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Research Paper

Non- linear stability of triangular librations points in circular restricted three body under radiating and oblate primaries in presence of resonance

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Abstract

The nonlinear stability of the triangular librations points is studied in the presence resonance considering both the primaries as radiating and oblate. The study is carried out for various values of radiation pressure and oblateness parameter in general and binary systems in particular. It is found that the normal forms of the Hamiltonian contains both the resonance cases; $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$. The case $\omega_1 = \omega_2$ corresponds to the boundary region of the stability for the system.It is investigated that for $\mu \le \mu_c = 0.0385209$; the motion is unstable for third order resonance but stable for fourth order resonance.

Keywords:*ER3BP, Hamiltonian Functions, Triangular Libration Points, Resonance*.

1. Introduction

The three body problem deals with three spherical masses which interact with each other through the gravitational interactions described by Newton's theory of gravity, and no restrictions are imposed on initial positions and velocities. In restricted three body problem, the two primaries have dominant masses and move around their common centre of mass; however, the third mass is very-very small and its gravitational influence on the primaries is negligible. In celestial mechanics the bodies are not strictly spherical but are regarded as spheroids with small degree of oblateness.
Manju and Choudhary (1985) studied the restricted problem of three bodiesby taking one of the bodies as radiating focusing on thenonlinear stability of the triangular libration points.Kumar and Choudhary (1987, 1988) investigated the stability of the triangular libration points for non-resonance as well as resonance case; taking both the bodies as radiating in circular restricted three body problems in presence of the third and fourth order resonance. Subba Rao and Sharma (1997) investigated the nonlinear stability of the triangular points by taking bigger primary as oblate and discussed three different ranges for critical values of mass ratios. Later on the study was extended by Markeev (2005), where the stability was studied focusing on the parametric resonance.
Many authors such as Danby (1964), Bennet (1965), Szebehely (1967) Markeev (1978), SelaruCucu-Dumitrescu (1995), Halan and Rana (2001) studied the behavior of equilibrium points in detail. The influence of the eccentricity of the orbits of the primaries with or without radiation pressure on the existence and stability of the equilibrium points was investigatedby Gyorgrey (1985), Grebenikov (1964), Kumar and Choudhary (1990), Markeellos (1992), Shaoo and Ishwar (2000), Roberts (2002), Floria (2004), Zimvoschikov and Thakai (2004), Ammar (2008), Erdi (2009), Kumar and Ishwar (2011).Singh and Umar (2012a), (2012b), Usha et. al. (2014), Narayan and Singh (2014 a, b, c,).
In the present paper an attempt has been made to analyze the existence of resonance and to study the nonlinear stability of infinitesimal in elliptical restricted three body problem near the resonance frequency satisfying $\omega_1 = \omega_2, \omega_1 = 2\omega_2, \omega_1 = 3\omega_2$ in special case when e=0.The study is conducted by considering both the primaries as radiating and oblate under the presence of resonance. This model can be easily applied for the exoplanets, Trojan asteroids and for binary systems (Achird, Luyten, α Cen AB, kruger-60, Xi- Bootis).
This paper is organized as follows:
Section 1 gives introduction; section 2 provides the equations of motion; while section 3 deals with characteristics roots and first order stability of the triangular equilibrium points. Existence of resonance and Normalisation for the cases $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$ is given in section 4. Section 5 and 6 focusses on Stability in the resonance case $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$ respectively. Finally, discussion and conclusion are drawn in sections 7.

2. Equations of motion

The differential equations of the motion of the infinitesimal mass in ellipticalrestricted three body problem under radiating primaries in pulsating system is given by [Narayan and Shrivastava (2014)] as:

$$x'' - 2y' = \varphi\,\Omega_x\ ;$$

$$y'' - 2x' = \varphi\,\Omega_y \qquad (2.1)$$

Where

$$\Omega=\frac{x^2+y^2}{2}+\frac{1}{1+3\left(\frac{A_1+A_2}{2}\right)}\times\left[\frac{(1-\mu)Q_1}{r_1}+\frac{\mu Q_2}{r_2}+\frac{(1-\mu)Q_1A_1}{2r_1^3}+\frac{\mu A_2Q_2}{2r_2^3}\right]$$

$$r_1^2=(x+\mu)^2+y^2 \tag{2.2}$$

$$r_2^2=(x-1+\mu)^2+y^2 \tag{2.3}$$

And

$$\varphi=\frac{1}{(1+e\cos f)}; \tag{2.4}$$

A_1, A_2 Are the oblateness parameter, Q_1, Q_2 Are the radiation pressure and f is a true anomaly of the primaries, e the eccentricity of the orbit and μ the mass ratio.

The coordinates of the triangular equilibrium points L_4 and L_5 are given by: [Narayan andShrivastava (2014)]

$$x_0=\frac{1}{2}-\mu+\frac{A_1}{2}-\frac{A_2}{2}+\frac{\beta_2}{3}-\frac{\beta_1}{3}-\frac{1}{2}A_1\beta_1+\frac{1}{2}A_2\beta_2;$$

$$y_o=\pm\frac{\sqrt{3}}{2}\left[1-\frac{A_1}{3}-\frac{A_2}{3}-\frac{2\beta_1}{9}-\frac{2\beta_2}{9}-\frac{1}{3}A_1\beta_1-\frac{1}{3}A_2\beta_2\right]. \tag{2.5}$$

3. Characteristics roots and first order stability of the triangular equilibrium points

The Hamiltonian as described by Narayan and Shrivastava (2014) is given by:

$$H=\frac{p_1^2+p_2^2}{2}+(p_1q_2-p_2q_1)+\frac{(q_1^2+q_2^2)e\cos f}{2(1+e\cos f)}-\frac{1}{(1+e\cos f)\left(1+3\frac{A_1+A_2}{2}\right)}\times\left\{\left\{\frac{(1-\mu)Q_1}{r_1}+\frac{\mu Q_2}{r_2}+\frac{(1-\mu)Q_1A_1}{2r_1^3}+\frac{\mu A_2Q_2}{2r_2^3}\right\}\right\} \tag{3.1}$$

For simplicity of calculations consider

$$Q=1-\beta_i\ ,i=1,2\ and\ \beta_i<1 \tag{3.2}$$

Equating the coefficients of 2nd 3rd, 4th order we get:

$$H_2=\frac{p_1^2+p_2^2}{2}+(P_1q_2-P_2q_1)+e\cos f\,\frac{(q_1^2+q_2^2)}{2(1+e\cos f)}-\frac{1}{(1+e\cos f)\left\{1+3\left(\frac{A_1+A_2}{2}\right)\right\}}\times\left[\begin{array}{l}\left(\begin{array}{l}-\frac{1}{8}+\frac{9}{8}A_1-\frac{9}{16}A_2-\frac{73}{24}A_1\beta_1\\+\frac{11}{48}A_2\beta_2-\frac{27}{16}\mu A_1+\frac{27}{16}\mu A_2\\+\frac{17}{8}\mu\beta_1-\frac{7}{9}\mu\beta_2+\frac{1}{2}\beta_2-\frac{13}{8}\beta_2\end{array}\right)q_1^2+\\ \sqrt{3}\left(\begin{array}{l}\frac{3}{4}+\frac{17}{4}A_1+\frac{7}{8}A_2+\frac{139}{24}A_1\beta_1\\+\frac{15}{8}A_2\beta_2-\frac{41}{8}\mu A_1-\frac{41}{8}\mu A_2\\-\frac{1}{6}\mu\beta_1+\mu\beta_2-\frac{3\sqrt{3}}{2}\mu-\frac{1}{3}\beta_2\\-\frac{7}{12}\beta_1\end{array}\right)q_1q_2+\\ \left(\begin{array}{l}\frac{5}{8}+\frac{27}{8}A_1+\frac{21}{16}A_2+\frac{107}{16}A_1\beta_1\\{}_1+\frac{11}{16}A_2\beta_2-\frac{33}{16}\mu A_1+\frac{33}{16}\mu A_2\\-\frac{7}{8}\mu\beta_1+\frac{11}{8}\mu\beta_2\\-\frac{1}{2}\beta_2+\frac{7}{8}\beta_1\end{array}\right)q_2^2\end{array}\right] \tag{3.3}$$

