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Holder inequalities for a subclass of univalent functions involving Dziok-Srivastava operator

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Abstract

In this paper, we introduce a new subclasses of univalent functions defined in the open unit disc involving Dziok-Srivastava Operator . The results on modified Hadamard product ,Holder inequalities and closure properties under integral transforms are discussed.

Keywords: *Analytic, coefficient bounds,convolution properties,convex functions, starlike functions, Holder inequality, univalent .*

1 Introduction

Denote by $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are normalized by $f(0) = 0 = f'(0) - 1$ and univalent in $\mathcal{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha (0 \leq \alpha < 1)$ in $\mathcal{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha (0 \leq \alpha < 1)$ in $\mathcal{U}$ satisfying the analytical criteria $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ and $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$ respectively.It is of interest to note that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

For given two functions $f, g \in \mathcal{A}$ where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. The Hadamard product (or convolution) $f(z) * g(z)$ is defined $f(z) * g(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathcal{U}$.

Also denote by $\mathcal{T}$ the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \;\; a_n \geq 0, \;\; z \in \mathcal{U} \tag{2}$$

studied extensively by Silverman[11].

For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$ the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \quad &:= \quad \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\; l, m \in N_0 \quad &:= \quad N \cup \{0\}; z \in U) \end{aligned} \tag{3}$$

where N denotes the set of all positive integers and $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1), & n \in N. \end{cases} \tag{4}$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial (for example see [5]).

Let $\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m) : \mathcal{A} \to \mathcal{A}$ be a linear operator defined by

$$\begin{aligned}\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m)f(z) &:= z\ {}_lF_m(\alpha_1, \alpha_2, \dots \alpha_l; \beta_1, \beta_2 \dots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Gamma_n\ a_n\ z^n \end{aligned} \tag{5}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \ \frac{1}{(n-1)!}, \tag{6}$$

(unless otherwise stated). For notational simplicity, we can use a shorter notation $\mathcal{H}_m^l[\alpha_1]$ for $\mathcal{H}(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m)$ in the sequel. The linear operator $\mathcal{H}_m^l[\alpha_1]$ is called Dziok-Srivastava operator (see [5]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [3], Ruscheweyh [10] and Owa-Srivastava [9]. Motivated by earlier works of Aouf et al.,[1] and Dziok and Raina[6] we define the following new subclass of $\mathcal{T}$ involving hypergeometric functions.

For $0 \leq \lambda \leq 1, 0 < \beta \leq 1, -1 \leq B < A \leq 1, 0 \leq \gamma \leq 1$,we let $\mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$ denote the subclass of $\mathcal{T}$ consisting of functions $f(z)$ of the form (2) satisfying the analytic condition.

$$\left| \frac{\frac{zF_\lambda'(z)}{F_\lambda(z)} - 1}{(B-A)\gamma\left[\frac{zF_\lambda'(z)}{F_\lambda(z)} - \alpha\right] - B\left[\frac{zF_\lambda'(z)}{F_\lambda(z)} - 1\right]} \right| < \beta, \qquad z \in U \tag{7}$$

where

$$\frac{zF_\lambda'(z)}{F_\lambda(z)} = \frac{z\mathcal{H}f'(z) + \lambda z^2 \mathcal{H}f''(z)}{(1-\lambda)\mathcal{H}f(z) + \lambda z \mathcal{H}f'(z)}, \qquad 0 \leq \lambda \leq 1 \tag{8}$$

and

$$\mathcal{H}f(z) = z + \sum_{n=2}^{\infty} a_n \Gamma_n z^n \tag{9}$$

where Γ_n is given by (6).

In our present investigation, we discuss some interesting properties of functions $f(z) \in \mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$ based on convolution. Further we discuss certain closure properties under integral transformation.

In the following theorem we obtain necessary and sufficient conditions for functions $f(z) \in \mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$.

