International Journal of Advanced Astronomy, *4 (1) (2016) 33-38*

International Journal of Advanced Astronomy

Website: www.sciencepubco.com/index.php/IJAA
doi: 10.14419/ijaa.v4i1.5834
Research paper

Stability of triangular equilibrium points in the Photogravitational elliptic restricted three body problem with Poynting-Robertson drag

Vivek Kumar Mishra*, J.P. Sharma, Bhola Ishwar

Department of Mathematics, B.R.A. Bihar University Muzaffarpur-842001
**Corresponding author E-mail:vivek.mishra2626@gmail.com*

Abstract

We have examined the stability of triangular equilibrium points in photogravitational elliptic restricted three-body problem with Poynting-Robertson drag. We suppose that smaller primary is an oblate spheroid. We have taken bigger primary as radiating. We have found the location of triangular equilibrium points and characteristic equation of the problem. We conclude that triangular equilibrium points remain unstable, different from classical case.

Keywords:*Triangular Points; Photogravitational; ERTBP; P-R Drag.*

1. Introduction

The elliptic restricted three-body problem (ERTBP) is a generalization of the classical restricted three-body problem (RTBP). It describes the three-dimensional motion of a small particle, called the third body (infinitesimal mass) under the gravitational attraction force of two finite bodies, called the primaries. They revolve in elliptic orbit in a plane around their common centre of mass. The infinitesimal mass moves in the plane of motion of the primaries and does not influence the motion of the primaries. ERTBP is more realistic than RTBP, because the orbits of a large number of celestial bodies are elliptic rather than circular. The stability of triangular points in the ERTBP was investigated by Danby [3]. Broucke [1] studied the stability of periodic orbits in the ERTBP.

Radiation and oblateness of the primaries also affect the motion of the infinitesimal mass. Many researchers studied the restricted problem taking into account one or both the primaries as oblate spheroids and radiating. Sahoo and Ishwar [11] examined stability of collinear points in the generalized photogravitational ERTBP. Zimovshchikov and Tkhai [14] investigated stability of libration points and resonance phenomenon in the photogravitational ERTBP. A.Narayan and C.R.Kumar [7] studied the effect of photogravitational and oblateness on the triangular Lagrangian points in ERTBP. J.Singh and A. Umar [12] studied the stability of triangular points in ERTBP under radiating and oblate primaries.

Due to radiation, a drag force known as Poynting-Robertson drag, also affect the motion of infinitesimal mass. Poynting [8] stated that the particle, such as, small meteors or cosmic dust were comparably affected by gravitational and light radiation force, as they approach luminous celestial bodies. He also suggested that infinitesimal body in solar orbit suffers a gradual loss of angular momentum and ultimately spiral into the Sun. Robertson [9] modified the theory of Poynting by considering only terms of first order in the ratio of velocity of the particle to that of light. The radiation force is given by.

$$F = F_p \left\{ \frac{\vec{R}}{R} - \frac{\vec{V}\vec{R}\vec{R}}{C\vec{R}R} - \frac{\vec{V}}{C} \right\}$$

The last two terms constitute Poynting-Robertson (P-R) effect. Wyatt and Whipple [13] have shown that P-R effect has been of very little significance. Chernikov [2] has dealt with the Sun-Planet-Particle model and conclude that due to P-R drag triangular points are unstable. Schuerman [10] studied the classical RTBP by including the radiation pressure and P-R effect. Murray [6] investigated location and stability of the five Lagranginan points in the CRTBP when infinitesimal mass is acted by a variety of drag forces. Liou and Zook [5] examined the effect of radiation pressure, P-R drag and Solar wind drag on dust grains trapped in mean motion resonance with Sun-Jupiter in RTBP. Kushvah and Ishwar [4] examined the linear stability of generalized photogravitational RTBP with P-R drag.

The present study aims to examine the motion of the infinitesimal mass in the ERTBP with radiation, oblateness and P-R drag. We suppose that bigger primary is radiating and smaller is an oblate spheroid. It will contribute a lot to understand the effects of eccentricity, radiation, oblateness and P-R drag on the celestial and stellar systems. The motion of a particle in the double stellar system may be of particular interest, because the system forms considerable part of all stellar systems. The results obtained will be useful to future space missions. The results may be applied for placement of large self-contained space colonies into stable equilibrium point at the L_4 or L_5 Lagrange points.

This paper is divided in five sections. Section (2) contains equations of motion of our problem. In section (3) and (4), we have found location of triangular equilibrium points and examined stability of triangular equilibrium points respectively while section (5) concludes the paper.

2. Equations of motion

We consider two bodies (primaries) of masses m_1 and m_2 with $m_1 > m_2$ moving in a plane around their common center of mass in elliptic orbit and a third body (infinitesimal mass) of mass m is moving in a plane of motion of the primaries. Equations of motion of our problem in rotating and pulsating co-ordinate system are given by (Sahoo and Ishwar) [11]

$$x'' - 2y' = \frac{\partial\Omega}{\partial x} + F_x = U_x \quad (1)$$
$$y'' + 2x' = \frac{\partial\Omega}{\partial y} + F_y = U_y \quad (2)$$
$$z'' = \frac{\partial\Omega}{\partial z} + F_z = U_z \quad (3)$$

