Inverse obstacle scattering with non-over-determined data

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Abstract

It is proved that the scattering amplitude $A(\beta, \alpha_0, k_0)$, known for all $\beta \in S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$, and fixed $\alpha_0 \in S^2$ and $k_0 > 0$, determines uniquely the surface S of the obstacle D and the boundary condition on S. The boundary condition on S is assumed to be the Dirichlet, or Neumann, or the impedance one. The uniqueness theorem for the solution of multidimensional inverse scattering problems with non-over-determined data was not known for many decades. A detailed proof of such a theorem is given in this paper for inverse scattering by obstacles for the first time. It follows from our results that the scattering solution vanishing on the boundary S of the obstacle cannot have closed surfaces of zeros in the exterior of the obstacle different from S. To have a uniqueness theorem for inverse scattering problems with non-over-determined data is of principal interest because these are the minimal scattering data that allow one to uniquely recover the scatterer.

Keywords: scattering theory; obstacle scattering; uniqueness theorem; non-over-determined scattering data.

1. Introduction

The uniqueness theorems for the solution of multidimensional inverse scattering problems with non-over-determined scattering data were not known since the origin of the inverse scattering theory, which goes, roughly speaking, to the middle of the last century. A detailed proof of such a theorem is given in this paper for inverse scattering by obstacles for the first time. To have a uniqueness theorem for inverse scattering problems with non-over-determined data is of principal interest because these are the minimal scattering data that allow one to uniquely recover the scatterer. In [8]–[12] such theorems are proved for the first time for inverse scattering by potentials. The result, presented in this paper will be presented in the author's monograph [11], where the ideas of its proof were outlined, see also [1]. In this paper the arguments are given in detail, parts of the presentation in [11] are used verbatim, two new theorems (Theorems 2 and 3) are formulated and proved, and it is pointed out that from these results it follows that the scattering solution vanishing on the boundary S of the obstacle cannot have closed surfaces of zeros different from S in the exterior of the obstacle. The results of this paper will be presented in [11], Chapter 2. Since these results solve the problem that has been unsolved for many decades, we think that their publication in a separate paper is warranted.

The data is called non-over-determined if it is a function of the same number of variables as the function to be determined from these data. In the case of the inverse scattering by an obstacle the unknown function describes the surface of this obstacle in $\mathbb{R}^3$, so it is a function of two variables. The non-over-determined scattering data is the scattering amplitude depending on a two-dimensional vector. The exact formulation of this inverse problem is given below.

Let us formulate the problem discussed in this paper. Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected C^2−smooth boundary S,

$D' := \mathbb{R}^3 \setminus D$ be the unbounded exterior domain and S^2 be the unit sphere in $\mathbb{R}^3$. The smoothness assumption on S can be weakened. Consider the scattering problem:

$$(\nabla^2 + k^2)u = 0 \quad in \quad D', \qquad \Gamma_j u|_S = 0, \qquad u = e^{ik\alpha \cdot x} + v, \quad (1)$$

where the scattered field v satisfies the radiation condition:

$$v_r - ikv = o\left(\frac{1}{r}\right), \quad r := |x| \to \infty. \quad (2)$$

Here $k > 0$ is a constant called the wave number and $\alpha \in S^2$ is a unit vector in the direction of the propagation of the incident plane wave $e^{ik\alpha \cdot x}$. The boundary conditions are assumed to be either the Dirichlet (Γ_1), or Neumann (Γ_2), or impedance (Γ_3) type:

$$\Gamma_1 u := u, \quad \Gamma_2 u := u_N, \quad \Gamma_3 u := u_N + hu, \quad (3)$$

where N is the unit normal to S pointing out of D, u_N is the normal derivative of u on S, $h = const$, Im$h \geq 0$, h is the boundary impedance, and the condition Im$h \geq 0$ guarantees the uniqueness of the solution to the scattering problem (1)-(2).

The scattering amplitude $A(\beta, \alpha, k)$ is defined by the following formula:

$$v = A(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \frac{x}{r} = \beta, \quad (4)$$

where $\alpha, \beta \in S^2$, β is the direction of the scattered wave, α is the direction of the incident wave.