Theorem 1.1. *A function $f(z)$ of the form (2) is in the class $\mathcal{HF}_\gamma^\lambda(\alpha, \beta, A, B)$ if and only if*

$$\sum_{n=2}^{\infty} C_n a_n \leq (1-\alpha)(B-A)\beta\gamma \tag{10}$$

where

$$C_n = (1 + n\lambda - \lambda)[(n-1)(1-\beta B) + \beta\gamma(B-A)(1-\alpha)]\Gamma_n \tag{11}$$

and Γ_n is defined by (6).

Proof. For $|z| = 1$, we have

$$\left| \frac{\frac{zF_\lambda'(z)}{F_\lambda(z)} - 1}{(B-A)\gamma\left[\frac{zF_\lambda'(z)}{F_\lambda(z)} - \alpha\right] - B\left[\frac{zF_\lambda'(z)}{F_\lambda(z)} - 1\right]} \right| < \beta, \qquad z \in \mathcal{U}.$$

It is suffices to show that

$$|zF_\lambda'(z) - F_\lambda(z)| - \beta\left|(B-A)\gamma\left[zF_\lambda'(z) - \alpha F_\lambda(z)\right] - B\left[zF_\lambda'(z) - F_\lambda(z)\right]\right| \tag{12}$$

$$= \left|\sum_{n=2}^{\infty}(1+n\lambda-\lambda)(n-1)a_n\Gamma_n z^{n-1}\right|$$
$$-\beta\left|(B-A)(1-\alpha)\gamma+\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(B-A)(n-\alpha)\gamma-B(n-1)]\Gamma_n a_n z^{n-1}]\right|$$
$$\leq \sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n a_n-\beta\gamma(B-A)(1-\alpha)$$
$$\leq 0, \text{by hypothesis.}$$

Thus by maximum modulus theorem $f \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha,\beta,A,B)$.

On the other hand suppose that

$$\left|\frac{\frac{zF_{\lambda}'(z)}{F_{\lambda}(z)}-1}{(B-A)\gamma\left[\frac{zF_{\lambda}'(z)}{F_{\lambda}(z)}-\alpha\right]-B\left[\frac{zF_{\lambda}'(z)}{F_{\lambda}(z)}-1\right]}\right|$$
$$= \left|\frac{\frac{\sum_{n=2}^{\infty}(1+n\lambda-\lambda)(n-1)a_n\Gamma_n z^n}{F_{\lambda}(z)}}{\frac{(B-A)\gamma[z(1-\alpha)+\sum_{n=2}^{\infty}(1+n\lambda-\lambda)(n-\alpha)a_n\Gamma_n z^n]-B[\sum_{n=2}^{\infty}(1+n\lambda-\lambda)(n-1)a_n\Gamma_n z^n]}{F_{\lambda}(z)}}\right| < \beta. \tag{13}$$

Since $\Re(z) < |z|$ for all z, we have

$$\Re\left(\frac{\sum_{n=2}^{\infty}(n-1)(1+n\lambda-\lambda)\Gamma_n a_n|z|^{n-1}}{(B-A)(1-\alpha)\gamma-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(B-A)(n-\alpha)\gamma-B(n-1)]\Gamma_n a_n|z|^{n-1}}\right) < \beta.$$

Choosing the value of z on the real axis so that $f'(z)$ is real and letting $z \to 1^-$, we obtain

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n a_n \leq \beta\gamma(B-A)(1-\alpha)$$

and hence the proof is complete. □

2 Convolution properties for functions in the class $\mathcal{HF}_{\gamma}^{\lambda}(\alpha,\beta,A,B)$.

In the following section, using the techniques of Schild and Silverman[12] we discuss some convolution properties for functions $f(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha,\beta,A,B)$.

