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Research Paper

First order normalization in the generalized photo gravitational non-planar restricted three body problems

Nirbhay Kumar Sinha

Assistant Professor, Deshbandhu College,University of Delhi, Delhi, India
E-mail: nirbhay.k.sinha@gmail.com

Abstract

In this paper, we normalised the second-order part of the Hamiltonian of the problem. The problem is generalised in the sense that fewer massive primary is supposed to be an oblate spheroid. By photogravitational we mean that both primaries are radiating. With the help of Mathematica, H_2 is normalised to $H_2 = \alpha_1\beta_1\omega_1 + \alpha_2\beta_2\omega_2$. The resulting motion is composed of elliptic motion with a short period ($2\pi/\omega_1$), completed by an oscillation along the ζ-axis with a short period ($2\pi/\omega_2$).

Keywords:*Normalisation; Photogravitational; Non-Planar; RTBP.*

1. Introduction

Normal forms are a standard tool in Hamiltonian mechanics to study the dynamics in a neighbourhood of invariant objects. Usually, these normal forms are obtained as divergent series, but their asymptotic character is what makes them useful. From theoretical point of view, they provide nonlinear approximations to the dynamics in a neighbourhood of the invariant object, that follows to obtain information about the real solutions of the system by taking the normal form up to a suitable finite order. The idea behind normal forms is to construct a transformation of phase-space that brings a given system of differential equations into the simplest possible form upto a certain order of accuracy.

From a more practical point of view, normal forms can be used as a computational method to obtain very accurate approximations to the dynamics in a neighbourhood of the selected invariant object, by neglecting the remainder. They have been applied, for example, to compute invariant manifolds or invariant tori. For this, it is necessary to compute the explicit expression of the normal form and of the canonical transformation that put the Hamiltonian into this reduced form. This computational formulation has special interest in some celestial mechanics model. It can be used to approximate the dynamics of some real world problems.

In this paper the first order normalisation have been studied analytically. We have examined the effect of gravitational potential, oblateness effect and radiation in out of plane equilibrium points in photogravitational restricted three body problem.

Sharma, R. K and Subba Rao, P.V (1976) discussed the three dimensional restricted three body problem with oblateness. Sokol'skii, A. G (1978) gave the proof of the stability of Lagrangian solutions for a critical mass ratio. Poschel, J. (1982) described the concept of integrability on Cantor sets for Hamiltonian systems. Meyer, K. R and Schmidt, D. S (1986) discussed the stability of the Lagrange Triangular Point and a theorem of Arnold. Celleti, A and Giorgilli, A (1991) examined the stability of the Lagrangian points in the spatial RTBP. Ito, H (1992) discussed the integrability of Hamiltonian systems and Birkhoff normal forms in the simple resonance case. Benettin, G et. al (1998) described stability of L_4 and L_5 in the spatial RTBP. Jorba, A (2001) gave numerical computation of the normal behaviour of invariant curves of n-dimensional maps. Oberti, P and Vienne, A (2003) dealt with a new theory improving the convergence of the solution around L_4 in RTBP. Duskos, C. N and Markellos, V. V (2006) described out of plane equilibrium points in the RTBP with oblateness. Kusvah, B. S et. al (2007) examined non-linear stability of L_4 in photogravitational RTBP with P-R drag.

Hence, we thought to examine non-linear stability of L_4 in out of plane photogravitational RTBP with one of the primaries as an oblate spheroid. We have given short introduction of the problem. For non-linear stability of L_4 in our problem, we first normalise H_2 in section 2. With the help of Mathematica, we have normalised H_2 of our problem as

$$H_2 = \alpha_1\beta_1\omega_1 + \alpha_2\beta_2\omega_2 \qquad \ldots(1)$$

The resulting motion is composed of elliptic motion in $\zeta\eta$-plane with a short period ($2\pi/\omega_1$), completed by an oscillation along the ζ-axis with a short period ($2\pi/\omega_2$).

2. Normalisation of H_2

The second order part of the Hamiltonian H_2 of our problem is given by

$$H_2 = \frac{P_1^2 + P_2^2}{2} + n\left(p_1 * z - p_2 * x\right) - ex^2 - f * z^2 - gxz$$

We suppose that J = {{0,0,1,0}, {0,0,0,1}, {–1,0,0,0}, {0,–1,0,0}}.

The transpose [J] = –J is true.

With the help of Mathematica, we proceed further for normalisation of H_2.

HessHqp = {{D[H_2,x]}, {D[H_2,z]}, {D[H_2,p_1]}, {D[H_2,p_2]}
= {{–2ex – gz – np_2}, {–gx – 2fz + np_1}, {nz + p_1}, {–nx + p_2}}

Matrix form is given by

$$\begin{pmatrix} -2ex-gz-np_2 \\ -gx-2fz+np_1 \\ nz+p_1 \\ -nx+p_2 \end{pmatrix}$$

HessH = {{-2e, -g, 0, -n}, {-g, -2f, n, 0}, {0, n, 1, 0}, {-n, 0, 0, 1}}

Matrix form is = $\begin{pmatrix} -2e & -g & 0 & -n \\ -g & -2f & n & 0 \\ 0 & n & 1 & 0 \\ -n & 0 & 0 & 1 \end{pmatrix}$

Let M = J. HessH

= {{0, n, 1, 0}, {-n, 0, 0, 1}, {2e, g, 0, n}, {g, 2f, -n, 0}}

Matrix form = $\begin{pmatrix} 0 & n & 1 & 0 \\ -n & 0 & 0 & 1 \\ 2e & g & 0 & n \\ g & 2f & -n & 0 \end{pmatrix}$

Characteristic Polynomial [M, λ]

$= 4ef - g^2 + 2en^2 + 2fn^2 + n^4 - 2e\lambda^2 - 2f\lambda^2 + 2n^2\lambda^2 + \lambda^4$