Equation (3.3) can be written as:

$$H_2=\frac{p_1^2+p_2^2}{2}+(p_1q_2-p_2q_1)+\frac{e\cos f}{2(1+e\cos f)\left\{1+3\left(\frac{A_1+A_2}{2}\right)\right\}}(q_1^2+q_2^2)-H_{20}q_1^2-H_{11}q_1q_2-H_{02}q_2^2 \tag{3.4}$$

Provided,

$$H_{20}=-\left(\begin{array}{l}\frac{-1}{8}+\frac{9}{8}A_1-\frac{9}{16}A_2-\frac{73}{24}A_1\beta_1\\+\frac{11}{48}A_2\beta_2-\frac{27}{16}\mu A_1+\frac{27}{16}\mu A_2\\+\frac{17}{8}\mu\beta_1-\frac{7}{9}\mu\beta_2+\frac{\beta_2}{2}-\frac{13\beta_1}{8}\end{array}\right) \tag{3.5}$$

$$H_{11}=\sqrt{3}\left(\begin{array}{l}\frac{3}{4}+\frac{17}{4}A_1+\frac{7}{8}A_2+\frac{139}{24}A_1\beta_1\\+\frac{15}{8}A_2\beta_2-\frac{41}{8}\mu A_1-\frac{41}{8}\mu A_2\\-\frac{1}{6}\mu\beta_1+\mu\beta_2-\frac{3\mu}{2}-\frac{\beta_2}{3}-\frac{7\beta_1}{12}\end{array}\right) \tag{3.6}$$

$$H_{02}=\left(\begin{array}{l}\frac{5}{8}+\frac{27}{8}A_1+\frac{21}{16}A_2+\frac{107}{16}A_1\beta_1\\+\frac{11}{16}A_2\beta_2-\frac{33}{8}\mu A_1+\frac{33}{16}\mu A_2\\-\frac{7}{8}\mu\beta_1+\frac{11}{8}\mu\beta_2-\frac{\beta_2}{2}+\frac{7\beta_1}{8}\end{array}\right) \tag{3.7}$$

$$
\begin{aligned}
H_3 = \frac{-1}{6(1+e\cos f)[1+3(A_1+A_2)/2]} \times \Bigg[& q_1^3 \times \Bigg(\frac{21}{8} + \frac{15A_1}{4} - \frac{295}{16}A_2 - \frac{93\beta_1}{4} - \frac{65\beta_2}{6} - \frac{365\beta_1 A_1}{24} - \frac{155\beta_1 A_2}{16} + \\
& \mu\left(-\frac{57}{8} - \frac{15A_1}{4} - \frac{305A_2}{16} - \frac{29\beta_1}{4} + \frac{51\beta_2}{8} - \frac{365\beta_1 A_1}{24} + \frac{85\beta_1 A_2}{16} - \frac{17\beta_2 A_1}{8} - \frac{5\beta_2 A_2}{3} \right) \Bigg) \\
& + 3q_1^2 q_2 \times \Bigg(\frac{-3}{8} + \frac{21A_1}{8} - \frac{61}{8}A_2 - \frac{113\beta_1}{24} + \frac{7\beta_2}{4} - \frac{53\beta_1 A_1}{24} + \frac{61\beta_1 A_2}{8} + \\
& \mu\sqrt{3}\left(\frac{17A_1}{16} + \frac{9A_2}{16} + \frac{47\beta_1}{6} - \frac{13\beta_2}{4} - \frac{53\beta_1 A_1}{24} - \frac{113\beta_1 A_2}{24} - \frac{91\beta_2 A_1}{16} - \frac{\beta_2 A_2}{48} \right) \Bigg) \\
& + 3q_1 q_2^2 \times \Bigg(\frac{-33}{8} - \frac{315A_1}{8} + \frac{5}{12}A_2 + \frac{149\beta_1}{24} + \frac{5\beta_2}{8} - \frac{265\beta_1 A_1}{8} - \frac{5\beta_1 A_2}{12} + \\
& \mu\left(\frac{33}{4} - \frac{245A_1}{8} + \frac{263A_2}{8} + \frac{77\beta_1}{48} - \frac{311\beta_2}{16} - \frac{55\beta_1 A_1}{4} + \frac{3805\beta_1 A_2}{96} + \frac{55\beta_2 A_1}{8} - \frac{1441\beta_2 A_2}{16} \right) \Bigg) + \\
& q_2^3 \times \Bigg(\frac{-9}{8} - \frac{51A_1}{4} - \frac{69}{16}A_2 - \frac{13\beta_1}{8} - \frac{11\beta_2}{4} + \frac{151\beta_1 A_1}{16} + \frac{65\beta_1 A_2}{16} + \\
& \mu\left(\frac{195A_1}{16} - \frac{33A_2}{2} + \frac{81\beta_1}{16} + \frac{31\beta_2}{8} - \frac{151\beta_1 A_1}{16} + \frac{117\beta_1 A_2}{32} - \frac{317\beta_2 A_1}{32} + \frac{5\beta_2 A_2}{4} \right) \Bigg) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
H_4 = \frac{-1}{24(1+e\cos f)} \times [1+3(A_1+A_2)/2] \times \Bigg[& q_1^4 \times \Bigg(\frac{-111}{16} - \frac{1155A_1}{16} + \frac{9815}{16}A_2 + \frac{397\beta_1}{2} - \frac{25\beta_2}{2} - \frac{1235\beta_1 A_1}{16} - \frac{1185\beta_1 A_2}{2} \\
& + \mu\left(-\frac{1575}{16} - \frac{395A_1}{48} - \frac{29545A_2}{48} + \frac{9269\beta_1}{16} + \frac{439\beta_2}{144} - \frac{1234\beta_1 A_1}{16} - \frac{6325\beta_1 A_2}{12} + \frac{6755\beta_2 A_1}{48} + \frac{605\beta_2 A_2}{48} \right) \Bigg) \\
& + 4\sqrt{3}q_1^3 q_2 \times \Bigg(\frac{-75}{16} - \frac{281A_1}{16} - \frac{21}{2}A_2 + \frac{5\beta_1}{16} + \frac{21\beta_2}{6} + \frac{351\beta_1 A_1}{16} + \frac{21\beta_1 A_2}{2} \\
& + \mu\left(\frac{75}{8} + \frac{5669A_1}{208} - \frac{117A_2}{8} - \frac{235\beta_1}{48} - \frac{6121\beta_2}{560} - \frac{7\beta_1 A_1}{2} + \frac{319\beta_1 A_2}{32} + \frac{2357\beta_2 A_1}{156} - \frac{83\beta_2 A_2}{4} \right) \Bigg) \\
& + 6q_1^2 q_2^2 \times \Bigg(\frac{123}{16} + \frac{4865A_1}{48} - \frac{105}{8}A_2 + \frac{235\beta_1}{12} + \frac{75\beta_2}{12} - \frac{4295\beta_1 A_1}{48} + \frac{105\beta_1 A_2}{8} \\
& + \mu\left(-\frac{123}{8} - \frac{5735A_1}{48} + \frac{1601A_2}{16} - \frac{421\beta_1}{48} - \frac{31\beta_2}{8} + \frac{4865\beta_1 A_1}{48} + \frac{245\beta_1 A_2}{32} + \frac{45\beta_2 A_1}{8} - \frac{539\beta_2 A_2}{4} \right) \Bigg) \Bigg]
\end{aligned}
$$