Let the functions $f_j(z)$ $(j=1,2)$ be defined by

$$f_j(z) = z-\sum_{n=2}^{\infty}a_{n,j}z^n \quad (a_{n,j}\geq 0; j=1,2) \tag{14}$$

then the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is given by

$$(f_1 * f_2)(z) = z-\sum_{n=2}^{\infty}a_{n,1}a_{n,2}z^n. \tag{15}$$

Theorem 2.1. *Let the function $f_1(z)$ defined by (14) be in the class $\mathcal{HF}_{\gamma}^{\lambda}(\xi_1,\beta,A,B)$ and the function $f_2(z)$ defined by (14) be in the class $\mathcal{HF}_{\gamma}^{\lambda}(\xi_2,\beta,A,B)$. If the sequence $\{C_n\}$ is non-decreasing then $(f_1*f_2)(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha^*,\beta,A,B)$ where*

$$\alpha^* = 1-\frac{(1-\xi_1)(1-\xi_2)[1-\beta B+\beta\gamma(B-A)]\beta\gamma(B-A)}{(1+\lambda)\Lambda(\beta,\gamma,\xi_1,2)\Lambda(\beta,\gamma,\xi_2,2)\Gamma_2-[\beta\gamma(B-A)]^2(1-\xi_1)(1-\xi_2)}, \tag{16}$$

$$\Lambda(\beta,\gamma,\xi_1,2) = [(1-\beta B)+\beta\gamma(B-A)(2-\xi_1)]$$

and

$$\Lambda(\beta,\gamma,\xi_2,2) = [(1-\beta B)+\beta\gamma(B-A)(2-\xi_2)].$$

Proof. In view of Theorem1.1 it is enough to show that

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha^*)]}{\beta\gamma(B-A)(1-\alpha^*)}\Gamma_n a_{n,1}a_{n,2}\leq 1. \tag{17}$$

where α^* is defined by (16).

Since $f_1\in\mathcal{HF}_{\gamma}^{\lambda}(\xi_1,\beta,A,B)$ we have

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)\Lambda(\beta,\gamma,\xi_1,n)}{\beta\gamma(B-A)(1-\xi_1)}\Gamma_n a_{n,1}\leq 1 \tag{18}$$

and for $f_2\in\mathcal{HF}_{\gamma}^{\lambda}(\xi_2,\beta,A,B)$ we have

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)\Lambda(\beta,\gamma,\xi_2,n)}{\beta\gamma(B-A)(1-\xi_2)}\Gamma_n a_{n,2}\leq 1 \tag{19}$$

where

$$\Lambda(\beta,\gamma,\xi_1,n)=[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_1)]$$

and

$$\Lambda(\beta,\gamma,\xi_2,n)=[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_2)].$$

On the other hand, under the hypothesis and by the Cauchy's-Schwarz inequality that

$$\sum_{n=2}^{\infty}\frac{[\Lambda(\beta,\gamma,\xi_1,n)]^{1/2}[\Lambda(\beta,\gamma,\xi_1,n)]^{1/2}}{\sqrt{(1-\xi_1)(1-\xi_2)}}(1+n\lambda-\lambda)\Gamma_n\sqrt{a_{n,1}a_{n,2}}\ \leq 1. \tag{20}$$

From (18) and (19), it follows that

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)^2\Lambda(\beta,\gamma,\xi_1,n)\Gamma_n\Lambda(\beta,\gamma,\xi_2,n)\Gamma_n}{[\beta\gamma(B-A)]^2(1-\xi_1)(1-\xi_2)}a_{n,1}a_{n,2}\leq 1. \tag{21}$$

Now we have to find largest α^* such that,

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha^*)]}{\beta\gamma(B-A)(1-\alpha^*)}\Gamma_n a_{n,1}a_{n,2}\\ \leq\ &\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[\Lambda(\beta,\gamma,\xi_1,n)]^{1/2}[\Lambda(\beta,\gamma,\xi_2,n)]^{1/2}\Gamma_n}{[\beta\gamma(B-A)]\sqrt{(1-\xi_1)(1-\xi_2)}}\sqrt{a_{n,1}a_{n,2}}.\end{aligned}$$

