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Research paper

A general solution to non-collinear equilibria in terms of largest root (κ) of confocal oblate spheroid

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Abstract

This paper deals with the existence of non-collinear equilibria in restricted three-body problem when less massive primary is an oblate spheroid and the potential of oblate spheroid is in terms of largest root of confocal oblate spheroid. This is found that the non-collinear equilibria are the solution of the equations $r_1 = n^{-2/3}$ and $\kappa = 1 - a^2$, where r_1 is the distance of the infinitesimal mass from more massive primary, n is mean-motion of primaries, a is semi axis of oblate spheroid and κ is the largest root of the equation of confocal oblate spheroid passes through the infinitesimal mass.

Keywords: *Celestial Mechanics; Restricted Three-Body Problem; Libration Points; Oblate Spheroid; Confocal Oblate Spheroid.*

1. Introduction

The restricted problem of three-body describes the motion of infinitesimal mass moving in the gravitational field of two massive primaries in the same plane or out of plane called two dimensional or three dimensional problem accordingly. The primaries are revolving around their center of mass either in circular or elliptical orbits under the influence of their mutual gravitational attraction. If the orbit of the primaries around their center of mass is elliptic, problem is said to be elliptic restricted three-body problem (ER3BP or ERTBP) and if the orbit of the primaries around their center of mass is circular, problem is said to be circular restricted three-body problem or restricted three-body problem, denoted by CR3BP or CRTBP or RTBP or R3BP.

The problem possesses five equilibrium points out of which three are collinear and two non-collinear. The collinear libration points are unstable while non-collinear are stable for the mass ratio $\mu \le 0.038520896505$ [3].Some studies related to the equilibrium points in R3BP or ER3BP, taken into account the oblateness and triaxiality of the primaries, Coriolis and Centrifugal forces, variation of the masses of the primaries and the infinitesimal mass etc. are discussed by Danby [2]; Szebehely [3]; Vidyakin [4]; Sharma [5]; Subbarao and Sharma [6]; Sharma et.al. [7]; Choudhary R. K. [8]; Cid R. et. al. [10]; El-Shaboury [11]; Bhatnagar et al. [12]; Selaru D. et.al. [13]; Markellos et al. [14]; Subbarao and Sharma [15]; Khanna and Bhatnagar [16, 17]; Roberts G.E. [18]; Oberti and Vienne [19]; Sosnytskyi [20]; Perdiou et. al. [21]; Arredondo et.al. [22]; Idrisi and Taqvi [23]; Idrisi [24]; Idrisi and Amjad [25].

In 1958, W. D. MacMillan [1] gave a theory to find out the potential of ellipsoid in terms of largest root of confocal ellipsoid. Let a, b, c be the semi-axes of the given oblate spheroid such that $a = b > c$, (x, y, z) the co-ordinates of the external point P and a', b', c' $(a' = b' > c')$ be the semi-axes of the confocal oblate spheroid passes through P.

Let the equation of the oblate spheroid be

$$\frac{\xi^2+\eta^2}{a^2}+\frac{\zeta^2}{c^2}=1 \tag{1}$$

where ξ, η, ζ are the principal axes of the oblate spheroid.

The equation of the confocal oblate spheroid whose semi-axes are a', b' and c' and passes through the point $P(x, y, z)$ is

$$\frac{x^2+y^2}{a'^2}+\frac{z^2}{c'^2}=1 \tag{2}$$

Since the oblate spheroids are confocal, we have $a'^2 = a^2 + \kappa$, $b'^2 = b^2 + \kappa$ and $c'^2 = c^2 + \kappa$. Therefore, the Equation (2) becomes

$$\frac{x^2+y^2}{a^2+\kappa}+\frac{z^2}{c^2+\kappa}=1 \tag{3}$$

The potential of the oblate spheroid with axes a and c and density σ at an external point $P(x, y, z)$ in terms of κ is given by (MacMillan [1])

$$V=\frac{2\pi\sigma a^2 c}{\sqrt{a^2-c^2}}\left(1-\frac{x^2+y^2-2z^2}{2(a^2-c^2)}\right)\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\kappa}}+\frac{\pi\sigma a^2 c\sqrt{c^2+\kappa}}{a^2-c^2}\frac{x^2+y^2}{a^2+\kappa}-\frac{\pi\sigma a^2 c}{a^2-c^2}\frac{2z^2}{\sqrt{c^2+\kappa}}, \tag{4}$$

where κ is defined in the Eqn. (3).

The aim of this paper is to find out non-collinear equilibria in restricted three-body problem when less massive primary is an oblate spheroid, using the potential in terms of κ (Eqn. 4) and also verify the results with previous studies such as [6] and [9].