(3.8)

$$\left[\begin{array}{l} +4\sqrt{3}q_1q_2^{\,3}\times \\ \left(\begin{array}{l} \frac{135}{16}+\frac{1531A_1}{16}+\frac{15}{4}A_2-\frac{69\beta_1}{16} \\ -\frac{21\beta_2}{5}+\frac{23\beta_1A_1}{2}-\frac{15\beta_1A_2}{4} \\ +\mu\left(\begin{array}{l} -\frac{135}{8}-\frac{337A_1}{2}-\frac{8997A_2}{112} \\ +\frac{259\beta_1}{16}+\frac{293\beta_2}{40}+\frac{1391\beta_1A_1}{16} \\ -\frac{3005\beta_1A_2}{32}+\frac{1315\beta_2A_1}{116}-\frac{175\beta_2A_2}{48} \end{array}\right) \end{array}\right) \end{array}\right.$$

$$\left.\begin{array}{l} +q_2^4\left(\begin{array}{l} \frac{9}{16}+\frac{1125A_1}{16}-\frac{1575}{32}A_2+\frac{1781\beta_1}{16} \\ -\frac{45\beta_2}{2}-\frac{1045\beta_1A_1}{8}+\frac{1575\beta_1A_2}{32} \\ +\mu\left(\begin{array}{l} -\frac{495A_1}{8}-\frac{3167A_2}{32}-\frac{1917\beta_1}{16} \\ +26\beta_2+\frac{1045\beta_1A_1}{8}-\frac{14765\beta_1A_2}{96}- \\ \frac{1325\beta_2A_1}{32}-\frac{6775\beta_2A_2}{48} \end{array}\right) \end{array}\right) \end{array}\right] \tag{3.9}$$

4. Existence of resonance

As discussed by Liapunov's (1956), the equilibrium points are stable ifH_2 is of positive definite form; otherwise stability can be investigated by Arnold's theorem [1963(a, b)] .In this theorem H_2 is normalized using linear transformation of variables as given by Markeev (1978). Restricting to H_2 alone the characteristics equation takes the following form:

$$\begin{vmatrix} \lambda^2-(1-2H_{20}) & -2\lambda-H_{11} \\ 2\lambda-H_{11} & \lambda^2-(1+2H_{02}) \end{vmatrix}=0 \tag{4.1}$$

Where H_{20}, H_{11}, H_{02} are given by equation [3.5-3.7]. After further calculations the characteristics equation reduces to the form:

$$\lambda^4+\lambda^2+\left\{\frac{27\mu(1-\mu)}{4}\right\}\times \left\{1-\frac{125}{6}A_1+\frac{40}{3}A_2+\frac{5}{6}\beta_1-\frac{35}{18}\beta_2\right\}=0 \tag{4.2}$$

If ω_1 and ω_2 be the frequencies then the equation can be written as:

$$\omega^4-\omega^2+\left\{\frac{27\mu(1-\mu)}{4}\right\}\times \left\{1-\frac{125}{6}A_1+\frac{40}{3}A_2+\frac{5}{6}\beta_1-\frac{35}{18}\beta_2\right\}=0 \tag{4.3}$$

The roots of equation (4.3) are given as:

$$\omega_{1,2}^2=\frac{1}{2}\left[1+\left\{\begin{array}{l}1-27\mu(1-\mu)\times \\ \left(\begin{array}{l}1-\frac{125}{6}A_1+\frac{40}{3}A_2+ \\ \frac{5}{6}\beta_1-\frac{35}{18}\beta_2\end{array}\right)\end{array}\right\}^{1/2}\right]$$

$$\omega_{3,4}^2=\frac{1}{2}\left[1-\left\{\begin{array}{l}1-27\mu(1-\mu)\times \\ \left(\begin{array}{l}1-\frac{125}{6}A_1+\frac{40}{3}A_2+ \\ \frac{5}{6}\beta_1-\frac{35}{18}\beta_2\end{array}\right)\end{array}\right\}^{1/2}\right] \tag{4.4}$$

Figs 1-5 show the value of ω_1 and ω_2 for different values of μ.

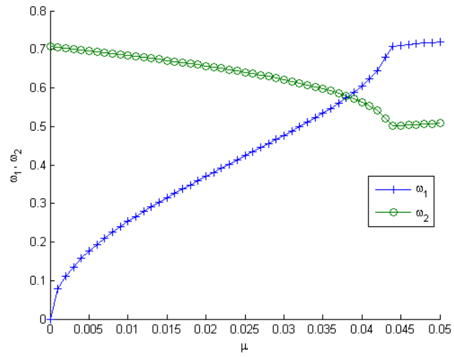

Fig.1: Correlation between μ and ω_1 and ω_2 with β_1=0.01, β_2=0.02, A_1=0.01, A_2=0.02

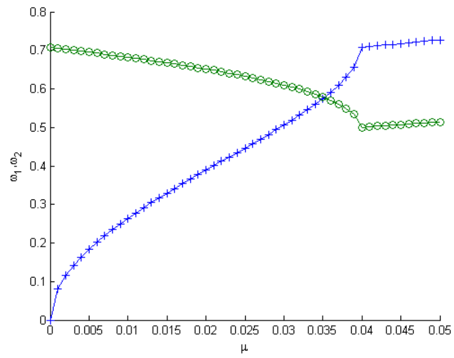

Fig.2: Correlation between μ and ω_1 and ω_2 with β_1=0.01, β_2= 0.02, A_1=0.02, A_2=0.03.

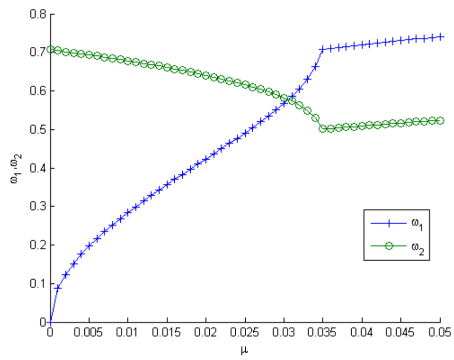

Fig. 3: Correlation between μ and ω_1 and ω_2 with β_1=0.01, β_2=0.02, A_1=0.02, A_2=0.035.