Or, equivalently that

$$\sqrt{a_{n,1}a_{n,2}}\quad\leq\quad\frac{1-\alpha^*}{\sqrt{(1-\xi_1)(1-\xi_2)}}\frac{[\Lambda(\beta,\gamma,\xi_1,n)]^{1/2}[\Lambda(\beta,\gamma,\xi_2,n)]^{1/2}}{[\Lambda(\beta,\gamma,\alpha^*,n)]},\ \ (n\geq 2)$$

where $\Lambda(\beta,\gamma,\alpha^*,n)=(1-B\beta)(n-1)+\beta\gamma(B-A)(n-\alpha^*)$.
In view of (20) it is sufficient to find largest α^* such that

$$\frac{[\beta\gamma(B-A)]\sqrt{(1-\xi_1)(1-\xi_2)}}{(1+n\lambda-\lambda)[\Lambda(\beta,\gamma,\xi_1,n)]^{1/2}[\Lambda(\beta,\gamma,\xi_2,n)]^{1/2}\Gamma_n}\leq\frac{1-\alpha^*}{\sqrt{(1-\xi_1)(1-\xi_2)}}\frac{[\Lambda(\beta,\gamma,\xi_1,n)]^{1/2}[\Lambda(\beta,\gamma,\xi_2,n)]^{1/2}}{[\Lambda(\beta,\gamma,\alpha^*,n)]}$$

which yields

$$\begin{aligned}&\alpha^*[(1+n\lambda-\lambda)\Lambda(\beta,\gamma,\xi_1,n)\Lambda(\beta,\gamma,\xi_2,n)\Gamma_n-[\beta\gamma(B-A)]^2(1-\xi_1)(1-\xi_2)]\\ &\quad\leq(1+n\lambda-\lambda)\Lambda(\beta,\gamma,\xi_1,n)\Lambda(\beta,\gamma,\xi_2,n)\Gamma_n-n[\beta\gamma(B-A)]^2(1-\xi_1)(1-\xi_2)\\ &\quad\ -\beta\gamma(n-1)(1-\beta B)(B-A)(1-\xi_1)(1-\xi_2).\end{aligned}$$

That is ,

$$\alpha^*\leq 1-\frac{[n[\beta\gamma(B-A)]^2-\beta\gamma(1-\beta B)(B-A)+[\beta\gamma(B-A)]^2](1-\xi_1)(1-\xi_2)}{(1+n\lambda-\lambda)\Lambda(\beta,\gamma,\xi_1,n)\Lambda(\beta,\gamma,\xi_2,n)\Gamma_n-[\beta\gamma(B-A)]^2(1-\xi_1)(1-\xi_2)}.$$

Let

$$\Phi(n)=\frac{[n[\beta\gamma(B-A)]^2-(n-1)\beta\gamma(1-\beta B)(B-A)+[\beta\gamma(B-A)]^2](1-\alpha_1)(1-\alpha_2)}{(1+n\lambda-\lambda)\Lambda(\beta,\gamma,\xi_1,n)\Lambda(\beta,\gamma,\xi_2,n)\Gamma_n-[\beta\gamma(B-A)]^2(1-\xi_1)(1-\xi_2)}.$$

Since $\Phi(n)$ is non decreasing function of $n(n\geq 2)$,then we have $\alpha^*\leq 1-\Phi(2)$.
That is ,

$$\alpha^*\leq 1-\frac{(1-\xi_1)(1-\xi_2)[1-\beta B+\beta\gamma(B-A)]\beta\gamma(B-A)}{(1+\lambda)\Lambda(\beta,\gamma,\xi_1,2)\Lambda(\beta,\gamma,\alpha_2,2)\Gamma_2-[\beta\gamma(B-A)]^2(1-\xi_1)(1-\xi_2)}$$

and hence the proof is complete. □

Remark 2.2. *When $\alpha_1=\alpha=\alpha_2$, we have*

$$\alpha^*\leq 1-\frac{(1-\alpha)^2[1-\beta B+\beta\gamma(B-A)]\beta\gamma(B-A)}{(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha)]^2\Gamma_2-[\beta\gamma(B-A)]^2(1-\alpha)^2} \tag{22}$$