For a bounded domain D one has $o(\frac{1}{r}) = O(\frac{1}{r^2})$ in formula (4). The function $A(\beta, \alpha, k)$, the scattering amplitude, can be measured experimentally. Let us call it *the scattering data*. It is known (see [2], p.25) that the solution to the scattering problem (1)-(2) does exist and is unique.

The inverse scattering problem (IP) consists of finding S and the boundary condition on S from the scattering data.

M.Schiffer was the first to prove in the sixties of the last century that if the boundary condition is the Dirichlet one then the surface S is uniquely determined by the scattering data $A(\beta, \alpha_0, k)$ known for a fixed $\alpha = \alpha_0$, all $\beta \in S^2$, and all $k \in (a, b)$, $0 \le a < b$.

M. Schiffer did not publish his proof. This proof can be found, for example, in [2], p.85, and the acknowledgement of M.Schiffer's contribution is on p.399 in [2].

A. G. Ramm was the first to prove that the scattering data $A(\beta, \alpha, k_0)$, known for all β in a solid angle, all α in a solid angle and a fixed $k = k_0 > 0$ determine uniquely the boundary S *and the boundary condition on S.* This condition was assumed of one of the three types Γ_j, $j = 1, 2$ or 3, (see [2], Chapter 2, for the proof of these results). By subindex zero fixed values of the parameters are denoted, for example, k_0, α_0. By a solid angle in this paper an open subset of S^2 is understood.

In [2], p.62, it is proved that for smooth bounded obstacles the scattering amplitude $A(\beta, \alpha, k)$ is an analytic function of β and α on the non-compact analytic variety

$M := \{z | z \in \mathbb{C}^3, z \cdot z = 1\}$, where $z \cdot z := \sum_{m=1}^3 z_m^2$. The unit sphere S^2 is a subset of M. If $A(\beta, \alpha, k)$ as a function of β is known on an open subset of S^2, it is uniquely extended to all of S^2 (and to all of M) by analyticity. The same is true if $A(\beta, \alpha, k)$ as a function of α is known on an open subset of S^2. By this reason one may assume that the scattering amplitude is known on all of S^2 if it is known in a solid angle, that is, on open subsets of S^2 as a function of α and β.

In papers [5] and [6] a new approach to a proof of the uniqueness theorems for inverse obstacle scattering problem (IP) was given. This approach is used in our paper.

In paper [4] the uniqueness theorem for IP with non-over-determined data was proved for strictly convex smooth obstacles. The proof in [4] was based on the location of resonances for a pair of such obstacles. These results are technically difficult to obtain and they hold for two strictly convex smooth obstacles with a positive distance between them.

The purpose of this paper is to prove the uniqueness theorem for IP with non-over-determined scattering data for arbitrary S. For simplicity the boundary is assumed C^2- smooth. By the boundary condition any of the three conditions Γ_j are understood below, but the argument is given for the Dirichlet condition for definiteness.

Theorem 1. *The surface S and the boundary condition on S are uniquely determined by the data $A(\beta)$ known in a solid angle.*

Theorem 2. *If $A_1(\beta) = A_2(\beta)$ for all β in a solid angle, then it is not possible that $D_1 \ne D_2$ and $D_1 \cap D_2 = \emptyset$.*

Theorem 3. *If $A_1(\beta) = A_2(\beta)$ for all β in a solid angle, then it is not possible that $D_1 \ne D_2$ and $D_1 \subset D_2$.*

Corollary. *It follows from Theorems 2 and 3 that the solution to problem* (1)*–* (2) *(the scattering solution) cannot have a closed surface of zeros except the surface S, the boundary of the obstacle.*

In Section 2 some auxiliary material is formulated and Theorems 1, 2, 3 are proved. The Corollary is an immediate consequence of these theorems. Theorem 1–3 and the Corollary are our main results.

Let us explain the logic of the proof of Theorem 1. Its proof is based on the assumption that there are two different obstacles, D_1 with the surface S_1 and D_2 with the surface S_2, that generate the same non-over-determined scattering data. This assumption leads to a contradiction which proves that $S_1 = S_2$. If it is proved that $S_1 = S_2$, then the type of the boundary condition (of one of the three types (4)) can be uniquely determined by calculating u or $\frac{u_N}{u}$ on S.

There are three cases to consider. The first case, when S_1 intersects S_2, is considered in Theorem 1. The second case, when S_1 does not intersect S_2 and does not lie inside S_2, is considered in Theorem2.

