffected by that of Coriolis force. It is also seen that the relativistic terms, oblateness, small perturbations in the centrifugal and Coriolis forces influence the critical mass ratio. It is also noticed that all the former three and the latter one possess destabilizing and stabilizing behavior respectively. However, the range of stability increases or decreases according to as p >0 or p<0 where p depends upon the relativistic, oblateness and small perturbations in the Coriolis and centrifugal forces.

Keywords: *Celestial Mechanics; Perturbation; Relativity; Triaxiality; R3BP.*

1. Introduction

The restricted three-body problem (R3BP) in which two massive bodies (primaries) revolve around their common centre of mass in circular orbits and third body of negligible mass moves in their gravitational field, is a simple problem and has been receiving considerable attention of scientists and astronomers because of its applications in the dynamics of the solar and stellar systems, lunar theory and artificial satellites. It possesses five equilibrium points: three collinear L_1, L_2, L_3 and two triangular L_4, L_5, where the gravitational and centrifugal forces just balance each other. The collinear points are unstable; while the triangular points are stable for mass ratio $\mu < 0.038520...$ Szebehely [1].Their Their stability occurs in spite of the fact that the potential energy has a maximum rather than a minimum at the latter points. The stability is actually achieved through the influence of the Coriolis force, because the coordinate system is rotating (Wintner [2]; Contopolous [3]). Various contributions (Szebehely[4]; Bhatnagar and Hallan[5]; AbdulRaheem and Singh[6]; Singh and Begha[7]; and Abouelmagd et al. [8]) have been made on the study of the restricted three-body problem under the effects of small perturbations in the centrifugal and Coriolis forces. Szebehely [4] investigated the stability of triangular points by keeping the centrifugal force constant and found that the Coriolis force is a stabilizing force.

AbdulRaheem and Singh [6] investigated the stability of equilibrium points when the primaries are radiating oblate spheroids, and small perturbations are given to the Coriolis and centrifugal forces. They observed that the Coriolis force has a stabilizing tendency; while the centrifugal force, radiation, and the oblateness of the primaries have destabilizing effects. However, the overall effect decreases the range of stability. Abouelmagd et al. [8] studied the existence of equilibrium points, their linear stability and periodic orbits around these points under the effects of oblateness of three participating bodies as well as small perturbations in the Coriolis and centrifugal forces. They found that the positions of the collinear points, and the y-coordinate of the triangular points are not affected by the small perturbation in the Coriolis force. While the x-coordinate of triangular points is neither affected by a perturbation in the Coriolis force nor the oblateness of the third body. Furthermore, the critical mass value and the elements of periodic orbits around the equilibrium points such as the semi-major and the semi-minor axes, the angular frequencies and corresponding periods may change by all the parameters of oblateness as well as the small perturbations in the Coriolis and centrifugal forces.

The bodies in the R3BP are strictly spherical in shape, but in nature, celestial bodies are not perfect spheres. They are either oblate or triaxial. The Earth, Jupiter, Saturn, Regulus, Neutron stars and black dwarfs are oblate. The Moon, Pluto and its moon Charon are triaxial. The lack of sphericity, triaxiality or oblateness of the celestial bodies causes large perturbations from a two-body orbit. The motions of artificial satellite are examples of this. The most striking example of perturbations arising from the solar system is the orbit of the fifth satellite of Jupiter, Amalthea. This planet is so oblate, and the satellite's orbit is so small that its line of apsides advances about 900^0 in a year (Moulton [9]). These inspired several researchers (SubbaRao and Sharma [10]; Elipe and Ferrer [11]; Khanna and Bhatnagar[12]; Singh[13]; Sharma et al.[14]) to include non-sphericity of the bodies in their studies of the R3BP.

The general theory of relativity was developed by Einstein a century ago. Since then, it has become the standard theory of gravity, especially important to the field of fundamental astrometry, astrophysics, cosmology and experimental gravitational physics.

Brumberg [15], [16] studied the relativistic n-body problem of three bodies in more details and collected most of the important results on relativistic celestial mechanics. He did not only obtain the equations of motion for the general problem of three bodies but also deduced the equations of motion for the restricted problem of three bodies.

Bhatnagar and Hallan [17] studied the existence and linear stability of the triangular points $L_{4,5}$ in the relativistic R3BP, and found that $L_{4,5}$ are always unstable in the whole range $0 \le \mu \le \frac{1}{2}$ in contrast to the classical R3BP where they are stable for $\mu < \mu_0$, where μ is the mass ratio, and $\mu_0 = 0.03852...$ is the Routh's value.

Douskos and Perdios [18] investigated the stability of the triangular points in the relativistic R3BP and contrary to the results of Bhatnagar and Hallan [17], they they obtained a region of linear stability in the parameter space $0 \le \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2}$ where $\mu_0 = 0.03852...$ is Routh's value.

Recently, some studies have been focused on the relativistic R3BP by taking the primaries as source of radiation or oblate spheroids or both and the small perturbations in the centrifugal and Coriolis forces.

Katour et al. [19] studied the positions of the triangular points within the framework of the post-Newtonian approximation when the masses of primaries are assumed to change under the effect of continuous radiation process and oblateness effects of the two primaries. New perturbed locations of the triangular points are computed.

Singh and Bello [20] investigated the effect of radiation pressure of the bigger primary in the relativistic R3BP, and they found that the positions of triangular points and their stability are affected by both the relativistic and radiation factors.