where the force function

$$U = \frac{1}{\sqrt{1-e^2}}\left[\frac{x^2+y^2}{2} + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} + W_1\left(\frac{(x+\mu)x'+yy'+zz'}{2r_1^2} - n\ \arctan\left(\frac{y}{x+\mu}\right)\right)\right\}\right]$$

$$\Omega = \frac{1}{\sqrt{1-e^2}}\left[\frac{x^2+y^2}{2} + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3}\right\}\right]$$

$$F = \frac{W_1}{n^2\sqrt{1-e^2}}\left\{\frac{(x+\mu)x'+yy'+zz'}{2r_1^2} n\ \arctan\left(\frac{y}{x+\mu}\right)\right\}$$

$$W_1 = \frac{(1-\mu)(1-q_1)}{c_d}.$$

Here dash ($'$) represents differenciation with respect to ecentric anomaly (E). The mean motion of our problem is given by

$$n^2 = \frac{(1+\frac{3A_2}{2})\sqrt{1+e^2}}{a(1-e^2)} \quad (4)$$
$$r_i = (x+x_i)^2 + y^2 + z^2 (i = 1,2) \quad (5)$$
$$x_1 = -\mu, x_2 = 1-\mu, \mu = \frac{m_2}{m_1+m_2} \quad (6)$$

Here, (x, y, z), $(x_1, 0,0)$ and $(x_2, 0,0)$ are the coordinate of m, m_1 and m_2 respectively. q_1 is mass reduction factor and W_1 is P-R drag due to bigger primary m_1. $A_2 = \frac{r_e^2 - r_p^2}{5r^2}$ is oblateness coefficient due to smaller primary m_2, where r_e, r_p represents equatorial radii and polar radii respectively. $r_i (i = 1,2)$ are the distance of the infinitesimal mass from m_1 and m_2 respectively. Semi-major axis and eccentricity of orbit are denoted by a and e respectively. c_d is dimensionless velocity of light. For all numerical calculations, we use a=0.80, e=0.20, $\mu = 0.00003$ and c_d=299792458.
F_x, F_y are the partial derivatives of drag force (Kushvah and Ishwar 2006 [4]) with respect to x and y respectively, which are purely functions of the particle's position and velocity. Now multiplying equations (1), (2) and (3) by $2x', 2y'$ and $2z'$ respectively and adding, we obtain

$$\frac{dC}{dt} = -2(x'F_x + y'F_y) \quad (7)$$

C=$2\Omega - x'^2 - y'^2$, the quantity C is Jacobi Integral. The zero velocity curves are given by C=2Ω.

3. Location of triangular equilibrium points

In order to find the Lagrangian equilibrium points, equations (1), (2) and (3) are solved with the condition that all derivatives are zero, that is

$$U_x = U_y = U_z = 0.$$

$$x - \frac{1}{n^2}\left\{\frac{(1-\mu)(x+\mu)q_1}{r_1^3} + \frac{\mu(x+\mu-1)}{r_2^3} + \frac{3\mu A_2(x+\mu-1)}{2r_2^5}\right\} - \frac{W_1}{n^2 r_1^2}\left(\frac{(x+\mu)}{r_1^2}[(x+\mu)x' + yy' + zz'] + x' - ny\right) = 0 \quad (8)$$

$$y - \frac{1}{n^2}\left\{\frac{(1-\mu)yq_1}{r_1^3} + \frac{\mu y}{r_2^3} + \frac{3\mu A_2 y}{2r_2^5}\right\} - \frac{W_1}{n^2 r_1^2}\left(\frac{y}{r_1^2}[(x+\mu)x' + yy' + zz'] + y' + n(x+\mu)\right) = 0 \quad (9)$$

$$\left\{\frac{(1-\mu)zq_1}{r_1^3} + \frac{\mu z}{r_2^3} + \frac{3\mu A_2 z}{2r_2^5}\right\} + \frac{W_1}{r_1^2}\left(\frac{z}{r_1^2}[(x+\mu)x' + yy' + zz'] + z'\right) = 0. \quad (10)$$

Now, for triangular equilibrium points $U_x = 0, U_y = 0, x \neq 0, y \neq 0$ and $z = 0$ because motion is in xy plane. Then from equations (8) and (9), we have.

$$x - \frac{1}{n^2}\left\{\frac{(1-\mu)(x+\mu)q_1}{r_1^3} + \frac{\mu(x+\mu-1)}{r_2^3} + \frac{3\mu A_2(x+\mu-1)}{2r_2^5}\right\} + \frac{W_1}{n^2 r_1^2} ny = 0 \quad (11)$$

$$y - \frac{1}{n^2}\left\{\frac{(1-\mu)yq_1}{r_1^3} + \frac{\mu y}{r_2^3} + \frac{3\mu A_2 y}{2r_2^5}\right\} - \frac{W_1}{n^2 r_1^2} n(x+\mu) = 0 \quad (12)$$

From equation (12), we have

$$\left[n^2 - \frac{(1-\mu)q_1}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3\mu A_2}{2r_2^5}\right] y_0 = \frac{W_1}{r_1^2}\ n(x+\mu) \quad (13)$$

y_0 is ordinate of photogravitational ERTBP, which is given by

$$y_0 = \pm\left[\delta^2(1-e^2) - \frac{1}{4}\left\{1 + 2\left(\delta^2 - a^{\frac{2}{3}}\right)(1-e^2)\right\}\right]^{\frac{1}{2}}$$

where $\delta = (aq_1)^{\frac{1}{3}}$.