The third case, when S_1 does not intersect S_2 and it lies inside S_2 is considered in Theorem 3.

Our results show that these cases cannot occur if the non-over-determined scattering data corresponding to S_1 and S_2 are the same. They also show that a scattering solution cannot have a closed surface of zeros except S.

2. Proofs of Theorems 1, 2 and 3

Let us formulate some lemmas which are proved by the author, except for Lemma 3, which was known. Lemma 3 was proved first by V.Kupradze in 1934 and then by I.Vekua, and independently by F.Rellich, in 1943, see a proof of Lemma 3 in the monograph [2], p.25, and also the references there to the papers of V.Kupradze, I.Vekua, and F.Rellich). Another proof of Lemma 3, based on a new idea, is given in paper [3].

Denote by $G(x, y, k)$ the Green's function corresponding to the scattering problem (1)-(2). The parameter $k > 0$ is assumed fixed in what follows. For definiteness we assume below the Dirichlet boundary condition, but our proof is valid for the Neumann and impedance boundary conditions as well. If there are two surfaces S_m, $m = 1, 2$, we denote by G_m the corresponding Green's functions of the Dirichlet Helmholtz operator in D'_m.

Lemma 1. ([2], p. 46) *One has:*

$$G(x, y, k) = g(|y|)u(x, \alpha, k) + O\left(\frac{1}{|y|^2}\right),$$
$$|y| \to \infty, \quad \frac{y}{|y|} = -\alpha. \tag{5}$$

Here $g(|y|) := \frac{e^{ik|y|}}{4\pi|y|}$, $u(x, \alpha, k)$ is the scattering solution, that is, the solution to problem (1)-(2), $O\left(\frac{1}{|y|^2}\right)$ is uniform with respect to $\alpha \in S^2$, and the notation $\gamma(r) := 4\pi g(r) = \frac{e^{ik|r|}}{|r|}$ is used below.

The solutions to equation (1) have the unique continuation property: If u solves equation (1) and vanishes on a set $\tilde{D} \subset D'$ of positive Lebesgue measure, then u vanishes everywhere in D'.

Formula (5) holds if y is replaced by the vector $-\tau\alpha + \eta$, where $\tau > 0$ is a scalar and η is an arbitrary fixed vector orthogonal to $\alpha \in S^2$, $\eta \cdot \alpha = 0$. If $\eta \cdot \alpha = 0$ and $y = -\tau\alpha + \eta$, then $\frac{|y|}{\tau} = 1 + O\left(\frac{1}{|\tau|^2}\right)$ as $\tau \to \infty$. The relation $|y| \to \infty$ is equivalent to the relation $\tau \to \infty$, and $g(|y|) = g(\tau)(1 + O(\frac{1}{|\tau|}))$.

Denote by $D_{12} := D_1 \cup D_2$, $D'_{12} := \mathbb{R}^3 \setminus D_{12}$, $S_{12} := \partial D_{12}$, $\tilde{S}_1 := S_{12} \setminus S_2$, that is, $\tilde{S}_1$ does not belong to D_2, $B'_R := \mathbb{R}^3 \setminus B_R$, $B_R := \{x : |x| \le R\}$. The number R is sufficiently large, so that $D_{12} \subset B_R$. Let S^{12} denote the intersection of S_1 and S_2. This set may have positive two-dimensional Lebesgue measure or it may have two-dimensional Lebesgue measure zero. In the first case let us denote by $L \subset S^{12}$ the line such that in an arbitrary small neighborhood of every point $s \in L$ there are points of S_1 and of S_2. The line L has two-dimensional Lebesgue measure equal to zero. Denote by S_ε the subset of points on S_{12} the distance from which to L is less than ε. The two-dimensional Lebesgue measure m_ε of S_ε tends to zero as $\varepsilon \to 0$.

A part of our proof is based on a global perturbation lemma, Lemma 2 below, which is proved in [6], see there formula (4). A similar lemma is proved for potential scattering in [7], see there formula (5.1.30). For convenience of the readers a short proof of Lemma 2 is given below.

Lemma 2. *One has:*

$$4\pi[A_1(\beta, \alpha, k) - A_2(\beta, \alpha, k)] = \int_{S_{12}} [u_1(s, \alpha, k)u_{2N}(s, -\beta, k)$$
$$-u_{1N}(s, \alpha, k)u_2(s, -\beta, k)]ds, \tag{6}$$

where the scattering amplitude $A_m(\beta, \alpha, k)$ corresponds to obstacle S_m, $m = 1, 2$.