Singh and Bello [21] examined the effect of a small perturbation in the centrifugal force in the relativistic R3BP and noticed that the positions and stability of the triangular points are affected by both the relativistic factor and a small perturbation in the centrifugal force. Singh and Bello [22] studied the effect of small perturbations in $\varepsilon_1, \varepsilon_2$ in centrifugal and Coriolis forces respectively in the relativistic R3BP without considering the coupling terms $\frac{\varepsilon_i}{c^2}(i = 1,2)$ where c is the speed of light. They observed that the stability region depends on the relativistic terms and small perturbations $\varepsilon_1, \varepsilon_2$; while the y-coordinate depends upon relativistic and centrifugal terms and the x-coordinate depends on relativistic terms only.

The classical problem of three bodies has been studied by considering various aspects such as the shape of the bodies, influence of the perturbations in the Coriolis and the centrifugal forces, etc. to make the problem more realistic. Some of the planets, like Saturn and Jupiter are sufficiently oblate. It has been seen that oblateness of the body plays an important role in the restricted three-body problem.

Hence, the idea of small perturbations in the Coriolis and centrifugal forces together with oblateness of the body raises a curiosity in our mind to study the stability of triangular points in the relativistic R3BP.

This paper is organized as follows: In Sect. 2, the equations governing the motion are presented; Sect. 3 describes the positions of triangular points, while their linear stability is analyzed in Sect.4; the discussion is given in Sect. 5, finally sect. 6 summarizes the results of this paper.

2. Equations of motion

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system (ξ, η) and dimensionless variables can be written as Brumberg [15] and Bhatnagar and Hallan [17].

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right) \\ \ddot{\eta} + 2n\dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \end{aligned} \tag{1}$$

With

$$\begin{aligned} W =\ & \frac{1}{2}(\xi^2+\eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \\ & \frac{1}{c^2}\Bigg[-\frac{3}{2}\left(1-\frac{1}{3}\mu(1-\mu)\right)(\xi^2+\eta^2) + \\ & \frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)\right\}^2 \\ & +\frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\left(\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)\right) - \\ & \frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{\mu^2}{\rho_2^2}\right) + \mu(1-\mu)\Bigg\{\left(4\dot{\eta}+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right) \\ & -\frac{\eta^2}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right) + \left(\frac{-1}{\rho_1\rho_2}+\frac{3\mu-2}{2\rho_1}+\frac{1-3\mu}{2\rho_2}\right)\Bigg\}\Bigg] \end{aligned} \tag{2}$$

$$n = 1 - \frac{3}{2c^2}\left(1-\frac{1}{3}\mu(1-\mu)\right) \tag{3}$$

$$\begin{aligned} \rho_1^2 &= (\xi+\mu)^2+\eta^2 \\ \rho_2^2 &= (\xi+\mu-1)^2+\eta^2 \end{aligned} \tag{4}$$

Where $0 < \mu \le \frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries, ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively; n is the mean motion of the primaries; c is the velocity of light. We now introduce small perturbations in the centrifugal and Coriolis forces by parameters $\psi = 1+\varepsilon_1$; $|\varepsilon_1| << 1, \phi = 1+\varepsilon_2; |\varepsilon_2| << 1$, and oblateness of the bigger primary by the parameter $A_1 = \frac{AE^2 - AP^2}{5R^2} << 1$ (McCuskey [23]) where AE and AP are the equatorial and polar radii of the bigger primary, and R is the distance between the primaries. Neglecting second and higher powers of A_1, we take equations of motion as:

$$\begin{aligned} \ddot{\xi} - 2\phi n\dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right) \\ \ddot{\eta} + 2\phi n\dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \end{aligned} \tag{5}$$

$$
\begin{aligned}
W =\ & \frac{1}{2}\psi\left(1+\frac{3}{2}A_1\right)(\xi^2+\eta^2)+\frac{(1-\mu)}{\rho_1}\left(1+\frac{3}{2\rho_1^2}A_1\right)+ \\
& \frac{\mu}{\rho_2}+\frac{1}{c^2}\left[-\frac{3}{2}\left\{1-\frac{1}{3}\mu(1-\mu)\right\}\psi\left(1+\frac{3}{4}A_1\right)(\xi^2+\eta^2)\right. \\
& +\frac{1}{8}\left\{\phi\left(\dot{\xi}^2+\dot{\eta}^2\right)+\left(2+\frac{3}{2}A_1\right)\left(\xi\dot{\eta}-\eta\dot{\xi}\right)\phi+\psi\left(1+\frac{3}{2}A_1\right)(\xi^2+\eta^2)\right\}^2 \\
& +\frac{3}{2}\left(\frac{1-\mu}{\rho_1}\left(1+\frac{3}{2\rho_1^2}A_1\right)+\frac{\mu}{\rho_2}\right)\left\{\begin{array}{l}\phi\left(\dot{\xi}^2+\dot{\eta}^2\right)+ \\ \left(2+\frac{3}{2}A_1\right)\phi\left(\xi\dot{\eta}-\eta\dot{\xi}\right)+ \\ \left(1+\frac{3}{2\rho_1^2}A_1\right)\psi(\xi^2+\eta^2)\end{array}\right\} \\
& -\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}\left(1+\frac{A_1}{2\rho_1^2}\right)^2+\frac{\mu^2}{\rho_2^2}\right)+ \\
& \mu(1-\mu)\left\{\left(\left(4+3A_1\right)\phi\dot{\eta}+\left(\frac{7}{2}+\frac{21}{4}A_1\right)\psi\xi\right)\left(\frac{1}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right)-\frac{1}{\rho_2}\right)\right. \\
& -\frac{1}{2}\left(1+\frac{3A_1}{2}\right)\psi\eta^2\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+ \\
& \left.\left.\left(1+\frac{3A_1}{2}\right)\psi\left(\frac{-1}{\rho_1\rho_2}+\frac{\mu}{2\rho_1}-\frac{1-\mu}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right)+\frac{1-3\mu}{2\rho_2}\right)\right\}\right]
\end{aligned}
\tag{6}
$$