2. Equations of motion

Let m_2 be an oblate spheroid whose axes are a, b and c ($a = b > c$) and m_1 a point mass ($m_1 > m_2$), are moving in the circular orbits around their center of mass O. An infinitesimal mass m_3 is moving in the plane of motion of m_1 and m_2. The distances of m_3 from m_1, m_2 and O are $\boldsymbol{r_1}$, $\boldsymbol{r_2}$ and $\boldsymbol{r}$ respectively. The principal axes of spheroid remains parallel to the synodic axes $Oxyz$ throughout the motion and the equatorial plane of m_2 is coincide with the plane of motion of m_1 and m_2. Let the line joining m_1 and m_2 be taken as X –axis and O their center of mass as origin. Let the line passing through O and perpendicular to OX and lying in the plane of motion m_1 and m_2 be the Y –axis. Let us consider a synodic system of co-ordinates $Oxyz$ initially coincide with the inertial system $OXYZ$, rotating with angular velocity ω about Z –axis (the z –axis is coincide with Z –axis). We wish to find the equations of motion of m_3 using the terminology of Szebehely (1967) in the synodic co-ordinate system and dimensionless variables *i.e.* the distance between the primaries is unity, the unit of time t is such that the gravitational constant $G = 1$ and the sum of the masses of the primaries is unity *i.e.* $m_1 + m_2 = 1$.

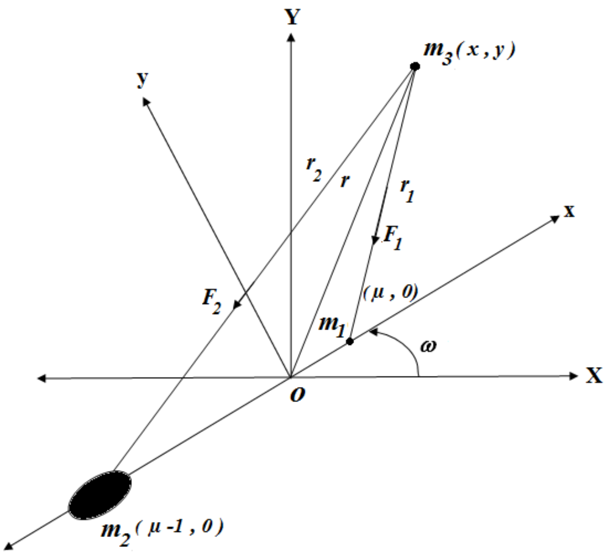

Fig. 1: The configuration of the R3BP when m_2 is an oblate spheroid

The potential of the oblate spheroid m_2 at $P(x, y)$ in our case is therefore given by,

$$-V = \frac{2\pi\sigma a^2 c}{\sqrt{a^2-c^2}}\left(1-\frac{x^2+y^2}{2(a^2-c^2)}\right)\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\kappa}} + \frac{\pi\sigma a^2 c\,\sqrt{c^2+\kappa}}{a^2-c^2}\,\frac{x^2+y^2}{a^2+\kappa}, \tag{5}$$

where κ is defined by the Eqn.

$$\frac{(x+1-\mu)^2+y^2}{a^2+\kappa}+\frac{z^2}{c^2+\kappa}=1 \tag{6}$$

and

$$\mu = \frac{m_2}{m_1+m_2} \le \frac{1}{2}.$$

The equations of the motion of m_3 in the synodic co-ordinate system and dimensionless variables are:

$$m_3[\partial^2\boldsymbol{r}/\partial t^2 + 2\boldsymbol{\omega}\times\partial\boldsymbol{r}/\partial t + \partial\boldsymbol{\omega}/\partial t\times\boldsymbol{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})] = \boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_2} \tag{7}$$

where
$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j}$,
$\boldsymbol{\omega} = n\,\boldsymbol{k}$,
$\boldsymbol{F}$ = Total force acting on $m_3 = \boldsymbol{F_1} + \boldsymbol{F_2}$,
$\boldsymbol{F_1}$ = Gravitational force exerted on m_3 due to m_1,
$\boldsymbol{F_2}$ = Gravitational force exerted on m_3 due to m_2.

We first calculate the L.H.S. of the Equation (8) in the Cartesian form as follows:
$\partial\boldsymbol{r}/\partial t = \dot{x}\,\boldsymbol{i} + \dot{y}\,\boldsymbol{j}$,
Relative acceleration $= \partial^2\boldsymbol{r}/\partial t^2 = \ddot{x}\,\boldsymbol{i} + \ddot{y}\,\boldsymbol{j}$,
$\boldsymbol{\omega}\times\boldsymbol{r} = -n\,y\,\boldsymbol{i} + n\,x\,\boldsymbol{j}$,
Coriolis acceleration $= \boldsymbol{\omega}\times\partial\boldsymbol{r}/\partial t = -n\,\dot{y}\,\boldsymbol{i} + n\,\dot{x}\,\boldsymbol{j}$,
Centrifugal acceleration $= \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = -n^2(x\,\boldsymbol{i} + y\,\boldsymbol{j})$,
$\partial\boldsymbol{\omega}/\partial t = 0$,
Euler's acceleration $= \partial\boldsymbol{\omega}/\partial t\times\boldsymbol{r} = \boldsymbol{0}$.