Proof. Denote by $G_m(x,y,k)$ the Green's function of the Dirichlet Helmholtz operator in D'_m, $m = 1,2$. Using Green's formula one obtains

$$G_1(x,y,k) - G_2(x,y,k)] = \int_{S_{12}} [G_1(s,x,k)G_{2N}(s,y,k) \\ - G_{1N}(s,x,k)G_2(s,y,k)]ds. \tag{7}$$

Pass in (7) to the limit $y \to \infty$, $\frac{y}{|y|} = \beta$, and use Lemma 1 to get:

$$u_1(x,-\beta,k) - u_2(x,-\beta,k)] = \int_{S_{12}} [G_1(s,x,k)u_{2N}(s,-\beta,k) \\ - G_{1N}(s,x,k)u_2(s,-\beta,k)]ds. \tag{8}$$

Use the formula

$$u_m(x,-\beta,k) = e^{-ik\beta\cdot x} + A_m(-\alpha,-\beta,k)\frac{e^{ik|x|}}{|x|} + O\big(\frac{1}{|x|^2}\big), \\ |x| \to \infty, \ \frac{x}{|x|} = -\alpha, \tag{9}$$

pass in equation (8) to the limit $x \to \infty$, $\frac{x}{|x|} = -\alpha$, use Lemma 1 and get

$$4\pi[A_1(-\alpha,-\beta,k) - A_2(-\alpha,-\beta,k)] = \\ \int_{S_{12}} [u_1(s,\alpha,k)u_{2N}(s,-\beta,k) - u_{1N}(s,\alpha,k)u_2(s,-\beta,k)]ds. \tag{10}$$

The desired relation (6) follows from (10) if one recalls the known reciprocity relation

$$A(-\alpha,-\beta,k) = A(\beta,\alpha,k),$$

which is proved, for example, in [2], pp. 53-54.
Lemma 2 is proved.   □

Remark 1. In (7) Green's formula is used. The surface S_{12} may be not smooth because it contains the intersection S^{12} of two smooth surfaces S_1 and S_2, and this intersection may be not smooth. However, the integrand in (7) is smooth up to the boundary S_{12} and is uniformly bounded because x and y belong to the exterior of D_{12}. The integral (7) is defined as the limit of the integral over $S_{12} \setminus S_\varepsilon$ as $\varepsilon \to 0$ (where S_ε was defined above Lemma 2). This limit does exist since m_ε, the two-dimensional Lebesgue measure of S_ε, tends to zero as $\varepsilon \to 0$ while the integrand is smooth and uniformly bounded on S_{12}. Consequently, the integral (7) is well defined. This argument also shows that formula (10) is valid for the domain D_{12} if the surfaces S_1 and S_2 are smooth and the functions u_1 and u_2 are smooth and uniformly bounded up to S_1 and S_2 respectively.

Lemma 3. ([2], p. 25) *If* $\lim_{r\to\infty} \int_{|x|=r} |v|^2 ds = 0$ *and v satisfies the Helmholtz equation* (1) *in* B'_R, *then $v = 0$ in* B'_R.

The following useful lemma is formulated below and is used in our proof. Its proof is given also in the monograph [11].
Lemma 4. *(lifting lemma) If* $A_1(\beta,\alpha,k) = A_2(\beta,\alpha,k)$ *for all* $\beta,\alpha \in S^2$, *then* $G_1(x,y,k) = G_2(x,y,k)$ *for all* $x,y \in D'_{12}$. *If* $A_1(\beta,\alpha_0,k) = A_2(\beta,\alpha_0,k)$ *for all* $\beta \in S^2$ *and a fixed* $\alpha = \alpha_0$, *then* $G_1(x,y_0,k) = G_2(x,y_0,k)$ *for all* $x \in D'_{12}$ *and* $y_0 = -\alpha_0\tau + \eta$, *where* $\tau > 0$ *is a number and η is an arbitrary fixed vector orthogonal to* α_0, $\alpha_0 \cdot \eta = 0$.