Theorem 2.3. *Let the function $f_j(z)(j=1,2)$ defined by (14) be in the class $\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$. If the sequence $\{C_n\}$ is non-decreasing. Then the function*

$$h(z)=z-\sum_{n=2}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n \tag{23}$$

belongs to the class $\mathcal{HF}_\gamma^\lambda(\delta,\beta,A,B)$ where

$$\delta=1-\frac{2(1-\alpha)^2[1-\beta B+\beta\gamma(B-A)]\beta\gamma(B-A)}{(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha)]^2\Gamma_2-2(1-\alpha)^2[\beta\gamma(B-A)]^2}.$$

Proof. By virtue of Theorem 1.1, it is sufficient prove that

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\delta)]\Gamma_n}{\beta\gamma(B-A)(1-\delta)}(a_{n,1}^2+a_{n.2}^2)\leq 1. \tag{24}$$

Since $f_j(z)(j=1,2)\in\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$ we have

$$\sum_{n=2}^{\infty}\left\{\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n}{\beta\gamma(B-A)(1-\alpha)}\right\}^2 a_{n,1}^2 \leq\sum_{n=2}^{\infty}\left\{\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n a_{n,1}}{\beta\gamma(B-A)(1-\alpha)}\right\}^2\leq 1 \tag{25}$$

and

$$\sum_{n=2}^{\infty}\left\{\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n}{\beta\gamma(B-A)(1-\alpha)}\right\}^2 a_{n,2}^2 \leq\sum_{n=2}^{\infty}\left\{\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n a_{n,2}}{\beta\gamma(B-A)(1-\alpha)}\right\}^2\leq 1. \tag{26}$$

It follows from (25) and (26) that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left\{\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n}{\beta\gamma(B-A)(1-\alpha)}\right\}^2(a_{n,1}^2+a_{n,2})\leq 1. \tag{27}$$

Therefore we need to find the largest δ, such that

$$\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\delta)]\Gamma_n}{\beta\gamma(B-A)(1-\delta)} \leq\frac{1}{2}\left[\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]\Gamma_n}{\beta\gamma(B-A)(1-\alpha)}\right]^2\quad(n\geq 2)$$

that is

$$\delta\leq 1-\frac{2(n-1)(1-\beta B)\beta\gamma(B-A)(1-\alpha)^2-2n[\beta\gamma(B-A)]^2(1-\alpha)^2+2[\beta\gamma(B-A)]^2(1-\alpha)^2}{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\delta)]^2\Gamma_n-2[\beta\gamma(B-A)]^2(1-\alpha)^2}.$$

Since

$$\Psi(n) = 1 - \frac{2(n-1)(1-\beta B)\beta\gamma(B-A)(1-\alpha)^2 - 2n[\beta\gamma(B-A)]^2(1-\alpha)^2 + 2[\beta\gamma(B-A)]^2(1-\alpha)^2}{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\delta)]^2\Gamma_n - 2[\beta\gamma(B-A)]^2(1-\alpha)^2}$$

is an increasing function of $n, (n \geq 2)$, we readily have

$$\delta \leq \Psi(2) = 1 - \frac{2(1-\alpha)^2[1-\beta B+\beta\gamma(B-A)]\beta\gamma(B-A)}{(1+\lambda)[1-\beta B+\beta\gamma(B-A)(2-\alpha)]^2\Gamma_2 - 2(1-\alpha)^2[\beta\gamma(B-A)]^2}, \tag{28}$$

which completes the proof. □

3 Hölder's inequality

Recently, Nishiwaki, Owa and Srivastava [8] have studied some results of Holder-type inequalities for a subclass of uniformly starlike functions. Now, we recall the generalization of the convolution due to Choi, Kim and Owa [4] as given below,

$$H_m(z) = z - \sum_{n=2}^{\infty}\left(\prod_{j=1}^{m} a_{n,j}^{p_j}\right) z^n \quad (p_j > 0, j = 1, 2, ..., m). \tag{29}$$