And n , the perturbed mean motion of the primaries is given by

$$
n=1+\frac{3}{4}A_1-\frac{3}{2c^2}\left(1-\frac{1}{3}\mu(1-\mu)\right) \tag{7}
$$

3. Locations of triangular points

The libration points are obtained from equations (5) after putting

$\dot{\xi}=\dot{\eta}=\ddot{\xi}=\ddot{\eta}=0.$

These points are the solutions of the equations

$\frac{\partial W}{\partial \xi}=0=\frac{\partial W}{\partial \eta}$ With $\dot{\xi}=\dot{\eta}=0.$

That is, substituting the values of $\psi=1+\varepsilon_1$ and $\phi=1+\varepsilon_2$ in the above equations and neglecting second and higher orders terms of $\varepsilon_1,\varepsilon_2,A_1$ and their products, we get

$$
\begin{aligned}
& \xi-\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}-\frac{\mu(\xi-1+\mu)}{\rho_2^3}+\frac{3}{2}A_1\left\{\xi-\frac{(1-\mu)(\xi+\mu)}{\rho_1^5}\right\}-\varepsilon_1\xi+\frac{1}{c^2}\left[-3\xi\left\{1-\frac{\mu(1-\mu)}{3}\right\}+\frac{1}{2}\xi(\xi^2+\eta^2)\right. \\
& -\frac{3}{2}(\xi^2+\eta^2)\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\}+3\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\xi+\frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4}+\frac{\mu^2(\xi-1+\mu)}{\rho_2^4} \\
& +\mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)+\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3}\right. \\
& \left.+\frac{(\xi+\mu)}{\rho_1^3\rho_2}-\frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right\}+\varepsilon_1\left\{-3\xi\left(1-\frac{\mu(1-\mu)}{3}\right)+\xi(\xi^2+\eta^2)\right. \\
& \left.-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\right)+3\xi\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\right\}+\mu(1-\mu)\varepsilon_1\left\{\frac{7}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right. \\
& +\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\left(\frac{(\xi+\mu)}{\rho_1^3\rho_2}+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3}-\frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}\right. \\
& \left.\left.-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right)\right\}+A_1\left\{-\frac{9}{4}\left(1-\frac{\mu(1-\mu)}{3}\right)\xi+\frac{3}{2}(\xi^2+\eta^2)\xi+\left(-\frac{9(1-\mu)(\xi+\mu)}{4\rho_1^5}-\frac{9(1-\mu)(\xi+\mu)}{4\rho_1^3}\right.\right. \\
& \left.\left.-\frac{9\mu(\xi-1+\mu)}{4\rho_2^3}\right)(\xi^2+\eta^2)+\left(\frac{3(1-\mu)}{2\rho_1^3}+\frac{9(1-\mu)}{2\rho_1}+\frac{9\mu}{2\rho_2}\right)\xi+\frac{2(1-\mu)^2(\xi+\mu)}{\rho_1^6}\right\}+\mu(1-\mu)A_1\left\{\frac{7}{4\rho_1^3}+\frac{21}{4}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right. \\
& +\frac{7}{2}\xi\left(-\frac{3(\xi+\mu)}{2\rho_1^5}\right)+\frac{21}{4}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{9}{4}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\frac{3(1-\mu)(\xi+\mu)}{2\rho_1^5} \\
& \left.\left.+\frac{3(\xi+\mu)}{2\rho_1^3\rho_2}+\frac{3(\xi-1+\mu)}{2\rho_1\rho_2^3}-\frac{3\mu(\xi+\mu)}{4\rho_1^3}+\frac{3(1-\mu)(\xi+\mu)}{2\rho_1^3}-\frac{3(1-3\mu)(\xi-1+\mu)}{4\rho_2^3}\right\}\right]=0
\end{aligned}
\tag{8}
$$

and $\eta F=0,$

Where,

$$
\begin{aligned}
F =\ & \left(1-\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}+\frac{3}{2}A_1\left(1-\frac{1-\mu}{\rho_1^5}\right)+\varepsilon_1\right)+ \\
& \frac{1}{c^2}\left[-3\left(1-\frac{\mu(1-\mu)}{3}\right)+\frac{1}{2}(\xi^2+\eta^2)+3\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\right. \\
& -\frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)+\left(\frac{(1-\mu)^2}{\rho_1^4}+\frac{\mu^2}{\rho_2^4}\right)+ \\
& \mu(1-\mu)\left\{\begin{array}{l}\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+ \\ \frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{(1-\mu)}{\rho_2^5}\right)\end{array}\right. \\
& \left.+\frac{1}{\rho_1\rho_2^3}+\frac{1}{\rho_1^3\rho_2}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right\}+ \\
& \varepsilon_1\left\{-3\left(1-\frac{1}{3}\mu(1-\mu)\right)+(\xi^2+\eta^2)+\right. \\
& \left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3}\right. \\
& \left.\left.+\frac{\mu}{\rho_2^3}\right)\right\}+\mu(1-\mu)\varepsilon_1\left\{\begin{array}{l}\frac{7}{2}\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+ \\ \frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{(1-\mu)}{\rho_2^5}\right)\end{array}\right. \\
& +\left(\frac{1}{\rho_1^3\rho_2}+\frac{1}{\rho_1\rho_2^3}-\frac{(3\mu-2)}{2\rho_1^3}\right. \\
& \left.\left.-\frac{(1-3\mu)}{2\rho_2^3}\right)\right\}+A_1\left\{-\frac{9}{4}\left(1-\frac{\mu(1-\mu)}{3}\right)+\frac{3}{2}(\xi^2+\eta^2)+\right. \\
& \left(\frac{-9(1-\mu)}{4\rho_1^5}-\frac{9(1-\mu)}{4\rho_1^3}-\frac{9\mu}{4\rho_2^3}\right)(\xi^2+\eta^2)+\frac{3(1-\mu)}{2\rho_1^3}+\frac{9(1-\mu)}{2\rho_1} \\
& \left.+\frac{9\mu}{2\rho_2}+\frac{2(1-\mu)^2}{\rho_1^6}\right\}+\mu(1-\mu)A_1\left\{\frac{21}{4}\xi\left(-\frac{1}{\rho_1^5}-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\right. \\
& \frac{3}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\frac{9}{4}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{1-\mu}{\rho_2^5}\right)+\frac{3}{2}\left(\frac{1-\mu)}{\rho_1^5}+\frac{1-\mu}{\rho_1^3}\right. \\
& \left.\left.\left.+\frac{1}{\rho_1^3\rho_2}+\frac{1}{\rho_1\rho_2^3}-\frac{\mu}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right)\right\}\right]
\end{aligned}
$$