Proof of Lemma 4. The function

$$w := w(x,y) := G_1(x,y,k) - G_2(x,y,k) \tag{11}$$

satisfies equation (1) in D'_{12} as a function of y and also as a function of x, and w satisfies the radiation condition as a function of y and also as a function of x. By Lemma 1 one has:

$$w = g(|y|)[u_1(x,\alpha,k) - u_2(x,\alpha,k)] + O\big(\frac{1}{|y|^2}\big), \\ |y| \to \infty, \ \alpha = -\frac{y}{|y|}. \tag{12}$$

Using formulas (1) and (4) one gets:

$$u_1(x,\alpha,k) - u_2(x,\alpha,k) = \gamma(|x|)[A_1(\beta,\alpha,k) - A_2(\beta,\alpha,k)] + \\ O\big(\frac{1}{|x|^2}\big), \quad |x| \to \infty, \ \beta = \frac{x}{|x|}, \tag{13}$$

because, for $m = 1,2$ and $\gamma(|x|) := \frac{e^{ik|x|}}{|x|}$, one has:

$$u_m(x,\alpha,k) = e^{ik\alpha\cdot x} + A_m(\beta,\alpha,k)\gamma(|x|) + O\big(\frac{1}{|x|^2}\big), \\ |x| \to \infty, \ \beta = \frac{x}{|x|}. \tag{14}$$

If $A_1(\beta,\alpha,k) = A_2(\beta,\alpha,k)$, then equation (13) implies

$$u_1(x,\alpha,k) - u_2(x,\alpha,k) = O\big(\frac{1}{|x|^2}\big). \tag{15}$$

Since $u_1(x,\alpha,k) - u_2(x,\alpha,k)$ solves equation (1) in D'_{12} and relation (15) holds, it follows from Lemma 3 that $u_1(x,\alpha,k) = u_2(x,\alpha,k)$ in B'_R. By the unique continuation property for the solutions to the Helmholtz equation (1), one concludes that $u_1 = u_2$ everywhere in D'_{12} and even in $\mathscr{D}^{12} := (D_1 \cap D_2)'$. Consequently, formula (12) yields

$$w(x,y) = O\big(\frac{1}{|y|^2}\big), \qquad |y| > |x| \geq R. \tag{16}$$

Since the function w solves the homogeneous Helmholtz equation (1) in the region $|y| > |x| \geq R$, it follows by Lemma 3 that $w = w(x,y) = 0$ in this region and, by the unique continuation property, $w = 0$ everywhere in D'_{12}. Thus, the first part of Lemma 4 is proved. Its second part deals with the case when $\alpha = \alpha_0$, where α_0 is fixed. Let us prove that if

$$A_1(\beta) := A_1(\beta,\alpha_0,k) = A_2(\beta,\alpha_0,k) := A_2(\beta) \qquad \forall \beta \in S^2, \tag{17}$$

then

$$w(x,y_0) = 0, \tag{18}$$

where $x \in D'_{12}$ is arbitrary, $y_0 = -\tau\alpha_0 + \eta$, $\alpha_0 \in S^2$ is fixed, $\tau > 0$ is a number and η is an arbitrary fixed vector orthogonal to α_0, $\eta \cdot \alpha_0 = 0$.
From (17) it follows that $u_1(x,\alpha_0) = u_2(x,\alpha_0)$ for all $x \in D'_{12}$ and even in $\mathscr{D}^{12} := (D_1 \cap D_2)'$. Let us derive a contradiction from the assumption that (18) is not valid, or, which is equivalent, that $S_1 \neq S_2$.
The Green's formula yields $G_1(x,y_0) = g(x,y_0) - \int_{S_1} g(x,s)G_{1N}(s,y_0)ds$, where $g(x,y_0) = \frac{e^{ik|x-y_0|}}{4\pi|x-y_0|}$, and a similar formula holds for G_2 with the integration over S_2. Consequently,

$$G_1(x,y_0) - G_2(x,y_0) = \int_{S_2} g(s,x)G_{2N}(s,y_0)ds - \\ \int_{S_1} g(s,x)G_{1N}(s,y_0)ds, \tag{19}$$

where ds is the surface area element.
Let $y_0 \to \infty$, $y_0/|y_0| = -\alpha_0$ and take into account that if $u_1(x,\alpha_0) = u_2(x,\alpha_0)$ for all $x \in D'_{12}$ then $u_1(x,\alpha_0) = u_2(x,\alpha_0) := u(x,\alpha_0)$ for all $x \in \mathscr{D}^{12} := (D_1 \cap D_2)'$ by the unique continuation principle. Therefore, equation (19) and Lemma 1 yield

$$\int_{S_2} g(s,x)u_N(s,\alpha_0)ds = \int_{S_1} g(s,x)u_N(s,\alpha_0)ds, \quad \forall x \in \mathscr{D}^{12}. \tag{20}$$