Further for functions $f_j(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)(j = 1, 2, ..., m)$ given by (15), the familiar Holder inequality assumes the following form

$$\sum_{n=2}^{\infty}\left(\prod_{j=1}^{m} a_{n,j}\right) \leq \prod_{j=1}^{m}\left(\sum_{n=2}^{\infty} a_{n,j}^{p_j}\right)^{\frac{1}{p_j}} \qquad (p_j > 1, j = 1, 2, ..., m; \sum_{j=i}^{m}\frac{1}{p_j} \geq 1). \tag{30}$$

Theorem 3.1. **(Holder's Inequality:)** *If* $f_j(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\xi_j, \beta, A, B), -1 \leq B < A \leq 1, 0 < \beta \leq 1, 0 \leq \lambda \leq 1, j = 1, 2, ..., m$ *then* $\mathcal{H}_m(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\xi, \beta, A, B)$ *with*

$$\xi \leq 1 - \frac{\prod_{j=i}^{m}(1-\xi_j)^{p_j} - [(1-\beta B)+\beta\gamma(B-A)][\beta\gamma(B-A)]^s}{(1+\lambda)^{s-1}\Gamma_n^{s-1}\prod_{j=i}^{m}[1-\beta B+\beta\gamma(B-A)(2-\xi_j)]^{p_j} - [\beta\gamma(B-A)]^s\prod_{j=i}^{m}(1-\xi_j)^{p_j}}.$$

where $(S = \sum_{j=i}^{m} p_j \geq 1; p_j \geq \frac{1}{q_j}(j = 1, 2, ..., m), q_j > 1(j = 1, 2, ..., m); \sum_{j=i}^{m}\frac{1}{q_j} \geq 1)$.

Proof. Let $f_j(z) \in \mathcal{HF}_{\gamma}^{\lambda}(\xi_j, \beta, A, B)(j = 1, 2, ..., m)$. Then we have

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]}{\beta\gamma(B-A)(1-\xi_j)}\Gamma_n a_{n,j} \leq 1$$

which in turn, implies that

$$\left(\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]}{\beta\gamma(B-A)(1-\xi_j)}\Gamma_n a_{n,j}\right)^{\frac{1}{q_j}} \leq 1$$

$$(q_j > 1 \quad (j = 1, 2, ..., m); \sum_{j=i}^{m}\frac{1}{q_j} = 1).$$

Applying the Holder inequality (30), we arrive at the following inequality

$$\sum_{n=2}^{\infty}\left[\sum_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]}{\beta\gamma(B-A)(1-\xi_j)}\Gamma_n\right)^{\frac{1}{q_j}} a_{n,j}^{\frac{1}{q_j}}\right] \leq 1.$$

Thus, we have to determine the largest ξ such that

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi)]}{\beta\gamma(B-A)(1-\xi)}\Gamma_n\left(\prod_{j=1}^{m} a_{n,j}^{p_j}\right) \leq 1$$

that is,

$$\sum_{n=2}^{\infty}\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi)]}{\beta\gamma(B-A)(1-\xi)}\Gamma_n\left(\prod_{j=1}^{m}a_{n,j}^{p_j}\right)$$
$$\leq \sum_{n=2}^{\infty}\left[\sum_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]}{\beta\gamma(B-A)(1-\xi_j)}\Gamma_n\right)^{\frac{1}{q_j}}a_{n,j}^{\frac{1}{q_j}}\right].$$

Therefore, we need to find the largest ξ such that

$$\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi)]}{\beta\gamma(B-A)(1-\xi)}\Gamma_n\left(\prod_{j=1}^{m}a_{n,j}^{p_j}\right)$$
$$\leq \prod_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]}{\beta\gamma(B-A)(1-\xi_j)}\Gamma_n\right)^{\frac{1}{q_j}}a_{n,j}^{\frac{1}{q_j}}.$$

Since,

$$\prod_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]}{\beta\gamma(B-A)(1-\xi_j)}\Gamma_n\right)^{p_j-\frac{1}{q_j}}a_{n,j}^{p_j-\frac{1}{q_j}}\leq 1,\,(p_j-\frac{1}{q_j}\geq 0, j=1,2,...,m)$$

we see that,

$$\prod_{j=i}^{m}a_{n,j}^{p_j-\frac{1}{q_j}}\leq\frac{1}{\prod_{j=i}^{m}\left(\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]}{\beta\gamma(B-A)(1-\xi_j)}\Gamma_n\right)^{p_j-\frac{1}{q_j}}}. \tag{31}$$