The triangular points are the solutions of equation (8) with $\eta \neq 0$. Since $\frac{1}{c^2} << 1$ and in the case $\frac{1}{c^2} \to 0$ and in the absence of small perturbations and oblateness $\left(i.e. \varepsilon_1 = \varepsilon_2 = A_1 = 0\right)$, one can obtain $\rho_1 = \rho_2 = 1$; we assume in the relativistic R3BP that $\rho_1 = 1 + x$ and $\rho_2 = 1 + y$ where $x, y << 1$, may be depending upon relativistic, perturbations and oblateness factors. Substituting these values in the equations (4), solving them for ξ, η and ignoring terms of second and higher powers of x and y, we get

$$\xi = x - y + \frac{1-2\mu}{2}$$

$$\eta = \pm\left(\frac{\sqrt{3}}{2} + \frac{x+y}{\sqrt{3}}\right)$$

Substituting the values of $\rho_1, \rho_2, \xi, \eta$ and neglecting of second and higher orders terms x^2, y^2, $\frac{x}{c^2}, \frac{y}{c^2}, \varepsilon_1, \varepsilon_2, A_1$ etc in equations (8) with $\eta \neq 0$, we have

$$\begin{aligned}
&\left\{\frac{3}{2}(1-\mu) + \frac{3}{2}A_1\left(\frac{5}{2} - \frac{3\mu}{2}\right)\right\}x - \\
&\left\{\frac{3\mu}{2} + \frac{3\mu}{2}A_1\right\}y - \frac{3\mu}{4}A_1 + \frac{1}{c^2}\left\{\left(-\frac{9\mu}{16} + \frac{27\mu^2}{16} - \frac{9\mu^3}{8}\right) + \right. \\
&\left(\frac{11}{8} - \frac{125\mu}{32} + \frac{183\mu^2}{32} - \frac{63\mu^3}{8}\right)A_1 + \\
&\left.\left(\frac{8\mu^3 - 18\mu^2 + 17\mu - 4}{16}\right)\varepsilon_1\right\} = 0
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\left\{3(1-\mu) + \frac{15}{2}(1-\mu)A_1\right\}x + 3\mu y + \frac{3\mu A_1}{2} + \\
&\frac{1}{c^2}\left\{\frac{21\left(\mu-\mu^2\right)}{8} + \left(\frac{11}{4} + \frac{53\mu}{16} - \frac{49\mu^2}{16} - \frac{3\mu^3}{2}\right)A_1\right. \\
&\left.+\left(\frac{-5\mu^2 + 5\mu + 2}{2}\right)\varepsilon_1\right\} = 0
\end{aligned}$$

Solving these equations for x and y, we obtain

$$\begin{aligned}
x &= -\frac{\mu(2+3\mu)}{8c^2} - \left(\frac{-44 + 51\mu - 22\mu^2 + 30\mu^3}{48(\mu-1)c^2}\right)A_1 + \\
&\left(\frac{-8\mu^3 + 8\mu^2 - 7\mu + 8}{24(\mu-1)c^2}\right)\varepsilon_1, \\
y &= -\frac{(1-\mu)(5-3\mu)}{8c^2} - \left(\frac{1}{2} + \frac{74 - 86\mu + 51\mu^2}{48c^2}\right)A_1 + \left(\frac{-8\mu^2 + 28\mu - 27}{24c^2}\right)\varepsilon_1
\end{aligned} \tag{10}$$

Thus, the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5 respectively are,

$$\begin{aligned}
\xi &= \frac{1-2\mu}{2}\left(1 + \frac{5}{4c^2}\right) + \\
&\left(\frac{1}{2} - \frac{\left(30 - 109\mu + 115\mu^2 - 21\mu^3\right)}{48(\mu-1)c^2}\right)A_1 + \left(\frac{-18\mu^2 + 33\mu - 14}{24(\mu-1)c^2}\right)\varepsilon_1, \\
\eta &= \pm\left\{\frac{\sqrt{3}}{2}\left(1 + \frac{1}{12c^2}\left(-5 + 6\mu - 6\mu^2\right)\right) + \right. \\
&\left(-\frac{\sqrt{3}}{6} - \frac{\sqrt{3}\left(-118 + 211\mu - 159\mu^2 + 81\mu^3\right)}{144(\mu-1)c^2}A_1\right) + \left(\frac{-2\sqrt{3}}{9}\right. \\
&\left.\left. + \frac{-6\mu^2 + 6\mu + 5}{54c^2} + \frac{\sqrt{3}\left(-10\mu^3 + 32\mu^2 - 51\mu + 30\right)}{72(\mu-1)c^2}\right)\varepsilon_1\right\}
\end{aligned} \tag{11}$$