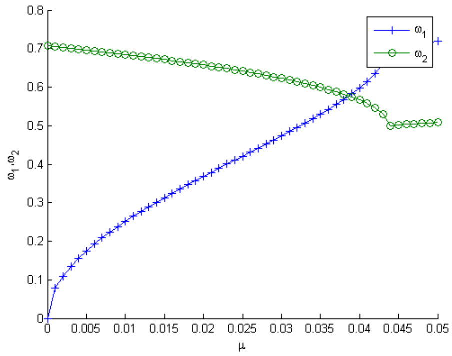

Fig. 4: Correlation between μ and ω_1 and ω_2 with $\beta_1=0.02$, $\beta_2=0.02$, $A_1=0.01$, $A_2=0.02$.

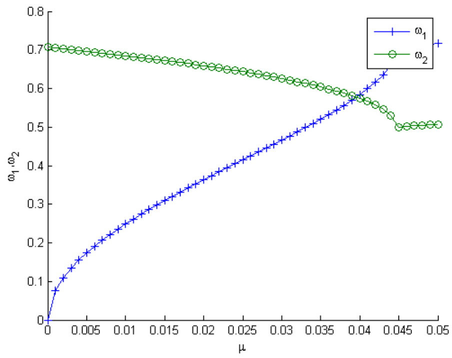

Fig. 5: Correlation between μ and ω_1 and ω_2 with $\beta_1=0.02$, $\beta_2=0.03$, $A_1=0.01$, $A_2=0.02$.

It is seen from Figs 1-5 that for various value of radiation pressure and varying oblateness ω_1 decreases with increasing μ whereas ω_2 increases, and becomes equal for $\mu=\mu_c$.
Let us consider the following cases:
Case1. When $\omega_1=\omega_2$

$$\lambda_{1,2}^2=\lambda_{3,4}^2$$

Then we have:

$$1-27\mu(1-\mu)\left(\begin{array}{l}1-\frac{125}{6}A_1+\frac{40}{3}A_2\\+\frac{5}{6}\beta_1-\frac{35}{18}\beta_2\end{array}\right)\geq 0 \qquad (4.5)$$

If the equality holds then we have:

$$27\left(1-\frac{125}{6}A_1+\frac{40}{3}A_2+\frac{5}{6}\beta_1-\frac{35}{18}\beta_2\right)\mu^2-27\left(1-\frac{125}{6}A_1+\frac{40}{3}A_2+\frac{5}{6}\beta_1-\frac{35}{18}\beta_2\right)\mu+1=0 \qquad (4.6)$$

i.e.

$$\mu=\frac{9\pm\sqrt{69}\left(\begin{array}{l}1-\frac{125}{69}A_1+\frac{80}{69}A_2+\\\frac{5}{69}\beta_1-\frac{35}{207}\beta_2\end{array}\right)}{18} \qquad (4.7)$$

Since $\mu\leq\frac{1}{2}$, the positive sign is inadmissible. Hence the region of stability in first approximation can be written as

$$0<\mu<\frac{\left\{9-\sqrt{69}\left(\begin{array}{l}1-\frac{125}{69}A_1+\frac{80}{69}A_2+\\\frac{5}{69}\beta_1-\frac{35}{207}\beta_2\end{array}\right)\right\}}{18} \qquad (4.8)$$

$$\mu_c=0.0385208965+0.83601A_1+0.535048A_2-0.0334405\beta_1+0.0780279\beta_2 \qquad (4.9)$$

The critical value (μ_c) usually corresponds to the boundary of the region of stability of the linear system.
Case2. When $\omega_1=2\omega_2$

$$i.e.\ \lambda_{1,2}^2=4\lambda_{3,4}^2$$

Solving for μ the resonance value is obtained as:

$$\mu_{c1}=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{16}{675\left(1-\frac{125A_1}{6}+\frac{40A_2}{3}+\frac{5\beta_1}{6}-\frac{35\beta_2}{18}\right)}} \qquad (4.10)$$

Case3. When $\omega_1=3\omega_2$

$$i.e.\ \lambda_{1,2}^2=9\lambda_{3,4}^2$$

The value of μ is given as:

$$\mu_{c2}=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{9}{675\left(1-\frac{125A_1}{6}+\frac{40A_2}{3}+\frac{5\beta_1}{6}-\frac{35\beta_2}{18}\right)}} \qquad (4.11)$$

The values of μ_{c1} and μ_{c2} are plotted for various values of oblateness parameters which is shown in Figs 6-15.It is clear from these figs. that resonance condition exists for cases $\omega_1=2\omega_2$ and $\omega_1=3\omega_2$ for different values of oblateness and radiation pressure.

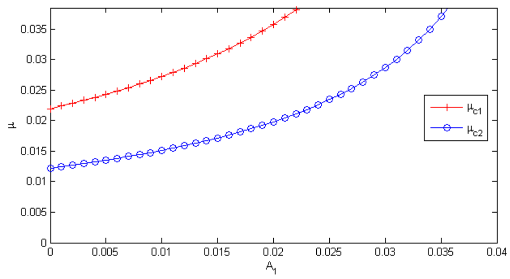

Fig. 6: μ_{C1}, μ_{C2} vs A_1 when $A_2=0.01$, $\beta_1=0.01$, $\beta_2=0.02$.

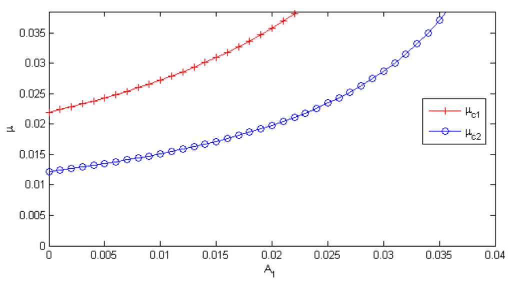

Fig. 7: μ_{C1}, μ_{C2} vs A_1 when $A_2=0.02$, $\beta_1=0.01$, $\beta_2=0.02$.

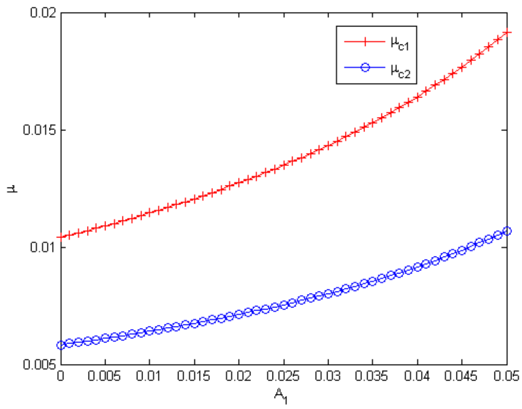

Fig. 8: μ_{C1}, μ_{C2} vs A_1 when A_2=0.1, β_1=0.01, β_2=0.02.

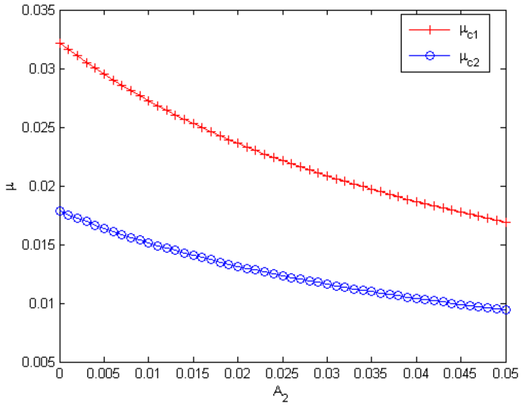

Fig. 9: μ_{C1}, μ_{C2} vs A_2 when A_1=0.01, β_1=0.01, β_2=0.02.