Collecting λ, we have

$= 4ef - g^2 + 2en^2 + 2fn^2 + n^4 + (-2e - 2f + 2n^2)\lambda^2 + \lambda^4$

We suppose A = {{-λ, n}, {-n, -λ}}

B = {{2e, g}, {g, 2f}}

Matrix form = $\begin{pmatrix} 2e & g \\ g & 2f \end{pmatrix}$

M = {{A, ID}, {B, A}}

= {{{{-λ, n}, {-n, λ}}, ID}, {{{2e, g}, {g, 2f}}, {{-λ, n}, {-n, λ}}}}

Let X = $\{x_1, x_2\}$, and Y = $\{x_3, x_4\}$ are the eigen vectors of M, then M { X, Y} = {0, 0}

i.e. AX + Y = 0 or Y = -AX

A.A = $\{\{-n^2 + \lambda^2, -2n\lambda\}, \{2n\lambda, -n^2 + \lambda^2\}\}$

and BX + AY = 0 or Y = $(B\text{-}A)^2$ X = 0

i.e. (B – A.A).X

$= \{(2e + n^2 - \lambda^2)\, x_1 + (g + 2n\lambda)x_2, (g - 2n\lambda)\, x_1 + (2f + n^2 - \lambda^2)x_2\}$

i.e. $\dfrac{x_1}{(g+2n\lambda)} = \dfrac{x_2}{-(2e+n^2-\lambda^2)}$

From Y = -A.X

$\{\lambda x_1 - nx_2, nx_1 - \lambda x_2\} = -A.\{(g + 2n\lambda), -(2e + n^2 - \lambda^2)\}$

$= \{\lambda(g + 2n\lambda) - n(-2e - n^2 + \lambda^2), n(g + 2n\lambda) + \lambda(-2e - n^2 + \lambda^2)\}$

$x_1 = (g + 2n\lambda),\ x_2 = -(2e + n^2 - \lambda^2)$

$x_3 = \lambda(g + 2n\lambda) - n(-2e - n^2 + \lambda^2)$

$x_4 = n(g + 2n\lambda) + \lambda(-2e - n^2 + \lambda^2)$

$x_1 = x_1/.\ \lambda \to I\omega = g + 2in\omega$

$x_2 = x_2/.\ \lambda \to I\omega = -2e - n^2 - \omega^2$

x_3 = Expand $[x_3/.\ \lambda \to I\omega] = 2en + n^3 + ig\omega - n\omega^2$

x_4 = Expand $[x_4/.\ \lambda \to I\omega] = gn - 2ie\omega + in^2\omega - i\omega^3$

$u = \{g, -2e - n^2 - \omega^2, 2en + n^3 - n\omega^2, gn\}$

$v = \{2n\omega, 0, g\omega, -2e\omega + n^2\omega - \omega^3\}$

$u_1 = \{g, -2e - n^2 - \omega_1^2, 2en + n^3 - n\omega_1^2, gn\}$

$u_2 = \{g, -2e - n^2 - \omega_2^2, 2en + n^3 - n\omega_2^2, gn\}$

$v_1 = \{2n\omega_1, 0, g\omega_1, -2e\omega_1 + n^2\omega_1 - \omega_1^3\}$

$v_2 = \{2n\omega_2, 0, g\omega_2, -2e\omega_2 + n^2\omega_2 - \omega_2^3\}$

ctmp = $\{\{g, -2e - n^2 - \omega_1^2, 2en + n^3 - n\omega_1^2, gn\}, \{g, -2e - n^2 - \omega_2^2, 2en + n^3 - n\omega_2^2, gn\},$

$\{2n\omega_1, 0, g\omega_1, -2e\omega_1 + n^2\omega_1 - \omega_1^3\}, \{2n\omega_2, 0, g\omega_2, -2e\omega_2 + n^2\omega_2 - \omega_2^3\}\}$

c = Transpose [ctmp]

$= \{\{g, g, 2n\omega_1, 2n\omega_2\}, \{-2e - n^2 - \omega_1^2, -2e - n^2 - \omega_2^2, 0, 0\},$

$\{2en + n^3 - n\omega_1^2, 2en + n^3 - n\omega_2^2, g\omega_1, g\omega_2\},$

$\{gn, gn, -2e\omega_1 + n^2\omega_1 - \omega_1^3, -2e\omega_2 + n^2\omega_2 - \omega_2^3\}\}$

Transpose [c] .J.c

$= \{\{g(2en + n^3 - n\omega_1^2) + g(-2en - n^3 + n\omega_1^2),$

$gn(-2e - n^2 - \omega_1^2) + g(-2en - n^3 + n\omega_1^2) - gn(-2e - n^2 - \omega_2^2) + g(2en + n^3 - n\omega_2^2),$

$g^2\omega_1 + 2n\omega_1(-2en - n^3 + n\omega_1^2) + (-2e - n^2 - \omega_1^2)(-2e\omega_1 + n^2\omega_1 - \omega_1^3),$

$g^2\omega_2 + 2n\omega_2(-2en - n^3 + n\omega_1^2) + (-2e - n^2 - \omega_1^2)(-2e\omega_2 + n^2\omega_2 - \omega_2^3)\},$

$\{-gn(-2e - n^2 - \omega_1^2) + g(2en + n^3 - n\omega_1^2) + gn(-2e - n^2 - \omega_2^2) + g(-2en - n^3 + n\omega_2^2),$

$g(2en + n^3 - n\omega_2^2) + g(-2en - n^3 + n\omega_2^2),$

$g^2\omega_1 + (-2e\omega_1 + n^2\omega_1 - \omega_1^3)(-2e - n^2 - \omega_2^2) + 2n\omega_1(-2en - n^3 + n\omega_2^2),$

$g^2\omega_2 + 2n\omega_2(-2en - n^3 + n\omega_2^2) + (-2e - n^2 - \omega_2^2)(-2e\omega_2 + n^2\omega_2 - \omega_2^3)\},$

$\{-g^2\omega_1 + 2n\omega_1(2en + n^3 - n\omega_1^2) + (-2e - n^2 - \omega_1^2)(2e\omega_1 - n^2\omega_1 + \omega_1^3),$

$-g^2\omega_1 + (2e\omega_1 - n^2\omega_1 + \omega_1^3)(-2e - n^2 - \omega_2^2) + 2n\omega_1(2en + n^3 - n\omega_2^2), 0, 0\},$

$\{-g^2\omega_2 + 2n\omega_2(2en + n^3 - n\omega_1^2) + (-2e - n^2 - \omega_1^2)(2e\omega_2 - n^2\omega_2 + \omega_2^3),$