Equations (11) and (12) are multiplied by y and (x+ μ) respectively and subtracting, we have

$$\left[n^2 - \frac{1}{r_2^3} - \frac{3A_2}{2r_2^5}\right]\mu y_0 = nW_1. \quad (14)$$

In photogravitational ERTBP, that is when oblateness and P-R drag is absent and bigger primary is radiating then

$$r_1 = \left(\frac{q_1}{n^2}\right)^{1/3} \quad (15)$$
$$r_2 = \frac{1}{n^{2/3}} \quad (16)$$

Now, we suppose due to P-R drag and oblateness perturbation in r_1 and r_2 are ϵ_1 and ϵ_2 (ϵ_1, $\epsilon_2 \ll 1$) respectively. Then

$$r_1 = \left(\frac{q_1}{n^2}\right)^{1/3} + \epsilon_1 \quad (17)$$
$$r_2 = \frac{1}{n^{2/3}} + \epsilon_2. \quad (18)$$

Considering only terms e^2 and A_2 and neglecting their product, equation (4) gives

$$n^2 = \frac{1}{a}\left(1 + \frac{3A_2}{2} + \frac{3e^2}{2}\right) \quad (19)$$

With the help of equation (19) with $A_2 = 0$, equations (17) and (18) gives

$$r_1 = (aq_1)^{1/3}\left(1 - \frac{e^2}{2}\right) + \epsilon_1 \quad (20)$$
$$r_2 = a^{1/3}\left(1 - \frac{e^2}{2}\right) + \epsilon_2 \quad (21)$$

With the help of equations (19), (20) and (21), we have from equations (13) and (14) (taken only first order terms)

$$\epsilon_1 = \frac{(aq_1)^{1/3} a^{1/2} W_1}{6(1-\mu)y_0}\left(1 + \frac{3A_2}{4} - \frac{5e^2}{4}\right)\left[\left\{(1+e^2) - a^{2/3}\right\}(aq_1)^{-2/3} - 1\right] - \frac{1}{2}A_2(aq_1)^{1/3} \quad (22)$$

$$\epsilon_2 = \frac{W_1 a^{5/6}}{3\mu y_0}\left(1 + \frac{3A_2}{4} - \frac{5A_2 a^{-2/3}}{2} - \frac{5e^2}{4}\right) - \frac{1}{2}A_2 a^{1/3}\left(1 - a^{-2/3}\right) \quad (23)$$

Substituting the values of ϵ_1 and ϵ_2 in equations (20) and (21) respectively, we have

$$r_1 = (aq_1)^{1/3}\left(1-\frac{e^2}{2}\right)+\frac{(aq_1)^{1/3}a^{1/2}W_1}{6(1-\mu)y_0}\left(1+\frac{3A_2}{4}-\frac{5e^2}{4}\right)\left[\left\{(1+e^2)-a^{2/3}\right\}(aq_1)^{-2/3}-1\right]-\frac{1}{2}A_2(aq_1)^{1/3} \tag{24}$$

$$r_2 = a^{1/3}\left(1-\frac{e^2}{2}\right)+\frac{W_1 a^{5/6}}{3\mu y_0}\left(1+\frac{3A_2}{4}-\frac{5A_2a^{-2/3}}{2}-\frac{5e^2}{4}\right)-\frac{1}{2}A_2a^{1/3}\left(1-a^{-2/3}\right) \tag{25}$$

Since, $(x+\mu)=\frac{r_1^2-r_2^2+1}{2}$, $y^2=r_1^2-(x+\mu)^2$ (26)

Using equations (24),(25) and (26) and solve for x, y, we have

$$x=\frac{1}{2}-\mu+\frac{1}{2}\left[(aq_1)^{2/3}(1-A_2-e^2)-a^{2/3}(1-A_2-e^2+A_2a^{-2/3}\right]+\frac{W_1a^{1/2}}{3y_0(1-\mu)\mu}\left\{\left(1+\frac{A_2}{4}-\frac{3e^2}{4}\right)\frac{\mu}{2}-\left(a^{2/3}+(aq_1)^{2/3}\right)\left(1+\frac{A_2}{4}-\frac{7e^2}{4}\right)\frac{\mu}{2}-(1-\mu)a^{2/3}\left(1+\frac{A_2}{4}-2A_2a^{-2/3}-\frac{7e^2}{4}\right)\right\} \tag{27}$$

$$y=\pm\Big[(aq_1)^{2/3}(1-A_2-e^2)-\frac{1}{4}\Big\{1+2\left((aq_1)^{2/3}-a^{2/3}\right)(1-e^2)+\left((aq_1)^{2/3}-a^{2/3}\right)^2(1-2e^2)-2A_2(1+(aq_1)^{2/3}-a^{2/3})^2\Big\}+\frac{W_1a^{1/2}}{3y_0(1-\mu)\mu}\Big\{\left(1+\frac{A_2}{4}-\frac{3e^2}{4}\right)\frac{\mu}{2}-\mu(aq_1)^{2/3}\left(1+\frac{A_2}{4}-\frac{7e^2}{4}\right)+\left((aq_1)^{4/3}-a^{4/3}\right)\left(1+\frac{A_2}{4}-\frac{11e^2}{4}\right)\frac{\mu}{2}+(1-\mu)a^{2/3}\left(1+\frac{A_2}{4}-2A_2a^{-2/3}-\frac{7e^2}{4}\right)+(1-\mu)a^{2/3}\left((aq_1)^{2/3}-a^{2/3}\right)\left(1+\frac{A_2}{4}-2A_2a^{-2/3}-\frac{11e^2}{4}\right)-\frac{\mu}{2}A_2\left(1+(aq_1)^{2/3}-a^{2/3}\right)\left((aq_1)^{2/3}+a^{2/3}-1\right)-(1-\mu)a^{2/3}\left(1+(aq_1)^{2/3}-a^{2/3}\right)A_2\Big\}\Big]^{1/2} \tag{28}$$