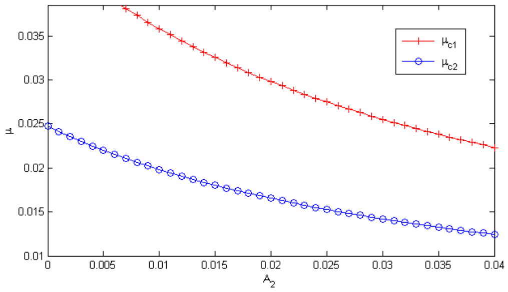

Fig. 10: μ_{C1}, μ_{C2} vs A_2 when A_1=0.02, β_1=0.01, β_2=0.02.

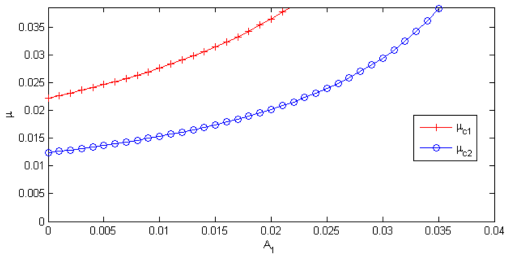

Fig.11: μ_{c1}, μ_{c2} vs A_1 when A_2=0.01, β_1=0.02, β_2=0.03.

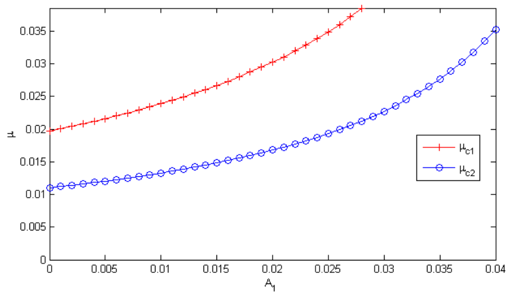

Fig. 12: μ_{C1}, μ_{C2} Vs A_1 when A_2=0.02, β_1=0.02, β_2=0.03.

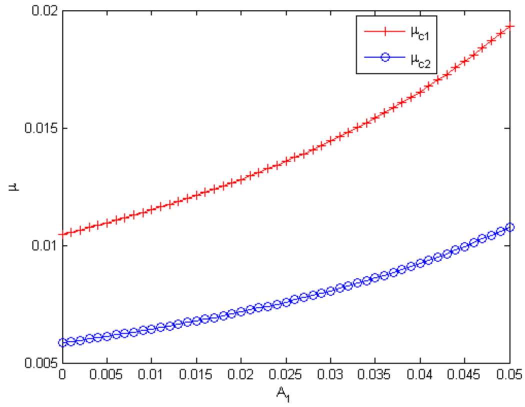

Fig .13: μ_{C1}, μ_{C2} vs A_1 when A_2=0.1, β_1=0.02, β_2=0.03.

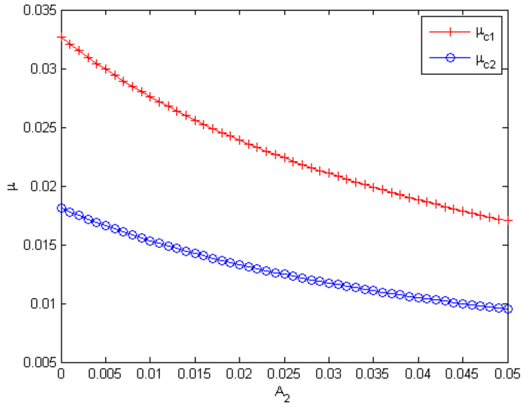

Fig .14: μ_{C1}, μ_{C2} vs A_2 when A_1=0.01, β_1=0.02, β_2=0.03.

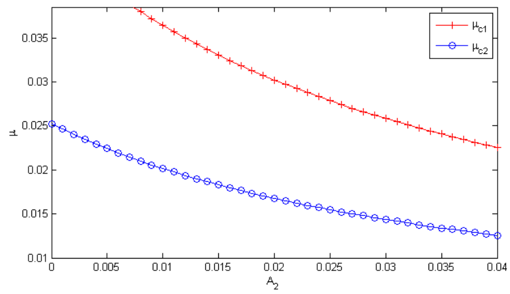

Fig .15: μ_{C1}, μ_{C2} vs A_2 when A_1=0.02, β_1=0.02, β_2=0.03.

Further, in order to discuss the stability of the triangular points, the Hamiltonian H is normalized up to fourth order by using suitable canonical transformation as:

$$(q_1, q_2, p_1, p_2) = (q'_1, q'_2, p'_1, p'_2)N \tag{4.12}$$

Where,

$$N = \begin{bmatrix} a_1 & a_1 c_1 & -a_1 c_1 & a_1\left(1-\omega_1^2 b_1\right) \\ a_2 & a_2 c_2 & -a_2 c_2 & a_2\left(1-\omega_2^2 b_2\right) \\ 0 & a_1 b_1 & a_1\left(1-b_1\right) & a_1 c_1 \\ 0 & -a_2 b_2 & -a_2\left(1-b_2\right) & -a_2 c_2 \end{bmatrix} \tag{4.13}$$

And

$$a_1 = \frac{1}{2}\left(\frac{2l_1}{\omega_1^2 - 1/2}\right)^{1/2};$$

$$a_2 = \frac{1}{2}\left(\frac{2l_2}{\omega_2^2 - 1/2}\right)^{1/2}$$

$$l_1 = 1 + \omega_1^2 + 2H_{02}; \quad l_2 = 1 + \omega_2^2 + 2H_{02}$$

$$b_1 = \frac{2}{l_1}; \qquad b_2 = \frac{2}{l_2}$$

$$c_1 = \frac{-H_{11}}{l_1}; c_2 = \frac{-H_{11}}{l_2} \tag{4.14}$$

The transformation (4.12) reduces the Hamiltonian to the following form:

$$H = \frac{1}{2}\left(p_1'^2 + \omega_1^2 q_1'^2\right) - \frac{1}{2}\left(p_2'^2 + \omega_2^2 q_2'^2\right) + \sum_{\alpha+\gamma=3}^{\infty} h_{\alpha_1,\alpha_2\beta_1,\beta_2} q_1'^{\alpha_1} q_2'^{\alpha_2} p_1'^{\gamma_2} p_2'^{\gamma_2} \tag{4.15}$$

Where, $\alpha=\alpha_1+\alpha_2$; $\gamma=\gamma_1+\gamma_2$

The coefficient of third and fourth order terms of $h'_{\alpha_1\alpha_2\gamma_1\gamma_2}$ and $h_{\alpha_1\alpha_2\gamma_1\gamma_2}$ is given in appendices [see (a)].

Again using the BirKhoffs transformation the Hamiltonian given by (4.15) reduces to the following form:

$$\bar{H} = i\left(\omega_1 q_1'' p_1'' + \omega_2 q_2'' p_2''\right) + \sum_{\alpha+\gamma=3}^{\infty} h'_{\alpha_1,\alpha_2\gamma_1,\gamma_2} q_1''^{\alpha_1} q_2''^{\alpha_2} p_1''^{\gamma_1} p_2''^{\gamma_2} \tag{4.16}$$

Where

$$h' = x_{\alpha_1,\alpha_2,\gamma_1,\gamma_2} + iy_{\alpha_1,\alpha_2,\gamma_1,\gamma_2} \tag{4.17}$$

The values of $x_{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$, and $y_{\alpha_1,\alpha_2,\gamma_1,\gamma_2}$ are given in appendices [see (b)].