$-g^2\omega_2 + 2n\omega_2(2en + n^3 - n\omega_2^2) + (-2e - n^2 - \omega_2^2)(2e\omega_2 - n^2\omega_2 + \omega_2^3), 0, 0\}\}$

Expand

$\{\{0, 0, 4e^2\omega_1 + g^2\omega_1 - 4en^2\omega_1 - 3n^4\omega_1 + 4e\,\omega_1^3 + 2n^2\,\omega_1^3 + \omega_1^5,$

$4e^2\omega_2 + g^2\omega_2 - 4en^2\omega_2 - 3n^4\omega_2 + 2e\,\omega_1^2\,\omega_2 + n^2\,\omega_1^2\,\omega_2 + 2e\,\omega_2^3 + n^2\,\omega_2^3 + \omega_1^2\,\omega_2^3\},$

$\{0, 0, 4e^2\omega_1 + g^2\omega_1 - 4en^2\omega_1 - 3n^4\omega_1 + 2e\,\omega_1^3 + n^2\,\omega_1^3 + 2e\omega_1\,\omega_2^2 + n^2\omega_1\,\omega_2^2 + \omega_1^3\,\omega_2^2,$

$4e^2\omega_2 + g^2\omega_2 - 4en^2\omega_2 - 3n^4\omega_2 + 4e\,\omega_2^3 + 2n^2\,\omega_2^3 + \omega_2^5\},$

$\{-4e^2\omega_1 - g^2\omega_1 + 4en^2\omega_1 + 3n^4\omega_1 - 4e\,\omega_1^3 - 2n^2\,\omega_1^3 - \omega_1^5,$

$-4e^2\omega_1 - g^2\omega_1 + 4en^2\omega_1 + 3n^4\omega_1 - 2e\,\omega_1^3 - n^2\,\omega_1^3 - 2e\omega_1\,\omega_2^2 - n^2\omega_1\,\omega_2^2 - \omega_1^3\,\omega_2^2, 0, 0,\},$

$\{-4e^2\omega_2 - g^2\omega_2 + 4en^2\omega_2 + 3n^4\omega_2 - 2e\,\omega_1^2\,\omega_2 - n^2\,\omega_1^2\,\omega_2 - 2e\,\omega_2^3 - n^2\,\omega_2^3 - \omega_1^2\,\omega_2^3,$

$\{-4e^2\omega_2 - g^2\omega_2 + 4en^2\omega_2 + 3n^4\omega_2 - 4e\,\omega_2^3 - 2n^2\,\omega_2^3 - \omega_2^5, 0, 0\}$

Simplify

$\{\{0, 0, \omega_1(4e^2\omega_1 + g^2 - 4en^2 - 3n^4 + 2(2e + n^2)\,\omega_1^2 + \omega_1^4),$

$\omega_2(4e^2 + g^2 - 4en^2 - 3n^4 + (2e + n^2)\,\omega_2^3 + \omega_1^2\,(2e + n^2 + \omega_2^2))\},$

$\{0, 0, \omega_1(4e^2 + g^2 - 4en^2 - 3n^4 + (2e + n^2)\,\omega_2^2 + \omega_1^2\,(2e + n^2 + \omega_2^2)),$

$\omega_2(4e^2 + g^2 - 4en^2 - 3n^4 + 2(2e + n^2)\,\omega_2^2 + \omega_2^4)\},$

$\{-\omega_1(4e^2 + g^2 - 4en^2 - 3n^4 + 2(2e + n^2)\,\omega_1^2 + \omega_1^4),$

$-\omega_1(4e^2 + g_2 - 4en^2 - 3n^4 + (2e + n^2)\,\omega_2^2 + \omega_1^2\,(2e + n^2 + \omega_2^2))\ 0, 0\},$

$\{-\omega_2(4e^2 + g^2 - 4en^2 - 3n^4 + (2e + n^2)\,\omega_2^2 + \omega_1^2\,(2e + n^2 + \omega_2^2)),$

$-\omega_2(4e^2 + g^2 - 4en^2 - 3n^4 + 2(2e + n^2)\,\omega_2^2 + \omega_2^4), 0, 0\}\}$

Using the followings

$4ef - g^2 + 2en^2 + 2fn^2 + n^4 + (-2e - 2f + 2n^2)\lambda^2 + \lambda^4\ /.\ \lambda \to I\omega$

$4ef - g^2 + 2en^2 + 2fn^2 + n^4 - (-2e - 2f + 2n^2)\omega^2 + \omega^4$

$n^4 = -(4ef - g^2 + 2en^2 + 2f n^2 - (-2e - 2f + 2n^2)\,\omega^2 + \omega^4)$

$\omega_1^2 + \omega_2^2 = (-2e - 2f + 2n^2),\ (\omega_1^2\ \omega_2^2) = 4ef - g^2 + 2en^2 + 2f n^2 + n^4$

$\{\{0,0,\ \omega_1\,(4e^2 + g^2 - 4en^2 - 3n^4 + 2(2e + n^2)\ \omega_1^2 + \omega_1^4),\ 0\},$

$\{0,\ 0,\ 0,\ \omega_2\,(4e^2 + g^2 - 4en^2 - 3n^4 + 2(2e + n^2)\ \omega_2^2 + \omega_2^4)\},$

$\{-\omega_1\,(4e^2 + g^2 - 4en^2 - 3n^4 + 2(2e + n^2)\ \omega_1^2 + \omega_1^4),\ 0,\ 0,\ 0\},$

$\{0,\ -\omega_2\,(4e^2 + g^2 - 4en^2 - 3n^4 + 2(2e + n^2)\ \omega_2^2 + \omega_2^4),\ 0,\ 0\}\}$

$\omega_1\,(4e^2 + g^2 - 4en^2 + 2(2e + n^2)\ \omega_1^2 + \omega_1^4 +$

$3(4ef - g^2 + 2en^2 + 2f n^2 - (-2e - 2f + 2n^2)\ \omega_1^2 + \omega_1^4$

Simplify

$2\omega_1\,(2e^2 - g^2 + 3fn^2 + e(6f + n^2) + (5e + 3f - 2n^2)\ \omega_1^2 + 2\ \omega_1^4$