Thus, the L.H.S. of the Equation (7) becomes
$m_3[(\ddot{x}\,\boldsymbol{i} + \ddot{y}\,\boldsymbol{j}) - 2\,n\,(\dot{y}\,\boldsymbol{i} - \dot{x}\,\boldsymbol{j}) - n^2(x\,\boldsymbol{i} + y\,\boldsymbol{j})]$

Now, we calculate the R.H.S. of the Equation (8) as follows:

Let the gravitational potential of m_1 and m_2 at m_3 be V_1 and V_2 respectively, therefore

$$-V_1 = \frac{G\,m_1\,m_3}{r_1},$$

$$-V_2 = \frac{3G\,m_2\,m_3}{4}\left[\frac{2}{\sqrt{a^2-c^2}}\left(1-\frac{x^2+y^2}{2(a^2-c^2)}\right)\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\kappa}} + \frac{\sqrt{c^2+\kappa}}{a^2-c^2}\,\frac{x^2+y^2}{a^2+\kappa}\right],$$

where

$$m_2 = \frac{4\pi a^2 c\sigma}{3},$$

Gravitational force exerted on m_3 due to m_1 and m_2 is given by
$\boldsymbol{F_1} = -(\partial V_1/\partial x\,\boldsymbol{i} + \partial V_1/\partial y\,\boldsymbol{j})$,
$\boldsymbol{F_2} = -(\partial V_2/\partial x\,\boldsymbol{i} + \partial V_2/\partial y\,\boldsymbol{j})$.

Thus, the Equation (7) becomes
$m_3[(\ddot{x}\,\boldsymbol{i} + \ddot{y}\,\boldsymbol{j}) - 2\,n\,(\dot{y}\,\boldsymbol{i} - \dot{x}\,\boldsymbol{j}) - n^2(x\,\boldsymbol{i} + y\,\boldsymbol{j})] = -(\partial V_1/\partial x + \partial V_2/\partial x)\,\boldsymbol{i} - (\partial V_1/\partial y + \partial V_2/\partial y)\,\boldsymbol{j}$
i. e.

$$\ddot{x} - 2n\dot{y} = n^2 x - \frac{1-\mu}{r_1^3}(x-\mu) - \frac{3\mu(x+1-\mu)}{2(a^2-c^2)}\times\left[\frac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\kappa}} - \frac{\sqrt{c^2+\kappa}}{(a^2+\kappa)}\right], \tag{8}$$

and

$$\ddot{y} + 2n\dot{x} = y\left[n^2 - \frac{1-\mu}{r_1^3} - \frac{3\mu}{2(a^2-c^2)}\times\left[\frac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\kappa}} - \frac{\sqrt{c^2+\kappa}}{(a^2+\kappa)}\right]\right], \tag{9}$$

where n is the mean-motion of the primaries.

$$1-\mu=\frac{m_1}{m_1+m_2},$$
$$r_1^2=(x-\mu)^2+y^2,$$
$$r_2^2=(x+1-\mu)^2+y^2.$$

Now, we define a function Ω such that

$$\Omega=\frac{n^2}{2}(x^2+y^2)+\frac{1-\mu}{r_1}+V \quad (10)$$

where V is defined in the Eqn. (5).
Hence, the Eqns. (8) and (9) become

$$\ddot{x}-2n\dot{y}=\Omega_x, \quad (11)$$
$$\ddot{y}+2n\dot{x}=\Omega_y, \quad (12)$$

where Ω_x and Ω_y are the partial derivatives of Ω with respect to x and y respectively.
The integral analogous to Jacobi integral is

$$(\dot{x}^2+\dot{y}^2)=2\Omega-C. \quad (13)$$

3. Calculation of the mean-motion *n* of the primaries

Since the primaries m_1 and m_2 are moving in the circular orbits around their center of mass O. Therefore, the mean-motion n of the primaries is given by,

$$n^2=\frac{F}{m_1\,m_2}$$

where F is the gravitational force acting on m_1 due to m_2.
From Eqn. (6), the value of the largest root κ at m_1 (μ, 0, 0) is $\kappa = 1 - a^2$.
Thus, the mean-motion n of the primaries is

$$n^2=\frac{3}{2(a^2-c^2)}\left[\frac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{a^2-c^2}-\sqrt{1-(a^2-c^2)}\right]$$