The position of triangular equilibrium points ($L_{4(5)}$) is given by equations (27) and (28) which are valid for $W_1 \ll 1, A_2 \ll 1$. Figure 1 shows that perturbations (ϵ_1, ϵ_2) in r_1 and r_2 will become zero, when $A_2 = 0$ and $q_1 = 1$. From figure 2, it is clear that x, y are increasing function of q_1 and decreasing function of A_2. Numerical values of coordinate (x,y) of triangular equilibrium points ($L_{4(5)}$) are given in table 1 and table 2, for different values of A_2 and q_1. For the numerical calculations we have taken μ =0.00003, e=0.2, a=0.8, c_d=299792458, $0 \le q_1 \le 1, 0 \le A_2 \le 1$.

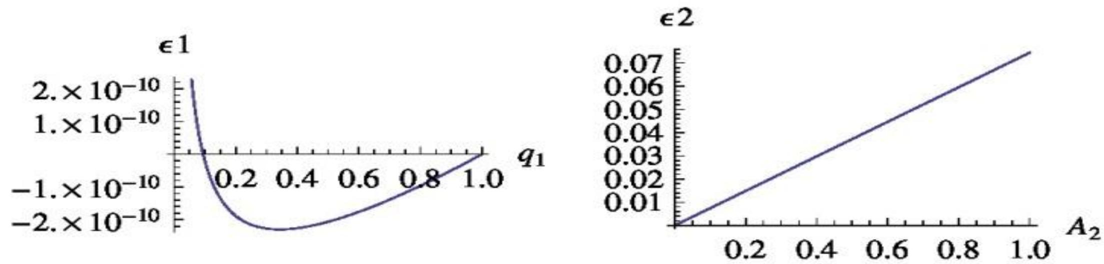

Fig. 1: The effect of A_2 and q_1 on ϵ_1 and ϵ_2. when $e = 0.2, a = 0.8, \mu = 0.00003, c_d = 299792458, 0 \le q_1 \le 1, 0 \le A_2 \le 1$

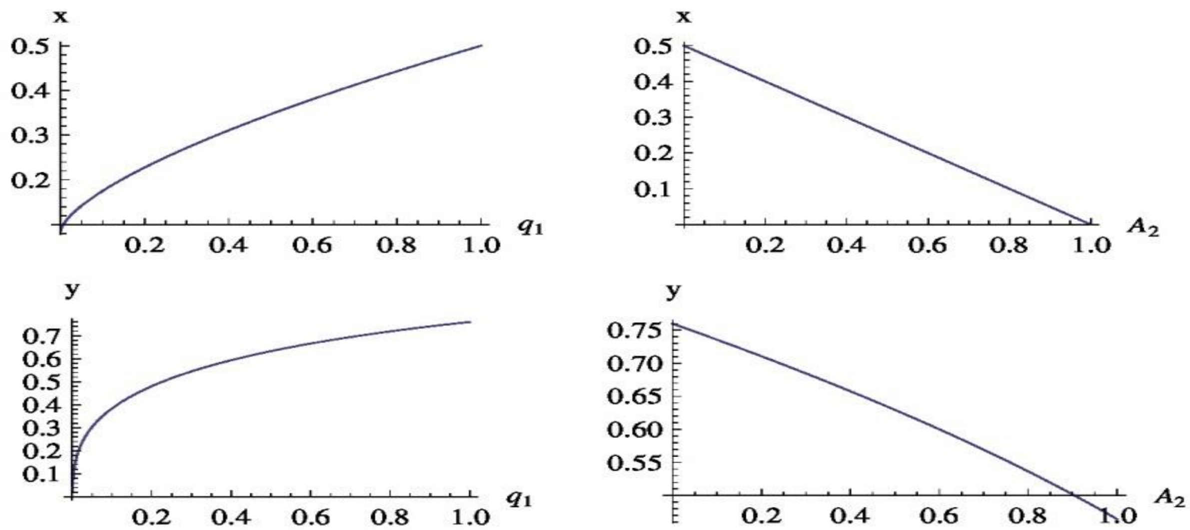

Fig. 2: The effect of A_2 and q_1 on x, y co-ordinates of triangular equilibrium points $L_{4(5)}$). when $e = 0.2, a = 0.8, \mu = 0.00003, c_d = 299792458, 0 \le q_1 \le 1, 0 \le A_2 \le 1$.

Table 1: x co-ordinate of L_4

A_2	$x_4: q_1 = 1$	$x_4: q_1 = 0.75$	$x_4: q_1 = 0.50$	$x_4: q_1 = 0.25$	$x_4: q_1 = 0$
0.00	0.49997	0.427771	0.346882	0.250438	Complex Number
0.25	0.37497	0.321576	0.261755	0.190431	Complex Number
0.50	0.24997	0.21538	0.176628	0.130425	Complex Number
0.75	0.12497	0.109185	0.0915006	0.0704191	Complex Number
1.00	-0.00003	0.00298943	0.00637354	0.010413	Complex Number

Notes: Numerical values of x coordinate of L_4 for different values of A_2 and q_1. $0 \le A_2, q_1 \le 1$.