$d\omega_1 = 2\omega_1\,(2e^2 - g^2 + efn^2 + e(6f + n^2) + (5e + 3f - 2n^2)\ \omega_1^2 + 2\ \omega_1^4)$

$= (2e^2 - g^2 + efn^2 + e(6f + n^2) + (5e + 3f - 2n^2)\ \omega_1^2 + 2\ \omega_1^4)$

$d\omega_2 = 2\omega_2\,(2e^2 - g^2 + efn^2 + e(6f + n^2) + (5e + 3f - 2n^2)\ \omega_2^2 + 2\ \omega_1^4)$

$= 2\omega_2\,(2e^2 - g^2 + efn^2 + e(6f + n^2) + (5e + 3f - 2n^2)\ \omega_2^2 + 2\ \omega_2^4)$

Transpose [c] .J .c = {{0, D}, {-D, 0}} where D = {{$d\omega_1$, 0}, {0, $d\omega_2$}}

Matrix form [{{0, D}, {-D, 0}}]

$$= \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

Matrix form [{{$d\omega_1$, 0}, {0, $d\omega_2$}}]

$$= \begin{pmatrix} d\omega_1 & 0 \\ 0 & d\omega_2 \end{pmatrix}$$

J = {{0, 0, 1, 0}, {0, 0, 0, 1}, {-1, 0, 0, 0}, {0, -1, 0, 0}}

$cd1 = \{\{g,\ g,\ 2n\omega_1,\ 2n\omega_2\},\ \{-2e - n^2 - \omega_1^2,\ -2e - n^2 - \omega_2^2,\ 0,\ 0\},$

$\{2en + n^3 - n\,\omega_1^2,\ 2en + n^3 - n\,\omega_2^2,\ g\omega_1,\ g\omega_2\},$

$\{gn,\ gn,\ -2e\omega_1 + n^2\omega_1 - \omega_1^3,\ -2e\omega_2 + n^2\omega_2 - \omega_2^3\}\}$

cinv = Inverse[c]

$= \{\{((-2e - n^2 - \omega_2^2)(g\ \omega_1^3\ \omega_2 - g\ \omega_1\ \omega_2^3))\ /\ (g^2\ \omega_1^5\ \omega_2 - 8en^2\ \omega_1^5\ \omega_2 - 4n^4\ \omega_1^5\ \omega_2 -$

$2g^2\ \omega_1^3\ \omega_2^3 + 16n^2\ \omega_1^3\ \omega_2^3 + 8n^4\ \omega_1^3\ \omega_2^3 + g^2\ \omega_1\ \omega_2^5 - 8en^2\ \omega_1\ \omega_2^5 - 4n^4\ \omega_1\ \omega_2^5),$

$(-g^2\ \omega_1^3\ \omega_2 + 4en^2\ \omega_1^3\ \omega_2 + 2n^4\ \omega_1^3\ \omega_2 + g^2\ \omega_1\ \omega_2^3 - 4en^2\ \omega_1\ \omega_2^3 - 2n^4\ \omega_1\ \omega_2^3 -$

$2n^2\ \omega_1^3\ \omega_2^3 + 2n^2\ \omega_1\ \omega_2^5)\ /\ (g^2\ \omega_1^5\ \omega_2 - 8en^2\ \omega_1^5\ \omega_2 - 4n^4\ \omega_1^5\ \omega_2 - 2g^2\ \omega_1^3\ \omega_2^3 +$

$16en^2\ \omega_1^3\ \omega_2^3 + 8n^4\ \omega_1^3\ \omega_2^3 + g^2\ \omega_1\ \omega_2^5 - 8en^2\ \omega_1\ \omega_2^5 - 4n^4\ \omega_1\ \omega_2^5),$

$-((-2e - n^2 - \omega_2^2)(2n\ \omega_1^3\ \omega_2 - 2n\ \omega_1\ \omega_2^3))\ /\ (g^2\ \omega_1^5\ \omega_2 - 8en^2\ \omega_1^5\ \omega_2 - 4n^4\ \omega_1^5\ \omega_2 -$

$2g^2\ \omega_1^3\ \omega_2^3 + 16en^2\ \omega_1^3\ \omega_2^3 + 8n^4\ \omega_1^3\ \omega_2^3 + g^2\ \omega_1\ \omega_2^5 - 8en^2\ \omega_1\ \omega_2^5 - 4n^4\ \omega_1\ \omega_2^5),\ 0\},$

$\{-((-2e - n^2 - \omega_1^2)(g\ \omega_1^3\ \omega_2 - g\ \omega_1\ \omega_2^3))\ /\ (g^2\ \omega_1^5\ \omega_2 - 8en^2\ \omega_1^5\ \omega_2 - 4n^4\ \omega_1^5\ \omega_2 -$

$2g^2\ \omega_1^3\ \omega_2^3 + 16en^2\ \omega_1^3\ \omega_2^3 + 8n^4\ \omega_1^3\ \omega_2^3 + g^2\ \omega_1\ \omega_2^5 - 8en^2\ \omega_1\ \omega_2^5 - 4n^4\ \omega_1\ \omega_2^5),$

$(g^2\ \omega_1^3\ \omega_2 - 4en^2\ \omega_1^3\ \omega_2 - 2n^4\ \omega_1^3\ \omega_2 + 2n^2\ \omega_1^5\ \omega_2 - g^2\ \omega_1\ \omega_2^3 + 4en^2\ \omega_1\ \omega_2^3 +$

$2n^4\ \omega_1\ \omega_2^3 - 2n^2\ \omega_1^3\ \omega_2^3)\ /\ (g^2\ \omega_1^5\ \omega_2 - 8en^2\ \omega_1^5\ \omega_2 - 4n^4\ \omega_1^5\ \omega_2 - 2g^2\ \omega_1^3\ \omega_2^3 +$

$16en^2\ \omega_1^3\ \omega_2^3 + 8n^4\ \omega_1^3\ \omega_2^3 + g^2\ \omega_1\ \omega_2^5 - 8en^2\ \omega_1\ \omega_2^5 - 4n^4\ \omega_1\ \omega_2^5),$