The right side of (20) is an infinitely smooth function when x passes the part of S_2 which lies outside of D_1 while the normal derivative of the left side has a jump $u_N(s,\alpha_0)$ in such a process. This is a contradiction unless $u_N(s,\alpha_0) = 0$ on S_2. However, $u = 0$ on S_2 and if $u_N(s,\alpha_0) = 0$ on S_2 then, by the uniqueness of the solution

to the Cauchy problem for the Helmholtz equation, one concludes that $u = 0$ in D_2'. This is impossible since $\lim_{x \to \infty} |u(x, \alpha_0)| = 1$. This contradiction proves that $D_1 = D_2 := D$, $S_1 = S_2 := S$, and $G_1(x, y_0) = G_2(x, y_0) := G(x, y_0)$, where G is the Green's function of the Dirichlet Helmholtz operator for the domain D', and G satisfies the radiation condition at infinity.

Thus, the proof of the relation $G_1(x, y_0) = G_2(x, y_0)$ is completed and the second part of Lemma 4 is proved.

Lemma 4 is proved. □

Lemma 4 is used for a proof of Theorem 1.

In Remark 2 (see below) a different proof of Theorems 1 and 3 is given.

Lemma 5. *One has*

$$\lim_{x \to t} G_{2N}(x, s, k) = \delta(s - t), \quad t \in S_2, \tag{21}$$

where $\delta(s-t)$ denotes the delta-function on S_2 and $x \to t$ denotes a limit along any straight line non-tangential to S_2.

Proof of Lemma 5. Let $f \in C(S_2)$ be arbitrary. Consider the following problem: W solves equation (1) in D_2', W satisfies the boundary condition $W = f$ on S_2, and W satisfies the radiation condition. The unique solution to this problem is given by the Green's formula:

$$W(x) = \int_{S_2} G_{2N}(x, s) f(s) ds. \tag{22}$$

Since $\lim_{x \to t \in S_2} w(x) = f(t)$ and $f \in C(S_2)$ is arbitrary, the conclusion of Lemma 5 follows.

Lemma 5 is proved. □

Let us point out the following implications:

$$G(x, y, k) \to u(x, \alpha, k) \to A(\beta, \alpha, k), \tag{23}$$

which hold by Lemma 1 and formula (14). The first arrow means that the knowledge of $G(x, y, k)$ determines uniquely the scattering solution $u(x, \alpha, k)$ for all $\alpha \in S^2$, and the second arrow means that the scattering solution $u(x, \alpha, k)$ determines uniquely the scattering amplitude $A(\beta, \alpha, k)$.

The reversed implications also hold:

$$A(\beta, \alpha, k) \to u(x, \alpha, k) \to G(x, y, k). \tag{24}$$

These implications follow from Lemmas 1, 3 , 4 and formula (14). Let us explain why the knowledge of $u(x, \alpha, k)$ determines uniquely $G(x, y, k)$. If there are two G_m, $m = 1, 2$, to which the same $u(x, \alpha, k)$ corresponds, then $w := G_1 - G_2$ solves equation (1) in D_{12}' and, by Lemma 1, $w = O(\frac{1}{|x|^2})$. Thus, by Lemma 3, $w = 0$, so $G_1 = G_2$ in D_{12}'. This implies, as in the proof of Theorem 1 below, that $D_1 = D_2 := D$.

Similar implications for $\alpha = \alpha_0$ fixed are formulated after the proof of Theorem 1.

Proof of Theorem 1. If $A_1(\beta) = A_2(\beta)$ for all β in a solid angle, then the same is true for all $\beta \in S^2$, so one may assume that $A_1(\beta) = A_2(\beta)$ for all $\beta \in S^2$.

Let us assume that $A_1(\beta) = A_2(\beta)$ for all β but $S_1 \neq S_2$. We want to derive from this assumption a contradiction. This contradiction will prove that the assumption $S_1 \neq S_2$ is false, so $S_1 = S_2$.