Table 2: y co-ordinate of L_4

A_2	$y_4: q_1 = 1$	$y_4: q_1 = 0.75$	$y_4: q_1 = 0.50$	$y_4: q_1 = 0.25$	$y_4: q_1 = 0$
0.00	0.759805	0.707049	0.633136	0.515451	Complex Number
0.25	0.697753	0.642103	0.568434	0.457167	Complex Number
0.50	0.629616	0.59801	0.495352	0.390275	Complex Number
0.75	0.553148	0.486879	0.409426	0.309236	Complex Number
1.00	0.464251	0.386559	0.299809	0.197330	Complex Number

Notes: Numerical values of y coordinate of L_4 for different values of A_2 and q_1. $0 \le A_2, q_1 \le 1$

4. Stability of triangular equilibrium points

Rewriting equations of motion, we have

$$x'' - 2y' = \frac{\partial \Omega}{\partial x} - \frac{W_1 N_1}{n^2 r_1^2 \sqrt{1-e^2}} = U_x \quad (29)$$

$$y'' + 2x' = \frac{\partial \Omega}{\partial y} - \frac{W_1 N_2}{n^2 r_1^2 \sqrt{1-e^2}} = U_y \quad (30)$$

$$z'' = \frac{\partial \Omega}{\partial z} - \frac{W_1 N_3}{n^2 r_1^2 \sqrt{1-e^2}} = U_z \quad (31)$$

where,

$$N_1 = \frac{(x+\mu)N}{r_1^2} + x' - ny,$$

$$N_2 = \frac{yN}{r_1^2} + y' + n(x+\mu)$$

$$N_3 = \frac{zN}{r_1^2} + z', N = (x+\mu)x' + yy' + zz'.$$

We suppose that α, β, γ be the small displacement from equilibrium points (x_*, y_*, z_*) then $x = x_* + \alpha, y = y_* + \beta, z = z_* + \gamma$. At equilibrium points (x_*, y_*, z_*), $x'_* = y'_* = z'_* = x''_* = y''_* = z''_* = 0$. Hence, the equations of motion corresponding to the system of equations (29), (30) and (31), by using Taylor's theorem, are written as

$$\alpha'' - 2\beta' = U_x^0 + \alpha U_{xx}^0 + \beta U_{xy}^0 + \gamma U_{xz}^0 + \alpha' U_{xx'}^0 + \beta' U_{xy'}^0 + \gamma' U_{xz'}^0 \quad (32)$$

$$\beta'' + 2\alpha' = U_y^0 + \alpha U_{yx}^0 + \beta U_{yy}^0 + \gamma U_{yz}^0 + \alpha' U_{yx'}^0 + \beta' U_{yy'}^0 + \gamma' U_{yz'}^0 \quad (33)$$

$$\gamma'' = U_z^0 + \alpha U_{zx}^0 + \beta U_{zy}^0 + \gamma U_{zz}^0 + \alpha' U_{zx'}^0 + \beta' U_{zy'}^0 + \gamma' U_{zz'}^0 \quad (34)$$

where superscript '0' indicates that the partial derivatives are to be evaluated at the equilibrium point (x_*, y_*, z_*). At $(x_*, y_*, z_*$, $U_x^0 = U_y^0 = U_z^0 = U_{xz}^0 = U_{yz}^0 = U_{zx}^0 = U_{zy}^0 = U_{xz'}^0 = U_{yz'}^0 = U_{zx'}^0 = U_{yz'}^0 = 0$. Hence system of equations (32), (33) and (34) are written as

$$\alpha'' - 2\beta' = \alpha U_{xx}^0 + \beta U_{xy}^0 + \alpha' U_{xx'}^0 + \beta' U_{xy'}^0 \quad (35)$$

$$\beta'' + 2\alpha' = \alpha U_{yx}^0 + \beta U_{yy}^0 + \alpha' U_{yx'}^0 + \beta' U_{yy'}^0 \quad (36)$$

$$\gamma'' = \gamma U_{zz}^0 + \gamma' U_{zz'}^0 . \quad (37)$$

The value of second order partial derivatives at $(x_*, y_*, 0)$ are

$$U_{xx}^0 = \frac{1}{\sqrt{1-e^2}}\left[1 - a\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)(f_* - 3J_{1*}) - \frac{2W_1 a^{1/2}(x_*+\mu)y_*}{r_{1*}^4}\left(1 - \frac{3A_2}{4} - \frac{3e^2}{4}\right)\right]$$

$$U_{xy}^0 = \frac{a}{\sqrt{1-e^2}}\left[3g_*\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right) + \frac{W_1 a^{-1/2}}{r_{1*}^2}\left(1 - \frac{3A_2}{4} - \frac{3e^2}{4}\right)\left(1 - \frac{2{y_*}^2}{r_{1*}^2}\right)\right]$$

$$U_{xx'}^0 = -\frac{W_1 a}{r_{1*}^2\sqrt{1-e^2}}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\left[1 + \frac{(x_*+\mu)^2}{r_{1*}^2}\right]$$