This last inequality (31) implies that

$$\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi)]}{\beta\gamma(B-A)(1-\xi)}\Gamma_n$$
$$\leq\frac{\prod_{j=i}^{m}(1+n\lambda-\lambda)^{p_j}[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)\Gamma_n]^{p_j}}{\prod_{j=i}^{m}[\beta\gamma(B-A)(1-\xi_j)]^{p_j}}$$

which implies

$$[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi)]\prod_{j=i}^{m}[\beta\gamma(B-A)]^{p_j-1}(1-\xi_j)^{p_j}$$
$$\leq\prod_{j=i}^{m}(1+n\lambda-\lambda)^{p_j-1}\Gamma_n^{p_j-1}[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]^{p_j}(1-\xi_j)^{p_j}$$

where,

$$\xi\leq 1-\frac{n\Upsilon_j+\Upsilon_j-(n-1)1-\beta B\prod_{j=i}^{m}[\beta\gamma(B-A)]^{p_j-1}(1-\xi_j)^{p_j}}{\prod_{j=i}^{m}(1+n\lambda-\lambda)^{p_j-1}\Gamma_n^{p_j-1}[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]^{p_j}-\Upsilon_j}$$

and $\Upsilon_j=\prod_{j=i}^{m}[\beta\gamma(B-A)]^{p_j}(1-\xi_j)^{p_j}$.

Let

$$\Phi(n)\leq 1-\frac{n\Upsilon_j+\Upsilon_j-(n-1)1-\beta B\prod_{j=i}^{m}[\beta\gamma(B-A)]^{p_j-1}(1-\xi_j)^{p_j}}{\prod_{j=i}^{m}(1+n\lambda-\lambda)^{p_j-1}\Gamma_n^{p_j-1}[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\xi_j)]^{p_j}-\Upsilon_j}$$

which is an increasing function in n, hence we have

$$\xi\leq\Phi(2)=1-\frac{\prod_{j=i}^{m}(1-\xi_j)^{p_j}-[(1-\beta B)+\beta\gamma(B-A)][\beta\gamma(B-A)]^{s}}{(1+\lambda)^{s-1}\Gamma_n^{s-1}\prod_{j=i}^{m}[1-\beta B+\beta\gamma(B-A)(2-\xi_j)]^{p_j}-[\beta\gamma(B-A)]^{s}\prod_{j=i}^{m}(1-\xi_j)^{p_j}}.$$

This completes the proof of the theorem. □

4 Closure properties under integral transforms

Integral transform of the class $\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$. For $f\in\mathcal{S}$, we define the integral transform

$$V_\lambda(f)(z)=\int_0^1\lambda(t)\frac{f(tz)}{t}dt$$

where λ is a real valued, non-negative weight function normalized so that $\int_0^1\lambda(t)dt=1$. Since special cases of $\lambda(t)$ are particularly interesting such as $\lambda(t)=(1+c)t^c, c>-1$, for which V_λ is known as Bernardi Operator [2], and

$$\lambda(t)=\frac{(c+1)^\delta}{\Gamma(\delta)}t^c\left(log\frac{1}{t}\right)^{\delta-1},c>-1,\delta\geq 0$$

which gives the Komatu operator[7].

