Global Journal of Mathematical Analysis, 1 (2) (2013) 29-36
©Science Publishing Corporation
www.sciencepubco.com/index.php/GJMA

λ_3-Statistical convergence of triple sequences on probabilistic normed space

Ayhan Esi

Adiyaman University, Department of Mathematics, 02040, Adiyaman, Turkey
E-mail: aesi23@hotmail.com

Abstract

The idea of λ-statistical convergence of single sequences was studied by Alotaibi [39] and double sequences was studied by Savas and Mohiuddine [3] in probabilistic normed spaces. The purpose of this paper is to study statistical convergence of triple sequences in probabilistic normed spaces and give some important theorems.

Keywords: *λ-sequence, probabilistic normed space, statistical convergence, t-norm.*

1 Introduction

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast [8] and Schoenberg [32] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated by Fridy [9], Šalát [31], Çakalli [5], Maio and Kocinac [19], Miller [21], Maddox [18], Leindler [17], Mursaleen and Alotaibi [24], Mursaleen and Edely [26], Mursaleen and Edely [28], and many others. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability, (see [6]).

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset of $\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \ \ exists.$$

A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \to \ell(S)$ and S denotes the set of all statistically convergent sequences.

The probabilistic metric space was introduced by Menger [20] which is an interesting and important generalization of the notion of a metric space. Karakus [14] studied the concept of statistical convergence in probabilistic normed spaces. The theory of probabilistic normed spaces was initiated and developed in [4], [33], [34], [35], [37] and further it was extended to random/probabilistic 2-normed spaces by Goleţ [11] using the concept of 2-norm which is defined by Gähler [10], and Gürdal and Pehlivan [13] and [12] studied statistical convergence in 2-Banach spaces.

Let $\lambda = (\lambda_n)$ be non-decreasing sequence of positive numbers tending to infinity such that

$$\lambda_{n+1} \leq \lambda_n + 1, \ \lambda_1 = 1.$$

The generalized de la Vallee-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$. Let $K \subseteq \mathbb{N}$ be a set of positive integers. Then

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : \ k \in K\}|$$

is said to be $\lambda-$density of K. In case $\lambda_n = n$, the $\lambda-$density reduces to natural density.

Mursaleen[23] introduced the $\lambda-$statistical convergence as follows: A sequence $x = (x_k)$ is said to be $\lambda-$*statistically convergent* or $S_\lambda-$convergent to ℓ if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : \ |x_k - \ell| \geq \varepsilon\}|.$$

The existing literature on statistical convergence and its generalizations appears to have been restricted to real or complex sequences, but in recent years these ideas have been also extended to the sequences in fuzzy normed [36] and intutionistic fuzzy normed spaces [15], [29], [30], and [16]. Further details on generalization of statistical convergence can be found in [25], [26], [27] and [28].

2 Preliminaries

Definition 2.1 *([20]) A triangular norm, briefly t-norm, is a binary operation on* $[0,1]$ *which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, that is, it is the continuous mapping* $\circledast : [0,1] \times [0,1] \to [0,1]$ *such that for all* $a, b, c \in [0,1]$ *:*

(1) $a \circledast 1 = a$,

(2) $a \circledast b = b \circledast a$,

(3) $c \circledast d \geq a \circledast b$ *if* $c \geq a$ *and* $d \geq b$,

(4) $(a \circledast b) \circledast c = a \circledast (b \circledast c)$.

Example 2.2 *The* $\circledast$ *operations* $a \circledast b = \max\{a + b - 1, 0\}$, $a \circledast b = a.b$ *and* $a \circledast b = \min\{a, b\}$ *on* $[0,1]$ *are t-norms.*

Definition 2.3 *([20]) A function* $f : \mathbb{R} \to \mathbb{R}_o^+$ *is called a distribution function if it is non-decreasing and left continuous with* $\inf_{t \in \mathbb{R}} f(t) = 0$ *and* $\sup_{t \in \mathbb{R}} f(t) = 1$*. We will denote the set of all distribution functions by* D*.*

Definition 2.4 *([33]) A triple* $(X, v, \circledast)$ *is called a probabilistic normed space or shortly PN-space if* X *is a real vector space,* v *is a mapping from* X *into* D *(for* $x \in X$*, the distribution function* $v(x)$ *is denoted by* v_x *and* $v_x(t)$ *is the value of* v_x *at* $t \in \mathbb{R}$*) and* $\circledast$ *is a t-norm satisfying the following conditions:*

(PN-1) $v_x(0) = 0$,

(PN-2) $v_x(t) = 1$ *for all* $t > 0$ *if and only if* $x = 0$,

(PN-3) $v_{\alpha x}(t) = v_x\left(\frac{t}{|\alpha|}\right)$ *for all* $\alpha \in R \backslash \{0\}$,

(PN-4) $v_{x+y}(s + t) \geq v_x(s) \circledast v_x(t)$ *for all* $x, y \in X$ *and* $s, t \in \mathbb{R}_o^+$.