4. Stability of L_4

Let (a,b) be the coordinates of the triangular points L_4
We set $\xi = a + \alpha, \eta = b + \beta, (\alpha, \beta << 1)$ in the equations (5).
First, we compute the terms on their R.H.S, neglecting second and higher order terms, we get

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi = a+\alpha, \eta = b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta}$$

Where,

$$\begin{aligned}
A &= \frac{3}{4}\left\{1 + \frac{1}{2c^2}\left(2 - 19\mu + 19\mu^2\right)\right\} + \\
&\left\{-\frac{3(8\mu-9)}{8} - \frac{\left(226 - 1036\mu + 1758\mu^2 - 1056\mu^3 + 87\mu^4\right)}{32(\mu-1)c^2}\right\}A_1 \\
&+\left\{\frac{5}{4} + \frac{-31 + 168\mu - 264\mu^2 + 126\mu^3}{8(\mu-1)c^2}\right\}\varepsilon_1,
\end{aligned}$$

$$\begin{aligned}
B &= \frac{3\sqrt{3}}{4}(1-2\mu)\left(1 - \frac{2}{3c^2}\right) + \\
&\left\{-\frac{\sqrt{3}(26\mu-19)}{8} - \frac{\sqrt{3}\left(-46 + 393\mu - 599\mu^2 - 135\mu^3 + 417\mu^4\right)}{96(\mu-1)c^2}\right\}A_1 \\
&+\left\{\frac{11\sqrt{3}(1-2\mu)}{12} - \frac{\sqrt{3}(34 - 279\mu + 635\mu^2 - 528\mu^3 + 132\mu^4)}{144(\mu-1)c^2}\right\}\varepsilon_1,
\end{aligned}$$

$$\begin{aligned}
C &= \frac{\sqrt{3}}{2c^2}(1-2\mu) + \left\{-\frac{\sqrt{3}(46\mu-35)}{24c^2}\right\}A_1 + \\
&\left\{\frac{5\sqrt{3}(1-2\mu)}{18c^2}\right\}\varepsilon_1 + \left\{\frac{\sqrt{3}(1-2\mu)}{2c^2}\right\}\varepsilon_2,
\end{aligned}$$

$$D=\frac{6-5\mu+5\mu^2}{2c^2}+\left\{\frac{\left(22-33\mu+45\mu^2\right)}{8c^2}\right\}A_1+$$
$$\left(\frac{4-15\mu+15\mu^2}{6c^2}\right)\varepsilon_1+\left(\frac{6-5\mu+5\mu^2}{2c^2}\right)\varepsilon_2.$$

Similarly, we obtain

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha,\eta=b+\beta}=E\alpha+B_1\beta+C_1\dot{\alpha}+D_1\dot{\beta}$$

Where,

$$E=\frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3c^2}\right)+$$
$$\left\{\begin{array}{l}-\frac{\sqrt{3}(26\mu-19)}{8}-\\ \frac{\sqrt{3}\left(-46+393\mu-599\mu^2-135\mu^3+417\mu^4\right)}{96(\mu-1)c^2}\end{array}\right\}A_1$$
$$+\left\{\begin{array}{l}\frac{11\sqrt{3}(1-2\mu)}{12}-\\ \frac{\sqrt{3}(34-279\mu+635\mu^2-528\mu^3+132\mu^4)}{144(\mu-1)c^2}\end{array}\right\}\varepsilon_1,$$

$$B_1=\frac{9}{4}\left\{1+\frac{7}{6c^2}\left(-2+3\mu-3\mu^2\right)\right\}+$$
$$\left\{\frac{33}{8}+\frac{\left(290-1292\mu+2028\mu^2-1221\mu^3+111\mu^4\right)}{96(\mu-1)c^2}\right\}A_1$$
$$+\left\{\frac{7}{4}+\frac{55-168\mu+216\mu^2-102\mu^3}{8c^2}\right\}\varepsilon_1,$$

$$C_1=\frac{1}{2c^2}\left(-4+\mu-\mu^2\right)+\left\{-\frac{\left(20-13\mu+9\mu^2\right)}{8c^2}\right\}A_1+$$
$$\left(\frac{-2+\mu-\mu^2}{2c^2}\right)\varepsilon_1+\left(\frac{-4+\mu-\mu^2}{2c^2}\right)\varepsilon_2,$$

$$D_1=-\frac{\sqrt{3}(1-2\mu)}{2c^2}+\left\{\frac{\sqrt{3}(46\mu-35)}{24c^2}\right\}A_1+$$
$$\left\{\frac{-5\sqrt{3}(1-2\mu)}{18c^2}\right\}\varepsilon_1+\left\{\frac{-\sqrt{3}(1-2\mu)}{2c^2}\right\}\varepsilon_2.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)_{\xi=a+\alpha,\eta=b+\beta}=A_2\dot{\alpha}+B_2\dot{\beta}+C_2\ddot{\alpha}+D_2\ddot{\beta}$$