$((-2e - n^2 - \omega_1^2)(2n\ \omega_1^3\ \omega_2 - 2n\ \omega_1\ \omega_2^3))\ /\ (g^2\ \omega_1^5\ \omega_2 - 8en^2\ \omega_1^5\ \omega_2 - 4n^4\ \omega_1^5\ \omega_2 -$

$2g^2\ \omega_1^3\ \omega_2^3 + 16en^2\ \omega_1^3\ \omega_2^3 + 8n^4\ \omega_1^3\ \omega_2^3 + g^2\ \omega_1\ \omega_2^5 - 8en^2\ \omega_1\ \omega_2^5 - 4n^4\ \omega_1\ \omega_2^5),\ 0\},$

$\{(8e^2n\ \omega_1^2\ \omega_2 + g^2n\ \omega_1^2\ \omega_2 - 2n^5\ \omega_1^2\ \omega_2 - 8e^2n\ \omega_2^3 - g^2n\ \omega_2^3 + 2n^5\ \omega_2^3 + 4en\ \omega_1^2\ \omega_2^3 +$

$2n^3\ \omega_1^2\ \omega_2^3 - 4en\ \omega_2^5 - 2n^3\ \omega_2^5)\ /\ (g^2\ \omega_1^5\ \omega_2 - 8en^2\ \omega_1^5\ \omega_2 - 4n^4\ \omega_1^5\ \omega_2 - 2g^2\ \omega_1^3\ \omega_2^3 +$

$16en^2\ \omega_1^3\ \omega_2^3 + 8n^4\ \omega_1^3\ \omega_2^3 + g^2\ \omega_1\ \omega_2^5 - 8en^2\ \omega_1\ \omega_2^5 - 4n^4\ \omega_1\ \omega_2^5),$

$(2egn\ \omega_1^2\ \omega_2 + g\,n^3\ \omega_1^2\ \omega_2 - 2egn\ \omega_2^3 - g\,n^3\ \omega_2^3 + gn\ \omega_1^2\ \omega_2^3 - gn\ \omega_2^5)\ /$

$(g^2\ \omega_1^5\ \omega_2 - 8en^2\ \omega_1^5\ \omega_2 - 4n^4\ \omega_1^5\ \omega_2 - 2g^2\ \omega_1^3\ \omega_2^3 +$

$16en^2\ \omega_1^3\ \omega_2^3 + 8n^4\ \omega_1^3\ \omega_2^3 + g^2\ \omega_1\ \omega_2^5 - 8en^2\ \omega_1\ \omega_2^5 - 4n^4\ \omega_1\ \omega_2^5),$

$(-2eg\ \omega_1^2\ \omega_2 - gn^2\ \omega_1^2\ \omega_2 + 2eg\ \omega_2^3 + gn^2\ \omega_2^3 - g\ \omega_1^2\ \omega_2^3 + g\ \omega_2^5)\ /$

$(g^2\ \omega_1^5\ \omega_2 - 8en^2\ \omega_1^5\ \omega_2 - 4n^4\ \omega_1^5\ \omega_2 - 2g^2\ \omega_1^3\ \omega_2^3 +$

$16en^2\ \omega_1^3\ \omega_2^3 + 8n^4\ \omega_1^3\ \omega_2^3 + g^2\ \omega_1\ \omega_2^5 - 8en^2\ \omega_1\ \omega_2^5 - 4n^4\ \omega_1\ \omega_2^5),$

$(-g^2\,\omega_1^2\,\omega_2 + 8en^2\,\omega_1^2\,\omega_2 + 4n^4\,\omega_1^2\,\omega_2 + g^2\,\omega_2^3 - 8en^2\,\omega_2^3 - 4n^4\,\omega_2^3)$ /

$(g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2 - 2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5)\}\}$,

$\{(-8e^2n\,\omega_1^3 - g^2n\,\omega_1^3 + 2n^5\,\omega_1^3 - 4en\,\omega_1^5 - 2n^3\,\omega_1^5 + 8e^2n\,\omega_1\,\omega_2^2 + g^2n\,\omega_1\,\omega_2^2 -$

$2n^5\,\omega_1\,\omega_2^2 + 4en\,\omega_1^3\,\omega_2^2 + 2n^3\,\omega_1^3\,\omega_2^2) / (g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2 -$

$2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5)$,

$(-2egn\,\omega_1^3 - gn^3\,\omega_1^3 - gn\,\omega_1^5 + 2egn\,\omega_1\,\omega_2^2 + gn^3\,\omega_1\,\omega_2^2 + gn\,\omega_1^3\,\omega_2^2)$ /

$(g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2 - 2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5)\}\}$,

$(2eg\,\omega_1^3 + gn^2\,\omega_1^3 + g\,\omega_1^5 - 2eg\,\omega_1\,\omega_2^2 - gn^2\,\omega_1\,\omega_2^2 - g\,\omega_1^3\,\omega_2^2) / (g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^3\,\omega_2 -$

$4n^4\,\omega_1^5\,\omega_2 - 2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5)$,

$(g^2\,\omega_1^3 - 8en^2\,\omega_1^3 - 4n^4\,\omega_1^3 - g^2\,\omega_1\,\omega_2^2 + 8en^2\,\omega_1\,\omega_2^2 + 4n^4\,\omega_1\,\omega_2^2)$ /

$(g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2 - 2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5)\}\}$,

Simplify

TRCHESSH2C = cinv.M.c

$$\left\{\left\{-\frac{g\left(2e+n^2+\omega_2^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\left(\omega_1^2-\omega_2^2\right)}, \frac{-g^2+4en^2+2n^4-2n^2\omega_2^2}{\left(g^2-4\left(2en^2+n^4\right)\right)\left(\omega_1^2-\omega_2^2\right)}, \frac{2n\left(2e+n^2+\omega_2^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\left(\omega_1^2-\omega_2^2\right)}, 0\right\}, \left\{\frac{g\left(2e+n^2+\omega_1^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\left(\omega_1^2-\omega_2^2\right)}, \frac{g^2-2\left(2en^2+n^4\right)+2n^2\omega_1^2}{\left(g^2-4\left(2en^2+n^4\right)\right)\left(\omega_1^2-\omega_2^2\right)}, -\frac{2n\left(2e+n^2+\omega_1^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\left(\omega_1^2-\omega_2^2\right)}, 0\right\},\right.$$