In terms of infinite series, mean-motion of the primaries is given by

$$n^2=1+\frac{3(a^2-c^2)}{10}+\frac{9(a^2-c^2)^2}{56}+\frac{5(a^2-c^2)^3}{48}+\frac{105(a^2-c^2)^4}{1408}+\ldots$$

Let $A = (a^2 - c^2) / 5 << 1$ be the oblateness factor, therefore

$$n^2=1+\frac{3}{2}A+\frac{45}{56}A^2+\frac{25}{48}A^3+\frac{525}{1408}A^4+\ldots$$

On neglecting the second and higher powers in A, we have

$$n^2=1+\frac{3A}{2}. \quad (14)$$

The results are agreed with [6]. For A = 0, the results are in conformity of the classical case of the restricted three-body problem [3]. From Fig. 2, as oblateness factor A increases, the mean-motion of the primaries also increases.

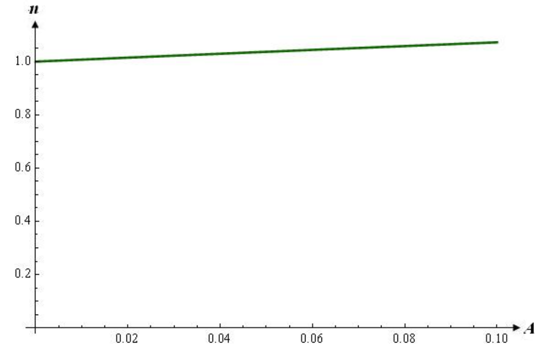

Fig. 2: *A* versus *n*

4. Non-collinear equilibrium solution

The non-collinear equilibria are the solution of the equations $\Omega_x = 0$ and $\Omega_y = 0$, $y \neq 0$ *i.e.*

$$n^2x-\frac{1-\mu}{r_1^3}(x-\mu)-\frac{3\mu(x+1-\mu)}{2(a^2-c^2)}\times\left[\frac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\kappa}}-\frac{\sqrt{c^2+\kappa}}{(a^2+\kappa)}\right]=0, \quad (15)$$

and

$$n^2-\frac{1-\mu}{r_1^3}-\frac{3\mu}{2(a^2-c^2)}\times\left[\frac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\kappa}}-\frac{\sqrt{c^2+\kappa}}{(a^2+\kappa)}\right]=0\ , \quad (16)$$

On eliminating the last terms from the Eqns. (15) and (16), we have

$$n^2-\frac{1}{r_1^3}=0\Rightarrow r_1=n^{-2/3}=1-\frac{A}{2}. \quad (17)$$

Again, on eliminating r_1 from Eqns. (15) and (16), we now get

$$n^2-\frac{3}{2(a^2-c^2)}\left[\frac{1}{\sqrt{a^2-c^2}}\times\sin^{-1}\sqrt{\frac{a^2-c^2}{a^2+\kappa}}-\frac{\sqrt{c^2+\kappa}}{(a^2+\kappa)}\right]=0 \quad (18)$$

On simplifying Eqn. (18) and considering only linear terms in $a^2 - c^2$, we have

$$n^2-\left(\frac{1}{(a^2+\kappa)^{3/2}}+\frac{3(a^2-c^2)}{10(a^2+\kappa)^{5/2}}+\ldots\right)=0.$$

Since $A = (a^2 - c^2) / 5 << 1$, therefore considering only linear terms in A, we get

$$n^2-\left(\frac{1}{(a^2+\kappa)^{3/2}}+\frac{3A}{2(a^2+\kappa)^{5/2}}\right)=0 \quad (19)$$

Obviously, $a^2 + \kappa = 1$, is the solution of Eqn. (19). Thus, the non-collinear equilibria are the solution of the Eqns. $r_1 = n^{-2/3}$ and $\kappa = 1 - a^2$ iff $r_2 = 1$ (Eqn. 6).

Hence, the coordinates of non-collinear equilibria $L_{4,5}$ are

$$x=\mu-\frac{1}{2}+\frac{A}{2}, \quad (20)$$

$$y = \pm\frac{\sqrt{3}}{2}\left(1-\frac{A}{3}\right). \tag{21}$$

Thus, there exist two non-collinear equilibria $L_{4,5}$ defined in Eqns. (20) and (21). The results are totally agree with [6] and [9]. Also, these points are linearly stable for critical mass parameter $\mu_c = \mu_o - 0.285001787791\,A$, where $\mu_o = 0.038520896505$ [6].

5. Conclusion

In the present paper the existence of non-collinear equilibria in restricted three-body problem considering less massive primary an oblate spheroid has been discussed when potential is given in terms of κ, the largest root of confocal spheroid passes through at an external point $P(x, y, z)$ given by W. D. MacMillan, 1958. The mean-motion of the primaries obtained is in conformity with [6] and [9]. Also, at non-collinear equilibria, the results are in agreement with [6] and [9]

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