Where,

$$A_2=\frac{\sqrt{3}}{2c^2}(1-2\mu)+\left\{-\frac{\sqrt{3}(46\mu-35)}{24c^2}\right\}A_1+$$
$$\left\{\frac{5\sqrt{3}(1-2\mu)}{18c^2}\right\}\varepsilon_1+\left\{\frac{\sqrt{3}(1-2\mu)}{2c^2}\right\}\varepsilon_2,$$

$$B_2=-\frac{1}{2c^2}\left(4-\mu+\mu^2\right)+\left\{-\frac{\left(20-13\mu+9\mu^2\right)}{8c^2}\right\}A_1+$$
$$\left(\frac{-2+\mu-\mu^2}{2c^2}\right)\varepsilon_1+\left(\frac{-4+\mu+\mu^2}{2c^2}\right)\varepsilon_2,$$

$$C_2=\frac{1}{4c^2}\left(17-2\mu+2\mu^2\right)+\left\{\frac{\left(23-10\mu+6\mu^2\right)}{8c^2}\right\}A_1+$$
$$\left(\frac{1-\mu+\mu^2}{2c^2}\right)\varepsilon_1+\left(\frac{10-\mu+\mu^2}{2c^2}\right)\varepsilon_2,$$

$$D_2=-\frac{\sqrt{3}}{4c^2}(1-2\mu)+\left\{\frac{\sqrt{3}(14\mu-13)}{24c^2}\right\}A_1+$$
$$\left\{\frac{\sqrt{3}(1-2\mu)}{9c^2}\right\}\varepsilon_1+\left\{\frac{\sqrt{3}(1-2\mu)}{2c^2}\right\}\varepsilon_2.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha,\eta=b+\beta}=A_3\dot{\alpha}+B_3\dot{\beta}+C_3\ddot{\alpha}+D_3\ddot{\beta}$$

Where,

$$A_3=\frac{1}{2c^2}\left(6-5\mu+5\mu^2\right)+\left\{\frac{\left(22-33\mu+45\mu^2\right)}{8c^2}\right\}A_1+$$
$$\left(\frac{4-15\mu+15\mu^2}{6c^2}\right)\varepsilon_1+\left(\frac{6-5\mu+5\mu^2}{2c^2}\right)\varepsilon_2,$$

$$B_3=-\frac{\sqrt{3}}{2c^2}(1-2\mu)+\left\{\frac{\sqrt{3}(46\mu-35)}{24c^2}\right\}A_1+$$
$$\left\{-\frac{5\sqrt{3}(1-2\mu)}{18c^2}\right\}\varepsilon_1-\left\{\frac{\sqrt{3}(1-2\mu)}{2c^2}\right\}\varepsilon_2,$$

$$C_3=-\frac{\sqrt{3}}{4c^2}(1-2\mu)+\left\{\frac{\sqrt{3}(14\mu-13)}{24c^2}\right\}A_1+$$
$$\left\{\frac{\sqrt{3}(1-2\mu)}{9c^2}\right\}\varepsilon_1-\left\{\frac{\sqrt{3}(1-2\mu)}{4c^2}\right\}\varepsilon_2,$$

$$D_3=\frac{3\left(5-2\mu+2\mu^2\right)}{4c^2}+\left\{\frac{\left(25-30\mu+18\mu^2\right)}{8c^2}\right\}A_1+$$
$$\left(\frac{7-3\mu+3\mu^2}{6c^2}\right)\varepsilon_1-\left\{\frac{\sqrt{3}(1-2\mu)}{4c^2}\right\}\varepsilon_2.$$

Thus, the variational equations of motion corresponding to (5), on making use of equation (7), can be expressed as

$$P_1\ddot{\alpha}+P_2\ddot{\beta}+P_3\dot{\alpha}+P_4\dot{\beta}+P_5\alpha+P_6\beta=0, \tag{12}$$

$$q_1\ddot{\alpha}+q_2\ddot{\beta}+q_3\dot{\alpha}+q_4\dot{\beta}+q_5\alpha+q_6\beta=0.$$

Where,

$P_1 = 1 + C_2, P_2 = D_2, P_3 = A_2 -$
$C\,, P_4 = \left\{ B_2 - 2\left(1 + \frac{3}{4}A_1 + \varepsilon_2 - \frac{3}{2c^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)(1+\varepsilon_2)\right) - D \right\},$
$P_5 = -A\;\;, P_6 = -B\,,$

$q_1 = C_3, q_2 = 1 +$
$D_3, q_3 = 2\left(1 + \frac{3}{4}A_1 + \varepsilon_2 - \frac{3}{2c^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)(1+\varepsilon_2)\right) -$
$C_1 + A_3, q_4 = B_3 - D_1,$
$q_5 = -E\,, q_6 = -B_1$

Then, the corresponding characteristic equation is

$$(P_1q_2 - P_2q_1)\lambda^4 + (P_1q_6 + P_5q_2 + P_3q_4 - P_6q_1 - P_2q_5 - P_4q_3)\lambda^2 + P_5q_6 - P_6q_5 = 0 \tag{13}$$

Substituting the values of $P_i, q_i, i = 1,2,...,6$ in (13) and neglecting second and higher powers of small quantities, the characteristic equation (13) becomes

$$\lambda^4 + b\lambda^2 + d = 0 \tag{14}$$

Where,

$$b = \left(1 - \frac{9}{c^2}\right) + \left\{ -\frac{3}{2} + 3\mu + \frac{\left(-80 - 108\mu + 105\mu^2 - 18\mu^3\right)}{8c^2} \right\} A_1 + \left\{ -3 + \frac{68 - 25\mu + 25\mu^2}{4c^2} \right\}\varepsilon_1 + \left\{ 8 + \frac{-147 + 30\mu - 30\mu^2}{2c^2} \right\}\varepsilon_2,$$

$$d = \frac{27\mu(1-\mu)}{4} + \frac{9\mu\left(-65 + 77\mu - 24\mu^2 + 12\mu^3\right)}{8c^2} + \left\{ \frac{117\mu(1-\mu)}{4} + \frac{3\left(80 - 7245\mu + 9624\mu^2 - 3366\mu^3 + 846\mu^4\right)}{64c^2} \right\} A_1 + \left\{ \frac{33\mu(1-\mu)}{2} - \frac{3\mu(1867 - 2082\mu + 540\mu^2 - 336\mu^3)}{32c^2} \right\}\varepsilon_1 + \left\{ \frac{-243\mu + 324\mu^2 - 162\mu^3 + 81\mu^4}{4c^2} \right\}\varepsilon_2.$$