$$\left\{\frac{n\left(8e^2+g^2-2n^4+2\left(2e+n^2\right)\omega_2^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\omega_1\left(\omega_1^2-\omega_2^2\right)}, \frac{gn\left(2e+n^2+\omega_2^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\omega_1\left(\omega_1^2-\omega_2^2\right)}, -\frac{g\left(2e+n^2+\omega_2^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\omega_1\left(\omega_1^2-\omega_2^2\right)}, \frac{1}{-\omega_1^3+\omega_1\omega_2^2}\right\},$$

$$\left.\left\{\frac{n\left(-8e^2-g^2+2n^4-2\left(2e+n^2\right)\omega_1^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\omega_2\left(\omega_1^2-\omega_2^2\right)}, \frac{gn\left(2e+n^2+\omega_1^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\omega_2\left(\omega_1^2-\omega_2^2\right)}, \frac{g\left(2e+n^2+\omega_1^2\right)}{\left(g^2-4\left(2en^2+n^4\right)\right)\omega_2\left(\omega_1^2-\omega_2^2\right)}, \frac{1}{\omega_1^2\omega_2-\omega_2^3}\right\}\right\}.$$

$\{\{0, n, 1, 0\}, \{-n, 0, 0, 1\}, \{2e, g, 0, n\}, \{g, 2f, -n, 0\}\}$.

$\{\{g, g, 2n\omega_1, 2n\omega_2\}, \{-2e - n^2 - \omega_1^2, -2e - n^2 - \omega_2^2, 0, 0\},$

$\{2en + n^3 - n\,\omega_1^2, 2en + n^3 - n\,\omega_2^2, g\omega_1, g\omega_2\},$

$\{gn, gn, -2e\omega_1 + n^2\omega_1 - \omega_1^3, -2e\omega_2 + n^2\omega_2 - \omega_2^3\}$

$= \{\{((2en + n^3 - n\,\omega_1^2)(-2e - n^2 - \omega_2^2)(g\,\omega_1^3\,\omega^2 - g\omega_1\,\omega_2^3)) /$

$(g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2 - 2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5) + (-2e - n^2 - \omega_1^2)$

$((n(-2e - n^2 - \omega_2^2)(g\,\omega_1^3\,\omega_2 - g\omega_1\,\omega_2^3)) / (g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2$

$- 2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5) -$

$(g(-2e - n^2 - \omega_2^2)(2n\,\omega_1^3\,\omega_2 - 2n\omega_1\,\omega_2^3)) / (g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2$

$- 2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5)) +$

$gn\,(-(n(-2e - n^2 - \omega_2^2)(2n\,\omega_1^3\,\omega_2 - 2n\omega_1\,\omega_2^3)) / (g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2$

$- 2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5) +$

$(-g^2\,\omega_1^3\,\omega_2 + 4en^2\,\omega_1^5\,\omega_2 + 2n^4\,\omega_1^3\,\omega_2 + g^2\,\omega_1\,\omega_2^3 - 4en^2\,\omega_1\,\omega_2^3 - 2n^4\,\omega_1\,\omega_2^3 -$

$2n^2\,\omega_1^3\,\omega_2^3 + 2n^2\,\omega_1\,\omega_2^5) / (g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2 - 2g^2\,\omega_1^3\,\omega_2^3 +$

$16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_1^5)) +$

$g(-(2e(-2e - n^2 - \omega_2^2)(2n\,\omega_1^3\,\omega_2 - 2n\omega_1\,\omega_2^3)) / (g^2\,\omega_1^5\,\omega_2 - 8en^2\,\omega_1^5\,\omega_2 - 4n^4\,\omega_1^5\,\omega_2$

$-2g^2\,\omega_1^3\,\omega_2^3 + 16en^2\,\omega_1^3\,\omega_2^3 + 8n^4\,\omega_1^3\,\omega_2^3 + g^2\,\omega_1\,\omega_2^5 - 8en^2\,\omega_1\,\omega_2^5 - 4n^4\,\omega_1\,\omega_2^5) -$

$(n(-g^2\omega_1^3\omega_2 + 4en^2\omega_1^3\omega_2^3\omega_2 + 2n^4\omega_1^3\omega_2 + g^2\omega_1\omega_2^3 - 4en^2\omega_1\omega_2^3$

$- 2n^4\omega_1\omega_2^3 -$

$2n^2\omega_1^3\omega_2^3 + 2n^2\omega_1\omega_2^5)) / (g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2 - 4n^4\omega_1^5\omega_2$

$-$

$2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3 + 8n^4\omega_1^3\omega_2^3 + g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5$

$- 4n^4\omega_1\omega_2^5)),$

$((-2e - n^2 - \omega_2^2)(2en + n^3 - n\omega_2^2)(g\omega_1^3\omega_2 - g\omega_1\omega_2^3)) /$

$(g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2 - 4n^4\omega_1^5\omega_2 - 2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3$

$+$

$8n^4\omega_1^3\omega_2^3 + g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5 - 4n^4\omega_1\omega_2^5) + (-2e - n^2 -$

$\omega_2^2)$

$((n(-2e - n^2 - \omega_2^2)(g\omega_1^3\omega_2 - g\omega_1\omega_2^3)) / (g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2 -$

$4n^4\omega_1^5\omega_2 -$

$2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3 + 8n^4\omega_1^3\omega_2^3 + g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5$

$- 4n^4\omega_1\omega_2^5) -$

$(g(-2e - n^2 - \omega_2^2)(2n\omega_1^3\omega_2 - 2n\omega_1\omega_2^3)) / (g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2$

$- 4n^4\omega_1^5\omega_2$

$- 2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3 + 8n^4\omega_1^3\omega_2^3 + g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5$

$- 4n^4\omega_1\omega_2^5)) +$

$(-2egn\omega_1^3 - gn^3\omega_1^3 - gn\omega_1^5 + 2egn\omega_1\omega_2^2 + gn^3\omega_1\omega_2^2 + gn\omega_1^3$

$\omega_2^2) /$

$(g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2 - 4n^4\omega_1^5\omega_2 - 2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3$

$+ 8n^4\omega_1^3\omega_2^3 +$

$g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5 - 4n^4\omega_1\omega_2^5)) + 2n\omega_2((g(g^2\omega_1^3 - 8en^2\omega_1^3$