If $A_1(\beta) = A_2(\beta)$ for all $\beta \in S^2$, then Lemma 4 yields the following conclusion:

$$G_1(x, y_0) = G_2(x, y_0), \qquad \forall x \in D_{12}', \tag{25}$$

where $k > 0$ and $\alpha_0 \in S^2$ are fixed and $y_0 = -\alpha_0 \tau + \eta$, $\tau > 0$, $\eta \cdot \alpha_0 = 0$, η is an arbitrary fixed vector orthogonal to α_0. This is the key point in the proof of Theorem 1. For definiteness we assume in the proof of Theorem 1 that S_1 intersects S_2. The other cases are considered in Theorems 2 and 3.

If $S_1 \neq S_2$ then one gets a contradiction: let y_0 approach a point $t \in S_2$ which does not belong to S_1 along the ray $-\tau\alpha_0 + \eta$. Then, on one hand, $G_1(x, t) = G_2(x, t) = 0$ for all $x \in D_{12}'$, and, on the other hand, $G_1(x, t) = O(\frac{1}{|x-t|})$, so that $|G_1(x, t)| \to \infty$ as $x \to t$. This contradiction proves that $S_1 = S_2$.

If $S_1 = S_2 := S$ then $D_1 = D_2 := D$ and $u_1(x, \alpha_0, k) = u_2(x, \alpha_0, k) := u(x, \alpha_0, k)$ for $x \in D'$, and, consequently, the boundary condition on S is uniquely determined: if $u|_S = 0$, then one has the Dirichlet boundary condition Γ_1, otherwise calculate $\frac{u_N}{u}$ on S. If this ratio vanishes, then one has the Neumann boundary condition Γ_2, otherwise one has the impedance boundary condition Γ_3, and the boundary impedance $h = -\frac{u_N}{u}$ on S, so the boundary condition is uniquely determined by the non-over-determined scattering data.

Theorem 1 is proved. □

One may give different proofs of Theorem 1. For example, if $S_1 \neq S_2$ and S_1 intersects S_2 then, by analytic continuation, the scattering solutions $u_m(x, \alpha_0)$, $m = 1, 2$, admit analytic continuation to the exterior of the domain $D^{12} = D_1 \cap D_2$. The boundary of this domain has edges. If a point t belongs to an edge, then the gradient of the solution to the homogeneous Helmholtz equation is singular when $x \to t$. On the other hand, this t belongs to a smooth boundary S_1 or S_2, so that the above gradient has to be smooth. This contradiction proves that $S_1 = S_2$ in the case when S_1 intersects S_2.

Let us formulate the implication similar to the one given before the proof of Theorem 1. If $y = y_0 = -\tau\alpha_0 + \eta$, $\tau > 0$ is an arbitrary number, α_0 is a fixed unit vector, and η is an arbitrary fixed vector orthogonal to α_0, then

$$G(x, y_0, k) \to u(x, \alpha_0, k) \to A(\beta, \alpha_0, k), \tag{26}$$

where α_0 is a free unit vector, that is, a vector whose initial point is arbitrary.

The reversed implications also hold:

$$A(\beta, \alpha_0, k) \to u(x, \alpha_0, k) \to G(x, y_0, k). \tag{27}$$

The first of these implications follows from Lemma 3 and the asymptotic of the scattering solution, while the second follows from Lemmas 1 and 4.

We have assumed implicitly that D_1 and D_2 have a common part but none of them is a subset of the other, that is, S_1 intersects S_2. Let us discuss the two remaining possibilities.

The first possibility is that $D_1 \neq D_2$ and $D_1 \cap D_2 = \emptyset$.