The other coefficients of third order terms are given in the appendices [see (c)].

5. Stability in the resonance case $\omega_1 = 2\omega_2$

In the resonance case $\omega_1 = 2\omega_2$, using the Birkhoffs transformation

$$(q''_j, p''_j) \to (Q_j, P_j)$$

It is not possible to cancel whole $\bar{H}_3$ of the Hamiltonian H .In this case $\bar{H}_3$ retains two resonant terms with coefficients h'_{1002} and h'_{0210}. Thus the normalised form of the Hamiltonian is given as;

$$\bar{H} = i(\omega_1 Q_1 P_1 + \omega_2 Q_2 P_2) + h'_{1002} Q_1 P_2^2 + h'_{0210} Q_2^2 P_1 + \ldots.. \tag{5.1}$$

Applying two canonical transformations as:

$$Q_1 = \frac{1}{(\omega_1)^{1/2}}(Q_1^0 - iP_1^0);$$

$$Q_2 = \frac{1}{(\omega_2)^{1/2}}(iQ_2^0 - P_2^0);$$

$$P_1 = \frac{(\omega_1)^{1/2}}{2}(-iQ_1^0 + P_1^0);$$

$$P_2 = \frac{(\omega_2)^{1/2}}{2}(iQ_2^0 - iP_2^0); \tag{5.2}$$

And

$$Q_1^0 = (2r_1)^{1/2}\sin(\varphi_1 - \theta_1);$$

$$P_1^0 = (2r_1)^{1/2}\cos(\varphi_1 - \theta_1); \quad Q_2^0 = (2r_2)^{1/2}\sin\varphi_2;$$

$$P_2^0 = (2r_1)^{1/2}\cos\varphi_2; \tag{5.3}$$

Where

$$\sin\theta_1 = \frac{y_{1002}}{(x_{1002}^2 + y_{1002}^2)^{1/2}};$$

$$\cos\theta_1 = \frac{x_{1002}}{(x_{1002}^2 + y_{1002}^2)^{1/2}}. \tag{5.4}$$

The Hamiltonian given by equation (5.1) is reduced to the following polar form as:

$$\bar{H} = 2\omega_2 r_1 - \omega_2 r_2 - \sqrt{(x_{1002}^2 + y_{1002}^2)\omega_2 r_2} \times \sqrt{r_1}\sin(\varphi_1 + \varphi_2) + o((r_1 + r_2)^2) \tag{5.5}$$

The nonlinear stability of the equilibrium points is determined by Markeev theorem (1968). The equilibrium points are stable, if

$(x_{1002}^2 + y_{1002}^2) = 0$; and $c_{20} + 2c_{11} + 4c_{02} \neq 0$

are true simultaneously. From table (1) and (2) it is clear that for no values of the radiation pressure and oblateness parameters the expression $(x_{1002}^2 + y_{1002}^2)$ vanishes. Hence it follows that the motion is unstable in the resonance case $\omega_1 = 2\omega_2$ in general and for the binary system (Achird, Luyten, α Cen AB, kruger-60, Xi- Bootis) , in particular.

Table 1: Values of $x_{1002}^2 + y_{1002}^2$, $c_{20} + 2c_{11} + 4c_{02}$, $x_{1003}^2 + y_{1003}^2$, $c_{20} + 3c_{11} + 9c_{02}$ and $3\omega_2(x_{1003}^2 + y_{1003}^2)^{1/2}$ for different Values of Radiation Pressure and Oblateness.

β_1	β_2	A_1	A_2	$(x_{1002}^2 + y_{1002}^2)$ $\times(10^3)$	$c_{20} + 2c_{11} + 4c_{02}$ $\times(10^4)$	$x_{1003}^2 + y_{1003}^2$ $\times(10^7)$	$c_{20} + 3c_{11} + 9c_{02}$ $\times(10^5)$	$3\omega_2 \times (x_{1003}^2 + y_{1003}^2)^{1/2}$ $\times(10^4)$
0.01	0.01	0.01	0.02	2.2500	7.3266	5.2698	1.7226	1.4048
0.01	0.01	0.02	0.03	2.1361	8.3939	5.7718	1.7957	1.4373
0.01	0.01	0.03	0.035	1.7953	4.4397	6.7146	8.4798	1.4328
0.01	0.02	0.01	0.02	1.9619	7.8852	4.8128	1.8300	1.3430
0.01	0.02	0.02	0.03	1.8444	8.7172	5.1042	1.8582	1.3516
0.01	0.02	0.03	0.035	1.5138	4.4578	5.3261	8.5033	0.3262
0.01	0.03	0.01	0.02	1.6974	8.4615	4.4190	1.9420	0.2739
0.01	0.03	0.02	0.03	1.5788	9.0441	4.5157	1.9222	0.3276
0.01	0.03	0.03	0.035	1.2597	4.4679	4.1075	0.85130	0.3252
0.01	0.04	0.01	0.02	1.4561	9.0566	4.0672	2.0590	0.2740

0.01	0.04	0.02	0.03	1.3390	9.3749	3.9877	1.9876	0.3276
0.01	0.04	0.03	0.035	1.0332	4.4687	3.0446	.85066	0.3241
0.02	0.01	0.01	0.02	2.7088	8.2907	6.8965	1.9799	0.2743
0.02	0.02	0.02	0.03	2.2455	9.8846	6.8744	2.1239	0.3286
0.02	0.03	0.03	0.035	1.5380	5.1843	5.1250	.99226	0.3279
0.02	0.04	0.01	0.02	1.8276e	10.310	5.5136	2.3712	0.2746
0.02	0.04	0.02	0.03	1.6797	10.658	5.4871	2.2749	0.3286
0.02	0.05	0.03	0.035	1.2858	5.2009	2.8704	.99480	0.3259
0.03	0.01	0.01	0.02	3.2131	9.2630	9.1260	2.2585	0.2748
0.03	0.02	0.02	0.03	2.6876	11.134	9.2175	2.4158	0.3296
0.03	0.03	0.03	0.035	1.8427	5.9342	6.4323	1.1436	0.3304
0.04	0.01	0.01	0.02	3.7635	10.221	12.143	2.5579	0.2754
0.04	0.02	0.02	0.03	3.1710	12.467	12.309	2.7359	0.3305
0.04	0.03	0.03	0.035	2.1736	6.7290	8.0892	1.3078	0.3326
0.02	0.02	0.01	0.02	2.3916	8.9410	6.3621	2.1046	0.2744
0.02	0.02	0.02	0.03	2.2455	9.8846	6.8744	2.1239	0.3286
0.02	0.02	0.03	0.035	1.8175	5.1596	6.5103	9.8878	0.3288

Table 2: Values of $x_{1002}^2 + y_{1002}^2$, $c_{20}+2c_{11}+4c_{02}$, $x_{1003}^2+y_{1003}^2$, $c_{20}+3c_{11}+9c_{02}$ and $3\omega_2(x_{1003}^2+y_{1003}^2)^{1/2}$ for the Binary Systems.