$- 4n^4\omega_1^3 -$

$g^2\omega_1\omega_2^2 + 8en^2\omega_1\omega_2^2 + 4n^4\omega_1\omega_2^2)) / (g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2 -$

$4n^4\omega_1^5\omega_2$

$-2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3 + 8n^4\omega_1^3\omega_2^3 + g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5$

$- 4n^4\omega_1\omega_2^5) +$

$(2e(2eg\omega_1^3 + gn^2\omega_1^3 + g\omega_1^5 - 2eg\omega_1\omega_2^2 - gn^2\omega_1\omega_2^2 - g\omega_1^3$

$\omega_2^2) /$

$(g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2 - 4n^4\omega_1^5\omega_2 - 2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3$

$+$

$8n^4\omega_1^3\omega_2^3 + g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5 - 4n^4\omega_1\omega_2^5) -$

$(n(-2egn\omega_1^3 - gn^3\omega_1^3 - gn\omega_1^5 + 2egn\omega_1\omega_2^2 + gn^3\omega_1\omega_2^2 + gn$

$\omega_1^3\omega_2^2)) /$

$(g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2 - 4n^4\omega_1^5\omega_2 - 2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3$

$+ 8n^4\omega_1^3\omega_2^3 +$

$g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5 - 4n^4\omega_1\omega_2^5) + g\omega_2(-(n(g^2\omega_1^3 - 8en^2\omega_1^3$

$- 4n_4\omega_1^3 -$

$g^2\omega_1\omega_2^2 + 8en^2\omega_1\omega_2^2 + 4n^4\omega_1\omega_2^2)) / (g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2 -$

$4n^4\omega_1^5\omega_2$

$-2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3 + 8n^4\omega_1^3\omega_2^3 + g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5$

$- 4n^4\omega_1\omega_2^5) +$

$(-8e^2n\omega_1^3 - g^2n\omega_1^3 + 2n^5\omega_1^3 - 4en\omega_1^5 - 2n^3\omega_1^5 + 8e^2n\omega_1\omega_2^2 +$

$g^2n\omega_1\omega_2^2 -$

$2n^5\omega_1\omega_2^2 + 4en\omega_1^3\omega_2^2 + 2n^3\omega_1^3\omega_2^2) / (g^2\omega_1^5\omega_2 - 8en^2\omega_1^5\omega_2 -$

$4n^4\omega_1^5\omega_2 -$

$2g^2\omega_1^3\omega_2^3 + 16en^2\omega_1^3\omega_2^3 + 8n^4\omega_1^3\omega_2^3 + g^2\omega_1\omega_2^5 - 8en^2\omega_1\omega_2^5$

$- 4n^4\omega_1\omega_2^5))\}\}$

Simplify

$$=\left\{\begin{array}{l}\{0,0,\omega_1,0\},\{0,0,0,\omega_2\}\\ \left\{\dfrac{-g^2+2fn^2+n^4+2e\left(2f+n^2\right)+\omega_1^2\left(2(e+f-n^2)+\omega_2^2\right)}{\omega_1^3-\omega_1\omega_2^2},\ \dfrac{-g^2+2fn^2+n^4+2e\left(2f+n^2\right)+2\left(e+f-n^2\right)\omega_2^2+\omega_2^4}{\omega_1^3-\omega_1\omega_2^2},0,0\right\},\\ \left\{\dfrac{-g^2+2fn^2+n^4+2e\left(2f+n^2\right)+2\left(e+f-n^2\right)\omega_1^2+\omega_1^4}{\omega_2\left(\omega_1^2-\omega_2^2\right)},\ \dfrac{-g^2+2fn^2+n^4+2e\left(2f+n^2\right)+2\left(e+f-n^2\right)+\omega_1^2)\omega_2^2}{\omega_2\left(\omega_1^2-\omega_2^2\right)},0,0\right\}\end{array}\right\}$$

$d\omega_1 = 2\omega_1$

$$\left(2e^2-g^2+3fn^2+e\left(6f+n^2\right)+\left(5e+3f-2n^2\right)\omega_1^2+2\omega_1^4\right)$$

$$n^4=-\left(4ef-g^2+2en^2+2fn^2-\left(-2e-2f+2n^2\right)\omega^2+\omega^4\right)$$

$$\left(\omega_1^2+\omega_2^2\right)=-2\left(e+f-n^2\right),\left(\omega_1^2\omega_2^2\right)=4ef-g^2+2en^2+2fn^2+n^4$$

$$4ef-g^2+2en^2+2fn^2+n^4+2\left(e+f-n^2\right)\omega^2+\omega^4=0$$

$$\left(e+f-n^2\right)\omega^2=-\frac{\left(4ef-g^2+2en^2+2fn^2+n^4+\omega^4\right)}{2}$$

Simplify

$$\left\{\begin{array}{l}\{0,0,\omega_1,0\},\ \{0,0,0,\omega_2\},\\ \left\{\begin{array}{l}\frac{1}{\omega_1^3-\omega_1\omega_2^2}\\ \left(-g^2+2fn^2+n^4+2e\left(2f+n^2\right)+2\left(e+f-n^2\right)\omega_1^2+4ef-g^2+2en^2+2fn^2+n^4\right),\\ 0,0,0\end{array}\right\},\\ \left\{-0,-\frac{1}{\omega_2\left(\omega_1^2-\omega_2^2\right)}\left(\begin{array}{l}-g^2+2fn^2+n^4+2e\left(2f+n^2\right)+\\ 2\left(e+f-n^2\right)\omega_1^2+4ef-g^2+2en^2+2fn^2+n^4\end{array}\right),0,0,0\right\},\\ -0,-\frac{1}{\omega_2\left(\omega_1^2-\omega_2^2\right)}\left(\begin{array}{l}-g^2+2fn^2+n^4+2e\left(2f+n^2\right)+\\ 2\left(e+f-n^2\right)\omega_2^2+4ef-g^2+2en^2+2fn^2+n^4\end{array}\right),0,0,\end{array}\right.$$