For $\frac{1}{c^2} \to 0$ and in the absence of small perturbations in the centrifugal and Coriolis forces and oblateness (14) (*i.e.* $\varepsilon_1 = \varepsilon_2 = A_1 = 0$), reduces to its well-known classical restricted problem form (see e.g. Szebehely [1]):

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0.$$

The discriminant of (14) is

$$\Delta = \left(\frac{-54}{c^2} - \frac{1269}{8c^2}A_1 - \frac{126}{c^2}\varepsilon_1 - \frac{81}{c^2}\varepsilon_2 \right)\mu^4 + \left(\frac{108}{c^2} + \frac{5013}{8c^2}A_1 + \frac{405}{2c^2}\varepsilon_1 - \frac{126}{c^2}\varepsilon_2 \right)\mu^3 + \Big(27 + 117A_1 + 66\varepsilon_1 - \frac{7113}{4c^2}A_1 - \frac{3073}{4c^2}\varepsilon_1 - \frac{354}{c^2}\varepsilon_2 - \frac{693}{2c^2}\Big)\mu^2 + \left(-27 - 111A_1 - 66\varepsilon_1 + \frac{20439}{16c^2}A_1 + \frac{5501}{8c^2}\varepsilon_1 + \frac{273}{c^2}\varepsilon_2 + \frac{585}{2c^2} \right)\mu + 1 - 3A_1 - 6\varepsilon_1 + 16\varepsilon_2 - \frac{8}{c^2}A_1 + \frac{88}{c^2}\varepsilon_1 - \frac{291}{c^2}\varepsilon_2 - \frac{18}{c^2}. \tag{15}$$

Its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \tag{16}$$

Where,

$$b = \left(1 - \frac{9}{c^2}\right) + \left\{ -\frac{3}{2} + 3\mu + \frac{\left(-80 - 108\mu + 105\mu^2 - 18\mu^3\right)}{8c^2} \right\} A_1 + \left\{ -3 + \frac{68 - 25\mu + 25\mu^2}{4c^2} \right\}\varepsilon_1 + \left\{ 8 + \frac{-147 + 30\mu - 30\mu^2}{2c^2} \right\}\varepsilon_2,$$

From (15), we have

$$\frac{d\Delta}{d\mu} = 4\left(\frac{-54}{c^2} - \frac{1269}{8c^2}A_1 - \frac{126}{c^2}\varepsilon_1 - \frac{81}{c^2}\varepsilon_2 \right)\mu^3 + 3\left(\frac{108}{c^2} + \frac{5013}{c^2}A_1 + \frac{405}{2c^2}\varepsilon_1 + \frac{162}{c^2}\varepsilon_2 \right)\mu^2 + 2\Big(27 + 117A_1 + 66\varepsilon_1 - \frac{7113}{4c^2}A_1 - \frac{3073}{4c^2}\varepsilon_1 - \frac{354}{c^2}\varepsilon_2 - \frac{693}{2c^2}\Big)\mu + \left(-27 - 111A_1 - 66\varepsilon_1 + \frac{20439}{16c^2}A_1 + \frac{5501}{8c^2}\varepsilon_1 + \frac{273}{c^2}\varepsilon_2 + \frac{585}{2c^2} \right) < 0$$

For

$$\mu \in \left(0, \frac{1}{2}\right) \tag{17}$$

From (17), it can be easily seen that Δ is monotone decreasing in $\left(0, \frac{1}{2}\right)$.

But

$$(\Delta)_{\mu=0} = 1 - \frac{18}{c^2} - 3A_1 - \frac{8}{c^2}A_1 - 6\varepsilon_1 + \frac{88}{c^2}\varepsilon_1 + 16\varepsilon_2 - \frac{291}{c^2}\varepsilon_2 > 0 \tag{18}$$

$$(\Delta)_{\mu=\frac{1}{2}} = -\frac{23}{4} + \frac{207}{4c^2} - \frac{117}{4}A + \frac{32585}{128c^2}A_1 + \frac{4115}{16c^2}\varepsilon_1 + 16\varepsilon_2 - \frac{3645}{16c^2}\varepsilon_2 < 0$$

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone decreasing and continuous, there is one value of μ, e.g. μ_c in the interval $\left(0,\frac{1}{2}\right)$ for which Δ vanishes.

Solving the equation $\Delta=0$, using (13), we obtain critical value of the mass parameter as

$$\mu_c=\frac{1}{2}-\frac{1}{18}\sqrt{69}-\frac{17\sqrt{69}}{486c^2}-\frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right)A_1+\left(\frac{-19733+15493\sqrt{69}}{536544c^2}\right)A_1+\frac{4(36\varepsilon_2-19\varepsilon_1)}{27\sqrt{69}}+\left(\frac{34155+175301\sqrt{69}}{804816c^2}\right)\varepsilon_1-\left(\frac{47\sqrt{69}}{81c^2}\right)\varepsilon_2 \tag{19}$$

There are three possible cases regarding the sign of the discriminant Δ

i) When $0\le\mu<\mu_c$, $\Delta>0$, the values of λ^2 given by (16) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.

ii) When, $\mu_c<\mu\le\frac{1}{2},\Delta<0$ the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.

iii) When, $\mu=\mu_c,\Delta=0$ the values of λ^2 given by (16) are the same. This induces instability of the triangular points.