Proof of Theorem 2. If $A_1(\beta) = A_2(\beta)$ in a solid angle, then $A_1(\beta) = A_2(\beta)$ in S^2. This implies that $u_1(x, \alpha_0) = u_2(x, \alpha_0)$ in D_{12}'. Since $u_1(x, \alpha_0)$ is defined in D_2 and satisfies there the Helmholtz equation (1), the unique continuation property implies that $u_2(x, \alpha_0, k)$ is defined in D_2 and satisfies there the Helmholtz equation. Consequently, $u_2(x, \alpha_0, k)$ is defined in $\mathbb{R}^3$, it is a smooth function that satisfies in $\mathbb{R}^3$ the Helmholtz equation, and the same is true for $u_1(x, \alpha_0, k)$. Therefore the scattered parts v_1 and v_2 of the scattering solutions u_1 and u_2 satisfy the Helmholtz equation (1) in $\mathbb{R}^3$ and the radiation condition. A function satisfying the radiation condition and the Helmholtz equation in $\mathbb{R}^3$ is equal to zero in $\mathbb{R}^3$. Therefore, $v_1 = v_2 = 0$ and $u_1 = u_2 = e^{ik\alpha_0 \cdot x}$ in $\mathbb{R}^3$. This is impossible since $u_m = 0$ on S_m, $m = 1, 2$, while $e^{ik\alpha_0 \cdot x} \neq 0$ on S_m.

Theorem 2 is proved. □

The second possibility is $D_1 \neq D_2$ and $D_1 \subset D_2$.

Proof of Theorem 3.

One proves Theorem 3 using Lemma 4. By Lemma 4 one has

$$G_1(x, y_0) = G_2(x, y_0) \quad \forall x \in D_2', \quad y_0 = -\tau\alpha_0 + \eta,$$
$$y_0 \in D_2', \quad \eta \cdot \alpha_0 = 0, \quad \tau \in (0, \infty). \tag{28}$$

Note that

$$\lim_{x \to y_0} |G_1(x, y_0)| = \infty \tag{29}$$

since both x and y_0 belong to D_1' and are away from S_1 if $D_1 \neq D_2$. On the other hand, if $y_0 \in S_2$, then $G_2(x, y_0) = 0$ for all $x \in D_2$, $x \neq y_0$ and

$$\lim_{x \to y_0} |G_2(x, y_0)| = 0. \tag{30}$$

This is a contradiction unless $D_1 = D_2$.
Theorem 3 is proved. $\qquad\qquad\square$

It follows from Theorems 1, 2 and 3 that Corollary holds: the solution to problem (1)– (2) *(the scattering solution) cannot have a closed surface of zeros except the surface S, the boundary of the obstacle.*
Remark 2. Let us give a new proof of Theorems 1 and 3. Assume first that $D_1 \subset D_2$, so that assumptions of Theorem 3 hold. Denote by u the analytic continuation of u_2 into $D_2 \setminus D_1$. This u is equal to u_1 in D_1'. Green's formula yields

$$u(x) = \int_{S_2} g(x, s) u_N(s) ds - \int_{S_1} g(x, s) u_N(s) ds, \tag{31}$$
$$x \in D_2 \setminus D_1.$$

$$u = u_0 - \int_{S_2} g(x, s) u_N(s) ds, \quad x \in D_2', \tag{32}$$

$$u = u_0 - \int_{S_1} g(x, s) u_N(s) ds, \quad x \in D_1'. \tag{33}$$

From (31) and (33) one derives

$$0 = u_0 - \int_{S_2} g(x, s) u_N(s) ds, \quad x \in D_2 \setminus D_1. \tag{34}$$

Denote $I^+ := \int_{S_2} g(x, s) u_N(s) ds, \quad x \in D_2 \setminus D_1$ and $I^- := \int_{S_2} g(x, s) u_N(s) ds, \quad x \in D_2'$. The functions u and u_0 are uniquely extended from D_2' into $D_2 \setminus D_1$ and integral I^- is extended uniquely from D_2' into $D_2 \setminus D_1$ as integral I^+. Equations (34) can be written as $I^+ = u_0$ in $D_2 \setminus D_1$. Equation (32) can be uniquely extended into $D_2 \setminus D_1$ and written there as $u = u_0 - I^+ = 0$. Therefore $u = 0$ in $D_2 \setminus D_1$. This is a contradiction since u is the scattering solution in D_1', it solves the elliptic Helmholtz equation in D_1' and if $u = 0$ in $D_2 \setminus D_1$ then $u = 0$ everywhere in D_1', which is impossible since $|u| \to 1$ as $|x| \to \infty$.
If the assumptions of Theorem 1 hold, then the argument is similar. The roles of S_2 and S_1 are played respectively by S_{12}, the boundary of D_{12}, and the boundary of the intersection $D^{12} = D_1 \cap D_2$. $\qquad\square$

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