Simplify

cinvMc =

$$\left\{\{0,0,\omega_1,0\},\{0,0,0,\omega_2\},\left\{\frac{\omega_1}{\omega_1^2-\omega_2^2},0,0,0,\right\}\left\{0,\frac{\omega_2}{-\omega_1^2+\omega_2^2},0,0\right\}\right\}$$

$$H_2=\frac{1}{2}\{P_1,P_2,Q_1,Q_2\}.cinvMc \qquad .\{P_1,P_2,Q_1,Q_2\}$$

$$=\frac{1}{2}\left(P_1Q_1\omega_1+P_2Q_2\omega_2+\frac{P_1Q_1\omega_1}{\omega_1^2-\omega_2^2}+\frac{P_2Q_2\omega_2}{-\omega_1^2+\omega_2^2}\right)$$

Expand

$$\frac{P_1Q_1\omega_1}{2\left(\omega_1^2-\omega_2^2\right)}+\frac{P_1Q_1\omega_1^3}{2\left(\omega_1^2-\omega_2^2\right)}-\frac{P_2Q_2\omega_2}{2\left(\omega_1^2-\omega_2^2\right)}+\frac{P_2Q_2\omega_1^2\omega_2}{2\left(\omega_1^2-\omega_2^2\right)}-\frac{P_1Q_1\omega_1\omega_2^2}{2\left(\omega_1^2-\omega_2^2\right)}-\frac{P_2Q_2\omega_2^3}{2\left(\omega_1^2-\omega_2^2\right)}$$

Simplify

$$\frac{P_2Q_2\omega_2\left(-1+\omega_1^2-\omega_2^2\right)+P_1Q_1\omega_1\left(1+\omega_1^2-\omega_2^2\right)}{2\left(\omega_1^2-\omega_2^2\right)}$$

$$=\frac{1}{2}\{P_1,P_2,Q_1,Q_2\}.$$

$$\left\{\begin{array}{l}\{0,0,\omega_1,0\},\{0,0,0,\omega_2\},\\ \left\{\frac{1}{\omega_1^3-\omega_1\omega_2^2}\left(\begin{array}{l}4ef-2g^2+2en^2+4fn^2+2n^4+2e\left(2f+n^2\right)\\ +2\left(e+f-n^2\right)\omega_1^2\end{array}\right),0,0,0\right\},\\ \left\{0,-\frac{1}{\omega_2\left(\omega_1^2-\omega_2^2\right)}\left(\begin{array}{l}4ef-2g^2+2en^2+4fn^2+2n^4+2e\left(2f+n^2\right)\\ +2\left(e+f-n^2\right)\omega_2^2\end{array}\right),0,0\right\}\end{array}\right\}$$

$$\{P_1,P_2,Q_1,Q_2\}$$

$$=\frac{1}{2}$$

$$\left\{\begin{array}{l}P_1Q_1\omega_1+P_2Q_2\omega_2-\frac{1}{\omega_2\left(\omega_1^2-\omega_2^2\right)}\\ \left(P_2Q_2\left(4ef-2g^2+2en^2+4fn^2+2n^4+2e\left(2f+n^2\right)+2\left(e+f-n^2\right)\omega_2^2\right)\right)+\\ \frac{1}{\omega_1^3-\omega_1\omega_2^2}\\ \left(P_1Q_1\left(4ef-2g^2+2en^2+4fn^2+2n^4+2e\left(2f+n^2\right)+2\left(e+f-n^2\right)\omega_2^2\right)\right)\end{array}\right\}$$

Simplify

$$\frac{1}{2\omega_1\omega_2\left(\omega_1^2-\omega_2^2\right)}$$

$$\left(\begin{array}{l}-P_2Q_2\omega_1\left(2\left(-g^2+2fn^2+n^4+2e\left(2f+n^2\right)\right)+\left(2\left(e+f-n^2\right)-\omega_1^2\right)\omega_2^2+\omega_2^4\right)+\\ P_1Q_1\omega_2\left(2\left(-g^2+2fn^2+n^4+2e\left(2f+n^2\right)\right)+\omega_1^4+\omega_1^2\left(2\left(e+f-n^2\right)-\omega_2^2\right)\right)\end{array}\right)$$

since

$$\left(2\left(-g^2+2fn^2+n^4+2e\left(2f+n^2\right)\right)+2\left(e+f-n^2\right)\omega_2^2-\omega_1^2\omega_2^2+\omega_2^4\right)=-2\omega_1^2\omega_2^2$$

so above

$$H_2=\frac{1}{2\omega_1\omega_2\left(\omega_1^2-\omega_2^2\right)}\left(-P_2Q_2\omega_1\left(-2\omega_1^2\omega_2^2\right)+P_1Q_1\omega_2\left(-2\omega_1^2\omega_2^2\right)\right)$$

$$=\frac{2P_2Q_2\omega_1^3\omega_2^2-2P_1Q_1\omega_1^2\omega_2^3}{2\omega_1\omega_2\left(\omega_1^2-\omega_2^2\right)}$$

Simplify

$$=\frac{\omega_1\omega_2\left(P_2Q_2\omega_1-P_1Q_1\omega_2\right)}{\omega_1^2-\omega_2^2}$$

Expand

$$=\frac{P_2Q_2\omega_1^2\omega_2}{\omega_1^2-\omega_2^2}-\frac{P_1Q_1\omega_1\omega_2^2}{\omega_1^2-\omega_2^2}$$

$$=\frac{P_2Q_2\omega_1^2\omega_2}{\omega_1^2-\omega_2^2}-\frac{P_1Q_1\omega_1\omega_2^2}{\omega_1^2-\omega_2^2}$$

$$P_1=\left(1-\frac{\omega_1}{\omega_2}\right)\alpha_1;Q_1=\left(1+\frac{\omega_1}{\omega_2}\right)\beta_1;P_2=\left(1-\frac{\omega_2}{\omega_1}\right)\alpha_2;Q_2=\left(1+\frac{\omega_2}{\omega_1}\right)\beta_2;$$

Change the scale as above

then $H_2=\frac{P_2Q_2\omega_1^2\omega_2}{\omega_1^2-\omega_2^2}-\frac{P_1Q_1\omega_1\omega_2^2}{\omega_1^2-\omega_2^2}$

H_2 = Simplify

$= \alpha_1\beta_1\omega_1 + \alpha_2\beta_2\omega_2$

which is required normal form for given Hamiltonian

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