$$U_{yx}^0 = \frac{a}{\sqrt{1-e^2}}\left[3g_*\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right) - \frac{W_1 a^{-1/2}}{r_{1*}^2}\left(1 - \frac{3A_2}{4} - \frac{3e^2}{4}\right)\left(1 - \frac{2(x_*+\mu)^2}{r_{1*}^2}\right)\right]$$

$$U_{yy}^0 = \frac{1}{\sqrt{1-e^2}}\left[1 - a\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)(f_* - 3J_{2*}) + \frac{2W_1 a^{1/2}(x_*+\mu)y_*}{r_{1*}^4}\left(1 - \frac{3A_2}{4} - \frac{3e^2}{4}\right)\right]$$

$$U_{yy'}^0 = -\frac{W_1 a}{r_{1*}^2\sqrt{1-e^2}}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\left[1 + \frac{{y_*}^2}{r_{2*}^1}\right]$$

$$U_{xy'}^0 = -\frac{W_1 a}{r_{1*}^4\sqrt{1-e^2}}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)(x_* + \mu)y_* = U_{yx'}^0$$

$$U_{zz}^0 = \frac{-a}{\sqrt{1-e^2}}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\left[\frac{(1-\mu)q_1}{r_{1*}^3} + \frac{\mu}{r_{2*}^3} + \frac{3\mu A_2}{2r_{2*}^5}\right]$$

$$U_{zz'}^0 = -\frac{W_1 a}{r_{1*}^2\sqrt{1-e^2}}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)$$

where

$$f_* = \frac{(1-\mu)q_1}{r_{1*}^3} + \frac{\mu}{r_{2*}^3} + \frac{3\mu A_2}{2r_{2*}^5}$$

$$g_* = \frac{q_1(1-\mu)(x_*+\mu)y_*}{r_{1*}^5} + \frac{\mu(x_*+\mu-1)y_*}{r_{2*}^5} + \frac{5\mu A_2(x_*+\mu-1)y_*}{2r_{2*}^7}$$

$$J_{1*} = \frac{q_1(1-\mu)(x_*+\mu)^2}{r_{1*}^5} + \frac{\mu(x_*+\mu-1)^2}{r_{2*}^5} + \frac{5\mu A_2(x_*+\mu-1)^2}{2r_{2*}^7}$$

$$J_{2*} = \frac{q_1(1-\mu){y_*}^2}{r_{1*}^5} + \frac{\mu {y_*}^2}{r_{2*}^5} + \frac{5\mu A_2 {y_*}^2}{2r_{2*}^7}$$

Star $(*)$ represents the value at equilibrium point (x_*, y_*, z_*). We suppose $\alpha = A_1 e^{\lambda t}, \beta = B_1 e^{\lambda t}$ and $\gamma = C_1 e^{\lambda t}$. Using these values, the system of equations (35), (36) and (37) are written as

$$(\lambda^2 - U_{xx}^0 - \lambda U_{xx'}^0)A_1 + \left[-\left(2 + U_{xy'}^0\right)\lambda - U_{xy}^0\right]B_1 = 0 \quad (38)$$

$$\left[\left(2 - U_{yx'}^0\right)\lambda - U_{yx}^0\right]A_1 + \left(\lambda^2 - \lambda U_{yy'}^0 - U_{yy}^0\right)B_1 = 0 \quad (39)$$

$$(\lambda^2 - \lambda U_{zz'}^0 - U_{zz}^0)C_1 = 0 \quad (40)$$

From equation (40), we have $\lambda = \frac{U_{zz'}^0 \pm \sqrt{\left(U_{zz'}^0\right)^2 + 4U_{zz}^0}}{2}$. After substituting values of partial derivatives, we have

$$\lambda = \frac{a}{2}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\left[\frac{-W_1}{r_{1*}^2\sqrt{1-e^2}} \pm \sqrt{\frac{W_1^2}{r_{1*}^4\sqrt{1-e^2}} - \frac{4f_*}{a\sqrt{1-e^2}}\left(1 + \frac{3A_2}{2} + \frac{3e^2}{2}\right)}\right]$$

Since λ consists always negative real part, hence motion is asymptotically stable in z direction. Now equations (38) and (39) have singular solution if

$$\begin{vmatrix} \lambda^2 - U_{xx}^0 - \lambda U_{xx'}^0 & -(2 + U_{xy'}^0)\lambda - U_{xy}^0 \\ (2 - U_{yx'}^0)\lambda - U_{yx}^0 & \lambda^2 - \lambda U_{yy'}^0 - U_{yy}^0 \end{vmatrix} = 0$$

ie

$$\lambda^4 + a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \quad (41)$$

This is the characteristic equation of the problem. a_0, a_1, a_2 and a_3 are coefficients, where

$$a_0 = -\left(U_{xx'}^0 + U_{yy'}^0\right)$$

$$a_1 = 4 - \left(U_{xx}^0 + U_{yy}^0\right) + 2\left(U_{xy'}^0 - U_{yx'}^0\right) + U_{xx'}^0 U_{yy'}^0 - U_{xy'}^0 U_{yx'}^0$$

$a_2 = U^0_{xx}U^0_{yy'} + U^0_{xx'}U^0_{yy} + 2\left(U^0_{xy} - U^0_{yx}\right) - U^0_{xy'}U^0_{yx} - U^0_{xy}U^0_{yx'}$
$a_3 = U^0_{xx}U^0_{yy} - U^0_{xy}U^0_{yx}$.