Example 2.5 *Suppose that* $(X, \|.\|)$ *is a normed space* $\mu \in D$ *with* $\mu(0) = 0$ *and* $\mu \neq h$*, where*

$$h(t) = \begin{cases} 0 & , \quad t \leq 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Define

$$v_x(t) = \begin{cases} h(t) & , \quad x = 0 \\ \mu\left(\frac{t}{\|x\|}\right) & , \quad x \neq 0 \end{cases},$$

where $x \in X, t \in \mathbb{R}$. *Then* $(X, v, \circledast)$ *is a PN-space. For example if we define the functions* μ *and* α *on* $\mathbb{R}$ *by*

$$\mu(x) = \begin{cases} 0 & , \quad x \le 0 \\ \frac{x}{1+x} & , \quad x > 0 \end{cases}, \quad \alpha(x) = \begin{cases} 0 & , \quad x \le 0 \\ e^{\frac{-1}{x}} & , \quad x > 0 \end{cases}$$

then we obtain the following well-known $\circledast$ *norms:*

$$v_x^{(1)}(t) = \begin{cases} h(t) & , \quad x = 0 \\ \frac{t}{t+\|x\|} & , \quad x \ne 0 \end{cases}, \quad v_x^{(2)}(t) = \begin{cases} h(t) & , \quad x = 0 \\ e^{\left(\frac{-\|x\|}{t}\right)} & , \quad x \ne 0 \end{cases}.$$

We recall the concepts of convergence and Cauchy sequences in a probabilistic normed space.

Definition 2.6 *([40]) Let* $(X, v, \circledast)$ *be a PN-space. Then a sequence* $x = (x_k)$ *is said to be convergent to* $l \in X$ *with respect to the probabilistic norm* v *if, for every* $\varepsilon > 0$ *and* $\theta \in (0, 1)$, *there exists a positive integer* k_o *such that* $v_{x_k - l}(\varepsilon) > 1 - \theta$ *whenever* $k \ge k_o$. *It is denoted by* $v - \lim x = L$ *or* $x_k \xrightarrow{v} L$ *as* $k \to \infty$.

Definition 2.7 *([40]) Let* $(X, v, \circledast)$ *be a PN-space. Then a sequence* $x = (x_k)$ *is called a Cauchy sequence with respect to the probabilistic norm* v *if, for every* $\varepsilon > 0$ *and* $\theta \in (0, 1)$, *there exists a positive integer* k_o *such that* $v_{x_k - x_l}(\varepsilon) > 1 - \theta$ *for all* $k, l \ge k_o$.

Definition 2.8 *([40]) Let* $(X, v, \circledast)$ *be a PN-space. Then a sequence* $x = (x_k)$ *is said to be bounded in* X, *if there is a* $r \in \mathbb{R}$ *such that* $v_{x_k}(r) > 1 - \theta$, *where* $\theta \in (0, 1)$. *We denote by* l_∞^v *the space of all bounded sequences in PN space.*

Definition 2.9 *([14]) Let* $(X, v, \circledast)$ *be a PN-space. Then a sequence* $x = (x_k)$ *is said to be statistically convergent to* $L \in X$ *with respect to the probabilistic norm* v *provided that for every* $\varepsilon > 0$ *and* $\gamma \in (0, 1)$

$$\delta\left(\{k \in \mathbb{N} : v_{x_k - L}(\varepsilon) \le 1 - \gamma\}\right) = 0,$$

or equivalently

$$\lim_n \frac{1}{n} \left|\{k \le n : \; v_{x_k - L}(\varepsilon) \le 1 - \gamma\}\right| = 0.$$

In this case we write $st_N - \lim x = l$.

In [12], Gürdal and Pehlivan studied statistical convergence in 2-normed spaces and in 2-Banach spaces in [13]. In fact, Mursaleen [22] studied the concept of statistical convergence of sequences in random 2-normed space. In [7], Esi and Özdemir introduced and studied the concept of generalized Δ^m-statistical convergence of sequences in probabilistic normed space. Recently in [1] Savaş and Esi and [2] Esi and Savaş introduced and studied the concepts of statistical convergence and lacunary statistical convergence of triple sequences on probabilistic normed space, respectively.

In this paper we define and study λ_3-statistical convergence in probabilistic normed space using by λ_3 sequences which is quite a new and interesting idea to work with. We show that some properties of λ_3-statistical convergence of real numbers also hold for sequences in probabilistic normed spaces. We find some relations related to λ_3-statistical convergent sequences in probabilistic normed spaces. Also we find out the relation between λ_3-statistical convergent and λ_3-statistical Cauchy sequences in this spaces.