Binary system	A_1	A_2	$(x_{1002}^2+y_{1002}^2)$ $\times(10^3)$	$c_{20}+2c_{11}+4c_{02}$ $\times(10^4)$	$x_{1003}^2+y_{1003}^2$ $\times(10^7)$	$c_{20}+3c_{11}+9c_{02}$ $\times(10^5)$	$3\omega_2(x_{1003}^2+y_{1003}^2)^{1/2}$ $\times(10^4)$
Achird	0.01	0.02	1.8595	3.9506	4.6624	1.4198	0.2731
	0.02	0.03	2.1416	7.3706	5.0439	1.5316	0.3268
	0.03	0.035	1.8595	3.9506	7.1162	7.2991	0.3257
Luyten	0.01	0.02	2.1181	5.9229	4.5846	1.3981	0.2730
	0.02	0.03	2.0325	7.0807	4.9378	1.5108	0.3267
	0.03	0.035	1.7759	3.7628	25.288	39.419	0.4297
α Cen AB	0.01	0.02	1.8148	6.9036	4.1875	1.6006	0.2733
	0.02	0.03	1.7160	7.7912	4.3966	1.6563	0.3270
	0.03	0.035	1.4423	3.9703	5.2780	7.5691	0.3248
Kruger-60	0.01	0.02	2.1205	5.9310	4.5909	1.4000	0.2730
	0.02	0.03	2.0348	7.0890	4.9466	1.5127	0.3268
	0.03	0.05	1.7774	3.7678	7.0671	7.1981	0.3255
Xi- Bootis	0.01	0.02	2.1632	6.0339	4.6994	1.4253	0.2731
	0.02	0.03	2.0751	7.2018	5.0934	1.5376	0.3269
	0.03	0.035	1.8081	3.8404	7.1639	7.3358	0.3258

6. Stability in resonance cases $\omega_1 = 3\omega_2$

In the resonance case $\omega_1 = 3\omega_2$, it is possible to cancel whole of $\bar{H}_3$ of theHamiltonian $\bar{H}$ but $\bar{H}_4$ retains the resonance terms and the terms with the same degree of canonical variables. Thus the normalized form of the Hamiltonian is written as:

$$\bar{H} = i(\omega_1 Q_1 P_1 + \omega_2 Q_2 P_2) - c_{20} Q_1^2 P_1^2 + c_{11} Q_1 P_1 Q_1 P_2 - c_{02} Q_2^2 P_2^2 + h'_{1003} Q_1 P_2^3 + h'_{0310} Q_2^3 P_1 + ... \tag{6.1}$$

Again, applying the canonical transformation by means of polar coordinates given as:

$$Q_1^0 = (2r_i)^{1/2}\sin(\varphi_i - \theta_i)$$

$$P_1^0 = (2r_i)^{1/2}\cos(\varphi_i - \theta_i);\ (i = 1, 2) \tag{6.2}$$

Where

$$\theta_2 = 0$$

$$\sin\theta_1 = \frac{x_{1003}}{(x_{1003}^2 + y_{1003}^2)^{1/2}};$$

$$\cos\theta_1 = \frac{-y_{1003}}{(x_{1003}^2 + y_{1003}^2)^{1/2}}. \tag{6.3}$$

The polar form of the normalized Hamiltonian reduces to the following form:

$$\bar{H} = 3\omega_2 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + \frac{\omega_2}{3}\times\{3(\sqrt{(x_{1003}^2 + y_{1003}^2)})\}.r_2(r_1 r_2)^{1/2} \sin(\varphi_1 + 3\varphi_2) + o((r_1 + r_2)^{5/2}) \tag{6.4}$$

Let us denote

$$a = c_{20} + 3c_{11} + 9c_{02};$$
$$d = 3\omega_2(x_{1003}^2 + y_{1003}^2)^{1/2}. \tag{6.5}$$

Now, according to Markeev theorem, if the following conditions are satisfied as:

i) $(x_{1003}^2 + y_{1003}^2) \neq 0$ *and* $d \geq |a|$ holds simultaneously, then the equilibrium points are unstable and if $d < |a|$ then it is stable.

ii) When $(x_{1003}^2 + y_{1003}^2) = 0$ *and* $d = 0$ are true simultaneously then also, it is stable.

iii) If $(x_{1003}^2 + y_{1003}^2) = 0$ *and* $d \neq 0$ then the stability in question is decided by the analysis of higher order terms in the normal form. The values of a and d are calculated for various values of radiation pressure and oblateness parameter in table 1.

From table (1) and (2) it is clear that for all the values of radiation pressures; $d < |a|$ .Hence, the triangular points are stable in fourth order resonance for the binary systems.

7. Discussion and conclusion

The nonlinear stability of the triangular equilibrium points is examined in ER3BP in presence of resonance under the radiating and oblate primaries in circular case. The value of μ has been plotted for different values of radiation pressure and oblateness parameter. It is noted that under the range $\mu < \mu_c$, $\omega_1 = \omega_2$ does not occur. It is clear for Fig1-3 that, for fixed value of radiation pressure and varying oblateness ω_1 decreases with increasing μ whereas ω_2 increases, and becomes equal for $\mu = \mu_c$. The same behavior is seen for fixed oblateness but varying radiation pressure which is obvious from Figs.4-5. It is found from Figs 6-15 that the value of μ_{c1} and μ_{c2} increases or decreases with increasing value of oblateness parameter A_1 and A_2. From the above mentioned graph it is also clear that the values of μ_{c1} and μ_{c2} is less than the critical value μ_c=0.0385209 for all the values of radiation pressure and oblateness parameter taken, into consideration.

Here, by application of Markeev's theorems, it is found that for $\mu \le \mu_c = 0.0385209$; the motion is unstable for third order resonance but stable for fourth order resonance; for different values of radiation and oblateness parameter and for the binary systems (Achird, Luyten, α Cen AB, kruger-60, Xi- Bootis).

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Appendices

(a) The coefficient of third and fourth order terms of $h'_{\alpha_1 \alpha_2 \beta_1 \beta_2}$ and $h_{\alpha_1 \alpha_2 \beta_1 \beta_2}$ can be given as:

$$h_{0030} = a_1^3 b_1^3 H_{0300};$$

$$h_{0030} = a_1^3 (H_{3000} + c_1 H_{2100} + c_1^2 H_{2100} + c_1^3 H_{0300};$$

$$h_{1020} = a_1^3 b_1^2 (H_{1200} + 3c_1 H_{0300});$$

$$x_{1003}^2 + y_{1003}^2$$

$$c_{20} + 3c_{11} + 9c_{02}$$

$$3\omega_2 (x_{1003}^2 + y_{1003}^2)^{1/2}$$

$$h_{1001} = -2a_1^2 a_2 b_1 b_2 (H_{1200} + 3c_1 H_{0300});$$

$$h_{1110}=2a_1^2a_2b_1[H_{2100}+(c_1+c_2)H_{1200}+3c_1c_2H_{0300}];$$

$$h_{0021}=-3a_1^2b_1^2b_2H_{0300};$$

$$h_{2100}=a_1^2a_2\ [3H_{3000}+(2c_1+c_2)H_{2100}+c_1(c_1+2c_2)H_{1200}+3c_1^2c_2H_{0300}];$$

$$h_{1002}=a_1b_1^2a_2^2(H_{1200}+3c_1H_{0300});$$

$$h_{0210}=a_1a_2^2b_1(H_{2100}+2c_2H_{1200}+3c_2^2H_{0300});$$

$$h_{0012}=3a_1a_2^2b_2^2b_1H_{0300};$$

$$h_{1200}=a_2^2a_1[3H_{3000}+(c_1+2c_2)H_{2100}+c_2(2c_1+c_2)H_{1200}+3c_1c_2^2H_{0300}];$$