At equilibrium point, we get

$$a_0 = \frac{3W_1\mathrm{a}}{r_{1*}^2\sqrt{1-e^2}}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right) \quad (42)$$

$$a_1 = 4 - \frac{\mathrm{a}}{\sqrt{1-e^2}}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\left[2 + f_* + \frac{3\mu A_2}{r_{2*}^5}\right] + \frac{2W_1^2\mathrm{a}}{(1-e^2)r_{1*}^4}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right) = b_0 + b_1 \quad (43)$$

$$b_0 = 4 - \frac{\mathrm{a}}{\sqrt{1-e^2}}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\left[2 + f_* + \frac{3\mu A_2}{r_{2*}^5}\right]$$

$$b_1 = \frac{2W_1^2\mathrm{a}}{(1-e^2)r_{1*}^4}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right) \quad (44)$$

$$a_2 = \frac{-a_1}{\sqrt{1-e^2}}\left[1 + \mathrm{a}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\left\{\frac{\mu A_2}{r_{2*}^5} + \frac{\mu}{r_{1*}^2 r_{2*}^5}\left(1 + \frac{5A_2}{2r_{2*}^2}\right)y_*^2\right\}\right]$$

$$a_3 = \frac{1}{1-e^2}\left[\left\{1 - \mathrm{a}f_*\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\right\}\left\{1 + \mathrm{a}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\left(2f_* + \frac{3\mu A_2}{r_{2*}^5}\right)\right\} + \frac{9\mathrm{a}^2(1-3A_2-3e^2)\mu(1-\mu)q_1}{r_{1*}^5 r_{2*}^5}\left(1 + \frac{5A_2}{2r_{2*}^2}\right)y_*^2 + \frac{6W_1\mathrm{a}^{3/2}}{r_{1*}^4}\left(1 - \frac{9A_2}{4} - \frac{9e^2}{4}\right)\left\{\frac{-\mu y_*}{r_{2*}^5}\left(1 + \frac{5A_2}{2r_{2*}^2}\right)((x_* + \mu)^2 + y_*^2 - (x_* + \mu)\right\} - \frac{W_1^2\mathrm{a}}{r_{1*}^4}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\right]. \quad (45)$$

We write the four roots of the classical characteristic equation as

$$\lambda_j = \pm zi \quad \text{where j=1,2,3,4}$$
$$z^2 = \frac{1}{2}\left\{1 \mp [1 - 27\mu(1-\mu)]^{1/2}\right\}. \quad (46)$$

Then

$$\lambda_{1,2} = \pm\left(\frac{27}{4}\mu(1-\mu)\right)^{1/2} i$$
$$\lambda_{3,4} = \pm\left(1 - \frac{27}{4}\mu(1-\mu)\right)^{1/2} i$$

$\lambda_{1,2}, \lambda_{3,4}$ represent the four roots of classical case where $i = \sqrt{-1}$. Due to P-R drag and oblateness, we suppose the solution of the characteristic equation (38) is of the form $\lambda = \lambda_j(1 + e_1 + e_2 i) = \pm[-e_2 + (1 + e_1)i]z$, where e_1, e_2 are small. We have in first order approximation

$$\lambda^2 = [-e_2 + (1 + e_1)i]^2 z^2 \quad (47)$$
$$\lambda^2 = [-(1 + 2e_1) - 2e_2 i]z^2 \quad (48)$$
$$\lambda^3 = \pm[3e_2 - (1 + 3e_1)i]z^3 \quad (49)$$
$$\lambda^4 = [1 + 4e_1 + 4e_2 i]z^4. \quad (50)$$

Substituting these values in equation (41), neglecting product of e_1, e_2 with $a_0, a_1, a_2, and\ a_3$, we get

$$e_1 = \frac{-a_3 + b_0 z^2 - z^4}{2z^2(2z^2 - b_0)} \quad (51)$$
$$e_2 = \frac{\mp a_2 z \pm a_0 z^3}{2z^2(2z^2 - b_0)} \quad (52)$$

If $e_2 \neq 0$, then the resulting motion of particle displaced from equilibrium points is asymptotically stable only when all the real part of λ are negative. For stability, we require Re $(\lambda) < 0$, $\mathrm{Re}(\lambda) = \frac{a_2 - a_0 z^2}{2(2z^2-1)}$. Taking positive sign from equation (46), we have

$$z^2 = 1 - \frac{27}{4}\mu(1-\mu) \quad (53)$$

We consider Re $(\lambda) < 0$, then

$$\left\{a_2 - a_0\left(1 - \frac{27}{4}\mu(1-\mu)\right)\right\}\left\{1 - \frac{27}{2}\mu(1-\mu)\right\}^{-1} < 0 \quad (54)$$

$$a_2 + \frac{27}{4}\mu(1-\mu)(2a_2 - a_0) < a_0 \quad (55)$$

Taking negative sign from equation (46), we have

$$0 < a_2 + \frac{27}{4}\mu(1-\mu)(2a_2 - a_0) \quad (56)$$

From equations (55) and (56), we get

$$0 < a_2 + \frac{27}{4}\mu(1-\mu)(2a_2 - a_0) < a_0$$
$$0 < a_2 < a_0 \text{ as } \mu \to 0 \text{ Murray [6]} \quad (57)$$