Hence, the stability region is

$$0\le\mu<\mu_0-\frac{17\sqrt{69}}{486c^2}-\frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right)A_1+\left(\frac{-19733+15493\sqrt{69}}{536544c^2}\right)A_1+\frac{4(36\varepsilon_2-19\varepsilon_1)}{27\sqrt{69}}+\left(\frac{34155+175301\sqrt{69}}{804816c^2}\right)\varepsilon_1-\left(\frac{47\sqrt{69}}{81c^2}\right)\varepsilon_2, \tag{20}$$

Where

$\mu_0=0.03852...$ Is Routh's value

Equation (20) can be written as:

$$0\le\mu<\mu_0+p \tag{21}$$

With

$$p=-\frac{17\sqrt{69}}{486c^2}-\frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right)A_1+\left(\frac{-19733+15493\sqrt{69}}{536544c^2}\right)A_1+\frac{4(36\varepsilon_2-19\varepsilon_1)}{27\sqrt{69}}+\left(\frac{34155+175301\sqrt{69}}{804816c^2}\right)\varepsilon_1-\left(\frac{47\sqrt{69}}{81c^2}\right)\varepsilon_2.$$

5. Discussion

Equation (1)-(7) describe the motion of a third body under the influence of oblateness of the bigger primary together with small perturbations in the Coriolis and centrifugal forces in the relativistic R3BP. Equations (11) give the positions of triangular equilibrium points, which are affected by the oblateness, relativistic factor and a small perturbation in the centrifugal force, but not that of Coriolis force because equation (11) is independent of the parameter ε_2.

Equations (19) gives the critical value of the mass parameter μ_c of the system which depends upon small perturbations $\varepsilon_1,\varepsilon_2$ given in the centrifugal and Coriolis forces, oblateness parameter A_1 and relativistic factor.

The critical value is used to determine the size of the region of stability of the triangular points and also helps in analyzing the behavior of the parameters involved therein. Equation (20) describes the region of stability. It is obvious from (20) that the relativistic term, oblateness coefficient and a small perturbation $\varepsilon_1>0$, in the centrifugal force all shrink the stability region independently; whereas the small perturbation in the centrifugal force expands it for $\varepsilon_1<0$ and that of the Coriolis force expands it for $\varepsilon_2>0$ and shrinks it for $\varepsilon_2<0$. This can be explained by the presence of negative coefficients of the formers and positive coefficient of the latter.

Even on considering the coupling terms $\frac{A_1}{c^2}$ and $\frac{\varepsilon_i}{c^2}(i=1,2)$ which are very small quantities, from mathematical points of view it can be observed from (20) that the joint effect of relativistic and oblateness and that of relativistic and a small perturbation $\varepsilon_1>0$ in the centrifugal force expand the size of region of stability; whereas the joint effect of relativistic and a small perturbation ε_2 in Coriolis force reduces it for $\varepsilon_2>0$ and expands it for $\varepsilon_2<0$. Similarly, the joint effect of relativistic term and a small perturbation ε_1 in the centrifugal force reduces it for $\varepsilon_1<0$. This is also as a result of the positive coefficients of the coupling terms $\frac{A_1}{c^2}$ and $\frac{\varepsilon_1}{c^2}$ and negative coefficient of the coupling term $\frac{\varepsilon_2}{c^2}$. However, the net effect is that the size of region of stability increases or decreases or remains unchanged according as $P>0$ or $P<0$ or $P=0$, respectively. In the absence of perturbations and oblateness $(\varepsilon_i=A_1=0,i=1,2)$, the results of the present study are in agreement with those of Douskos and Perdios [18]and disagree with those of Bhatnagar and Hallan [17].In the absence of a small perturbation in the Coriolis force and oblateness $(\varepsilon_2=A_1=0,)$, the results of this study coincide with those of Singh and Bello [20].In the absence of the coupling terms and oblateness $\left(\frac{\varepsilon_i}{c^2}\to0,A_1=0,i=1,2\right)$, the present results of the present study are in accordance with those of Singh and Bello [22].

In the absence of relativistic terms and centrifugal force $\left(\frac{1}{c^2}\to0,\varepsilon_1=0\right)$, the results coincide with those of Szebehely [4].In the absence of relativistic terms and oblateness $\left(\frac{1}{c^2}\to0,A_1=0\right)$, our results are in agreement with those of Bhatnagar and Hallan [5].In the absence of relativistic terms and perturbations $\left(\frac{1}{c^2}\to0,\varepsilon_i=0,i=1,2\right)$, the results of the present study coincide with those of SubbaRao and Sharma [10].In the absence of relativistic terms, our results are in accordance with those of Abouelmagd et al. [8] when the bigger primary is oblate and the mixed effect $A_1\varepsilon_i\ (i=1,2)$ is ignored in their study.

6. Conclusion

By considering a bigger primary as an oblate spheroid body under the influence of small perturbations in the Coriolis and centrifugal forces in the relativistic R3BP, we have determined the positions of the triangular points and investigated their linear stability. It is found that the effect of relativistic terms, oblateness and a small change in the centrifugal force on these positions are quite prominent. It may also be seen that relativistic terms, oblateness and a small change in the centrifugal force all reduce the size of region of stability independently, where a small perturbation in the Coriolis force expands it.

We have observed the expressions for A, D, A_2, C_2 in Bhatnagar and Hallan [17] differ from the present study when the oblateness and small perturbations in the Coriolis and centrifugal forces are absent $\left(i.e.\ \ \varepsilon_i = A_1 = 0, i = 1,2\right)$. Consequently, the expressions P_1, P_3, P_4, P_5 and the characteristic equation are also different. This led them (Bhatnagar and Hallan 1998) to infer that the triangular points are unstable, contrary to Douskos and Perdios and our results. It is important to note that the results of the present study differ from those of Singh and Bello [22] in the sense that they did not include the oblateness parameter A_1 and coupling terms $\frac{\varepsilon_i}{c^2}\left(i = 1,2\right)$ in their study.

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