$$h_{0111}=-2a_1^2a_2^2b_1b_2(H_{1200}+3c_2H_{0300});$$

$$h_{1101}=-2a_2^2a_1b_2[H_{2100}+(c_1+c_2)H_{1200}+3c_1c_2H_{0300}];$$

$$h_{0120}=a_1^2a_2b_1^2(H_{1200}+3c_2H_{0300});$$

$$h_{0201}=-a_2^3b_2(H_{2100}+2c_2H_{1200}+3c_2^2H_{0300});$$

$$h_{0102}=a_2^3b_2^2(H_{1200}+3c_2H_{0300});$$

$$h_{0003}=-a_2^3b_2^3H_{0300};$$

$$h_{0300}=a_2^3(H_{3000}+c_2H_{2100}+c_2^2H_{1200}+c_2^3H_{0300});$$

$$h_{0040}=a_1^4b_1^4H_{0400};$$

$$h_{4000}=a_1^4(H_{4000}+c_1H_{3100}+c_1^2H_{2200}+c_1^3H_{1300}+c_1^4H_{0400});$$

$$h_{2020}=a_1^4b_1^2(H_{2200}+3c_1H_{1300}+6c_1^2H_{0400});$$

$$h_{0022}=6a_1^2a_2^2b_1^2b_2^2H_{0400};$$

$$h_{2200}=a_1^2a_2^2[6H_{4000}+3(c_1+c_2)H_{3100}+(c_1^2+4c_1c_2+c_2^2)H_{2200}+3c_1c_2(c_1+c_2)H_{1300}+6c_1^2c_2^2H_{0400}];$$

$$h_{2002}=a_1^2a_2^2b_2^2(H_{2200}+3c_1H_{1300}+6c_1^2H_{0400});$$

$$h_{0004}=a_2^4b_2^4H_{0400};$$

$$h_{0400}=a_2^4(H_{4000}+c_2H_{3100}+c_2^2H_{2200}+c_2^3H_{1300}+c_2^4H_{0400});$$

$$h_{0202}=a_2^4b_2^2(H_{2200}+3c_2H_{1300}+6c_2^2H_{0400});$$

$$h_{0013}=-a_1a_2^3b_2^3H_{0400};$$

$$h_{1102}=a_1a_2^3[4H_{4000}+(c_1+3c_2)H_{3100}+4c_1c_2^3H_{0400}+2c_2(c_1+c_2)H_{2200}+c_2^2(3c_1+c_2)H_{1300}];$$

$$h_{0211}=-2a_2^3a_1b_1b_2[H_{2200}+3c_2H_{1300}+6c_2^2H_{0400}];$$

$$h_{0112}=3a_1a_2^3b_1b_2^2(H_{1300}+4c_1H_{0400});$$

$$h_{1003}=-a_1a_2^3b_2^3(H_{1300}+4c_1H_{0400});$$

$$h_{1201}=-a_1a_2^3b_2[3H_{3100}+2(c_1+2c_2)H_{2200}+3c_2(2c_1+c_2)H_{1300}+12c_1c_2^2H_{0400}];$$

$$h_{0310}=a_1a_2^3b_1(H_{3100}+2c_2H_{2200}+3c_2^2H_{1300}+c_2^2H_{0400});$$

(b)

$$x_{0030}=h_{0030}-\frac{1}{\omega_1^2}h_{2010};$$

$$y_{0030}=\frac{1}{\omega_1}h_{1020}-\frac{1}{\omega_1^3}h_{3000};$$

$$x_{1020}=-\frac{1}{2}h_{1020}-\frac{3}{2\omega_1^2}h_{3000};$$

$$y_{1020}=\frac{3\omega_1}{2}h_{0030}+\frac{1}{2\omega_1}h_{2010};$$

$$x_{0120}=-\frac{\omega_2}{2}h_{0021}+\frac{1}{2\omega_1}h_{1110}+\frac{\omega_2}{2\omega_1}h_{2001};$$

$$y_{0120}=-\frac{1}{2}h_{0120}-\frac{\omega_2}{2\omega_1}h_{1011}+\frac{1}{2\omega_1^2}h_{2100};$$

$$x_{1011}=-\omega_1h_{0021}-\frac{1}{\omega_1}h_{2001};$$

$$y_{0021} = h_{0021} + \frac{1}{\omega_1\omega_2} h_{1110} - \frac{1}{\omega_1^2} h_{2001};$$

$$x_{1002} = -\frac{\omega_1}{2\omega_2} h_{0111} - \frac{1}{2} h_{1002} + \frac{1}{2\omega_2^2} h_{1200};$$

$$y_{1002} = -\frac{\omega_1}{2} h_{0012} + \frac{\omega_1}{2\omega_1^2} h_{0210} + \frac{1}{2\omega_2^2} h_{1101};$$

$$x_{0012} = -h_{0012} + \frac{1}{\omega_2^2} h_{0210} - \frac{1}{\omega_1\omega_2} h_{1101};$$

$$y_{0012} = \frac{1}{\omega_2} h_{0111} - \frac{1}{\omega_1} h_{1002} + \frac{1}{\omega_1\omega_2^2} h_{1200};$$

$$x_{0111} = \frac{\omega_2}{\omega_1} h_{1002} + \frac{1}{\omega_1\omega_2} h_{1200};\ y_{0111} = -\omega_2 h_{0012} - \frac{1}{\omega_2} h_{0210};$$

$$x_{0201} = \frac{-\omega_2}{4} h_{0102} - \frac{3}{4\omega_2} h_{0300};\ y_{0201} = \frac{-3\omega_2^2}{4} h_{0003} + \frac{1}{4} h_{0201};$$

$$x_{0003} = \frac{-1}{\omega_2} h_{0102} + \frac{1}{\omega_2^3} h_{0300};$$

$$y_{0003} = -h_{0003} + \frac{1}{\omega_2^2} h_{0201};$$

(c)

$$\begin{aligned} x_{1003} = {} & \frac{1}{2}\omega_1 h_{0013} + \frac{1}{2\omega_2^3} h_{1300} - \frac{1}{2\omega_2} h_{1102} \\ & - \frac{\omega_1}{2\omega_2^2} h_{0211} - \frac{9}{5}(x_{0120}x_{0012} + y_{0120}y_{0012}) \\ & - \frac{1}{\omega_2}(x_{1002}y_{1011} + x_{1011}y_{1002}) \\ & + \frac{4}{\omega_2^2}(x_{1002}x_{0201} + y_{1002}y_{0201}) \\ & - \frac{3}{2}(x_{0003}x_{0111} + y_{0003}y_{0111}); \end{aligned}$$

$$\begin{aligned} y_{1003} = {} & \frac{-1}{2\omega_2}\omega_1 h_{0112} - \frac{1}{2} h_{1003} + \frac{1}{2\omega_2^2} h_{1201} \\ & + \frac{\omega_1}{2\omega_2^2} h_{0310} - \frac{9}{5}(x_{0120}y_{0012} - y_{0120}x_{0012}) \\ & - \frac{1}{\omega_2}(y_{1002}x_{1011} - x_{1011}x_{1002}) \\ & + \frac{4}{\omega_2^2}(y_{1002}x_{0201} - x_{1002}y_{0201}) \\ & + \frac{3}{2}(y_{0003}x_{0111} - x_{0003}y_{0111}). \end{aligned}$$