Inequality (57) is necessary condition for stability of triangular equilibrium points $L_{4(5)}$. When $W_1 \neq 0, A_2 \neq 0$ and $n^2 = \frac{1}{a}\left(1 + \frac{3A_2}{2} + \frac{3e^2}{2}\right)$

$$a_0 = \frac{3W_1\mathrm{a}}{r_{1*}^2\sqrt{1-e^2}}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right) \quad (58)$$
$$a_2 = \frac{-a_1}{\sqrt{1-e^2}}\left[1 + \mathrm{a}\left(1 - \frac{3A_2}{2} - \frac{3e^2}{2}\right)\left\{\frac{\mu A_2}{r_{2*}^5} + \frac{\mu}{r_{1*}^2 r_{2*}^5}\left(1 + \frac{5A_2}{2r_{2*}^2}\right)\right\}\right] \quad (59)$$

Since $W_1 > 0$ hence $a_0 > 0$. From equation (59) it is clear a_2 is always negative that is $a_2 < 0$. If we consider P-R drag and oblateness then $a_2 < 0$. This does not satisfy the necessary condition of stability, hence motion is unstable in linear sense.

5. Conclusion

We have studied the effect of P-R drag, radiation and oblateness on triangular equilibrium points in elliptic restricted three body problem. We have found that locations of triangular points are different from classical case. If we put $W_1 = 0, A_2 = 0, q_1 = 1, a = 1$ and $e = 0$ in equations (27) and (28), then we get $x = \frac{1}{2} - \mu, y = \pm\frac{\sqrt{3}}{2}$. These are coordinate of classical RTBP. In this case $27\mu(1-\mu) < 1$, that is $\mu < 0.03852089$. If we do not consider P-R drag ($W_1 = 0$) then $L_{4(5)}$ coincide with Singh and Umar [12]. With $a = 1$, $e = 0$ and $A_2 = 0$ triangular equilibrium points are same as Schuerman [10]. Kushvah [4] studied same problem in circular case. He found that triangular equilibrium points are unstable in his problem. When we put $a = 1$ and $e = 0$ in equation (27) and (28), triangular equilibrium points are similar as in Kushvah [4]. From figure 2, we find that x and y both are increasing function of q_1 and decreasing function of A_2. Numerical values of x and y are given in table 1 and table 2. At last with the help of Murray [6], we conclude that triangular equilibrium points are unstable in linear sense in our problem.

Acknowledgement

We are thankful to D.S.T. Government of India, New Delhi for sanctioning the project SR/S4/MS: 728/11, dated:15/06/2013 on this topic and IUCAA, Pune for providing local hospitality and facility of library and computer centre for research work.

References

[1] Broucke,R.,(1969): Stability of periodic orbits in the elliptic restricted three body problem, AIAA Journal, 7(6), 1003-1009 http://dx.doi.org/10.2514/3.5267.

[2] Chernikov Yu. A., (1970): The photogravitational restricted three body problems, Soviet Astronomy-AJ, 14(1), 176-181.

[3] Danby,J.M.A.,(1964): Stability of the triangular points in the elliptic restricted problem of three bodies, The Astronomical Journal,69(2), 165-172.http://dx.doi.org/10.1086/109254.

[4] Kushvah,B.S., Ishwar, B.,(2006): Linear stability of generalized photogravitational restricted three body problem with Poynting-Robertson drag,Journal of Dynamical system & Geometric Theories 4(1),79-86.

[5] Liou J. C., Zook, H.A. and Jackson, A.A. (1995b): Radiation pressure,Poynting-Robertson drag and solar wind drag in the restricted three body problem, Icarus 116,186-201. http://dx.doi.org/10.1006/icar.1995.1120.

[6] Murray,C.D.,(1994): Dynamical effects of drag in the circular restricted three body problem:Location and stability of the Lagrangian equilibrium points, Icarus 112,465-484. http://dx.doi.org/10.1006/icar.1994.1198.

[7] Narayan. A. and Kumar,C.R. (2011): Effects of photogravitational and oblateness on the triangular Lagrangian points in the elliptical restricted three body problem,Int.J.Pure and Appl.Math,68,201-224.

[8] Poynting, J.H., (1903): Radiation in the solar system: its effect on temperature and its pressure on small bodies, MNRAS, 64,525-552. http://dx.doi.org/10.1093/mnras/64.1.1a.

[9] Robertson.H.P. (1937): Dynamical effects of radiation in the solar system, MNRAS, 97, 423-438. http://dx.doi.org/10.1093/mnras/97.6.423.

[10] Schuerman, D. (1980): The restricted three body problem including radiation pressure, Astrophys. J., 238, 337–342 http://dx.doi.org/10.1086/157989.

[11] Sahoo, S.K.,Ishwar,B.,(2000): Stability of collinear equilibrium points in the generalized photogravitational elliptic restricted three body problem,Bulletin of the Astronomical Society of India 28,579-586.

[12] Singh, J. and Umar, A.(2012):Motion in the photogravitational elliptic restricted three body problem under an oblate primary, The Astronomical Journal, 143:109,(22pp).

[13] Wyatt S.P and Whipple F.L. (1950): The Poynting-Robertson effect on meteor orbits,American Astron.Soc.APJ.111,134-141. http://dx.doi.org/10.1086/145244.

[14] Zimovshchikov, A.S., Tkhai. (2004): Instability of libration points and resonance phenomena in the photogravitational elliptic restricted three body problem, V.N.: Sol. Syst. Res. 38, 155-164. http://dx.doi.org/10.1023/B:SOLS.0000022826.31475.a7.