Theorem 4.1. *Let $f(z)\in\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$. Then $V_\lambda(f(z))\in\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$.*

Proof. By definition, we have

$$V_\lambda(f)(z)=\frac{(c+1)^\delta}{\Gamma(\delta)}\int_0^1(-1)^{\delta-1}t^c(logt)^{\delta-1}\left[z-\sum_{n=2}^\infty a_nz^nt^{n-1}\right]dt$$

$$V_\lambda(f)(z)=\frac{(-1)^{\delta-1}(c+1)^\delta}{\Gamma(\delta)}\lim_{r\to o^+}\left[\int_r^1 t^c(logt)^{\delta-1}\left[z-\sum_{n=2}^\infty a_nz^nt^{n-1}\right]dt\right]$$

By simple computation, we get

$$V_\lambda(f)(z)=z-\sum_{n=2}^\infty\left(\frac{c+1}{c+n}\right)^\delta a_nz^n.$$

We need to prove that $V_\lambda(f(z))\in\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$, it is enough to prove

$$\sum_{n=2}^\infty\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]}{\beta\gamma(B-A)(1-\alpha)}\left(\frac{c+1}{c+n}\right)^\delta\Gamma_na_n\leq 1 \tag{32}$$

on the other hand by Theorem 1.1, $f(z)\in\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$ if and only if

$$\sum_{n=2}^\infty\frac{(1+n\lambda-\lambda)[(n-1)(1-\beta B)+\beta\gamma(B-A)(n-\alpha)]}{\beta\gamma(B-A)(1-\alpha)}\Gamma_na_n\leq 1.$$

Hence $\frac{c+1}{c+n}<1$. Therefore (32) holds and the proof is complete.
The above theorem yields the following two special cases. □

Theorem 4.2. *1)If $f(z)$ is starlike of order γ then $V_\lambda f(z)$ is also starlike of order α.*
2) If $f(z)$ is convex of order γ then $V_\lambda f(z)$ is also convex of order α.

Theorem 4.3. *Let $f\in\mathcal{HF}_\gamma^\lambda(\alpha,\beta,A,B)$. Then $V_\lambda f(z)$ is starlike of order $0\leq\xi<1$ in $|z|<R_1$, where*

$$R_1=\inf_n\left\{\frac{(1-\xi)C_n}{(n-\xi)(B-A)(1-\alpha)\beta\gamma}\right\}^{\frac{1}{n-1}} \tag{33}$$

where C_n is defined by (11).

Proof. It is sufficient to prove

$$\left|\frac{z(V_\lambda f(z))'}{V_\lambda f(z)}-1\right|<1-\xi\quad for\quad |z|<R_1,$$

where R_1 is given by (33)

$$\left|\frac{z(V_\lambda f(z))'}{V_\lambda f(z)} - 1\right| \quad \leq \quad \frac{\sum_{n=2}^{\infty}(n-1)\left(\frac{c+1}{c+n}\right)^{\delta} a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n|z|^{n-1}}$$

Thus

$$\left|\frac{z(V_\lambda f(z))'}{V_\lambda f(z)} - 1\right| < 1-\xi \qquad \text{if,} \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta}\left(\frac{n-\xi}{1-\xi}\right) a_n|z|^{n-1} \leq 1. \tag{34}$$

But by Theorem 1.1, we have

$$\sum_{n=2}^{\infty}\frac{C_n}{(B-A)(1-\alpha)\beta\gamma}a_n \leq 1 \tag{35}$$

Comparing (34) and (35), we have $\left(\frac{c+1}{c+n}\right)^{\delta}\left(\frac{n-\xi}{1-\xi}\right)|z|^{n-1} \leq \frac{C_n}{(B-A)(1-\alpha)\beta\gamma}$.Thus

$$|z| \leq \left\{\left(\frac{c+n}{c+1}\right)^{\delta}\frac{(1-\xi)C_n}{(n-\xi)(B-A)(1-\alpha)\beta\gamma}\right\}^{\frac{1}{n-1}}.$$

Therefore, the proof is complete.We state the following theorem without proof. □

Theorem 4.4. *Let $f \in \mathcal{HF}_{\gamma}^{\lambda}(\alpha, \beta, A, B)$. Then $V_\lambda f(z)$ is convex of order $0 \leq \xi < 1$ in $|z| < R_2$, where*

$$R_2 = \inf_n\left\{\left(\frac{c+n}{c+1}\right)^{\delta}\frac{(1-\xi)C_n}{n(n-\xi)(B-A)(1-\alpha)\beta\gamma}\right\}^{\frac{1}{n-1}} \tag{36}$$

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