Definition 2.10 *([38]) A subset* K *of* $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ *is said to be natural density* δ_3 *if*

$$\delta_3(K) = \lim_{n,m,p} \frac{K(n,m,p)}{nmp} \text{ exists}$$

where $K(n,m,p)$ *denote the number of* (j,k,l) *in* K *such that*$j \le n, k \le m$ *and* $l \le p$.

Definition 2.11 *([38]) A real triple sequence* $x = (x_{jkl})$ *is said to be statistically convergent to the number* L *if for each* $\varepsilon > 0$

$$\delta_3\left(\{(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \; |x_{jkl} - L| \ge \varepsilon\}\right) = 0.$$

In this case we write $st_3 - \lim x = L$.

3 Main results

First we define the concept of λ_3-density.

Definition 3.1 *Let* $\lambda = (\lambda_n)$, $\mu = (\mu_m)$ *and* $w = (w_p)$ *be three non-decreasing sequences of positive real numbers each tending to infinity and such that* $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 0$; $\mu_{m+1} \leq \mu_m + 1$, $\mu_1 = 0$ *and* $w_{p+1} \leq w_p + 1$, $w_1 = 0$. *Let* $K \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. *The number*

$$\delta_{\lambda_3}(K) = \lim_{n,m,p} \frac{1}{\lambda_{nmp}} \left|\{j \in I_n, k \in J_m, l \in K_p : \ (j,k,l) \in K\}\right|$$

is said to be λ_3-*density of* K, *provided that the limit exists, where* $\lambda_{nmp} = \lambda_n \lambda_m \lambda_p$ *and* $I_n = [n - \lambda_n + 1, n]$, $J_m = [m - \lambda_m + 1, m]$, $K_p = [p - \lambda_p + 1, p]$.

We now ready to define the triple λ_3-statistical convergence.

Definition 3.2 *A triple sequence* $x = (x_{jkl})$ *is said to be triple* λ_3-*statistically convergent or* $S_{\lambda_3}-$*convergent to* L *if for every* $\varepsilon > 0$

$$\lim_{n,m,p} \frac{1}{\lambda_{nmp}} \left|\{j \in I_n, k \in J_m, l \in K_p : \ |x_{jkl} - L| \geq \varepsilon\}\right| = 0.$$

In this case we write $s_{\lambda_3} - \lim x = L$ *or* $x_{jkl} \to L\,(s_{\lambda_3})$ *and denote the set of all triple* $s_{\lambda_3}-$*statistically convergent sequences by* s_{λ_3}. *If* $\lambda_{nmp} = nmp$ *for all n,m,p then the set of* $s_{\lambda_3}-$*statistically convergent sequences reduces to the space* st_3.

Now we define the $s_{\lambda_3}-$convergence in PN-space.

Definition 3.3 *Let* $(X, v, *)$ *be a PN-space. We say that a triple sequence* $x = (x_{jkl})$ *is said to be* $s_{\lambda_3}-$*convergence to* L *in probabilistic normed space* X *(for short,* $s_{\lambda_3}^{(PN)}$ *− convergent), if for every* $\varepsilon > 0$ *and* $\eta \in (0,1)$,

$$\delta_{\lambda_3}\left(\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) \leq 1 - \eta\}\right) = 0$$

or equivalently

$$\delta_{\lambda_3}\left(\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) > 1 - \eta\}\right) = 1.$$

In this case we write $x_{jkl} \to L\left(s_{\lambda_3}^{(PN)}\right)$ *or* $s_{\lambda_3}^{(PN)} - \lim x = L$ *and denote the set of all* $s_{\lambda_3}^{(PN)}$ *− convergent triple sequences in probabilistic normed space by* $\left(s_{\lambda_3}^{(PN)}\right)_v$.

Theorem 3.4 *Let* $(X, v, *)$ *be a PN-space. If a sequence* $x = (x_{jkl})$ *is a triple* λ_3-*statistical convergent in probabilistic normed space* X, *then* $s_{\lambda_3}^{(PN)} - \lim x$ *is unique.*

Proof. Suppose that $s_{\lambda_3}^{(PN)} - \lim x = L_1$ and $s_{\lambda_3}^{(PN)} - \lim x = L_2$ $(L_1 \neq L_2)$. Let $\varepsilon > 0$ and $t > 0$. Choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) \geq 1 - t$. Then we define the following sets:

$$K_1(\eta, \varepsilon) = \left\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L_1}\left(\frac{\varepsilon}{2}\right) \leq 1 - \eta\right\},$$

$$K_1(\eta, \varepsilon) = \left\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L_2}\left(\frac{\varepsilon}{2}\right) \leq 1 - \eta\right\}.$$

Since $s_{\lambda_3}^{(PN)} - \lim x = L_1$ and $s_{\lambda_3}^{(PN)} - \lim x = L_2$ we have $\delta_{\lambda_3}(K_1(\eta, \varepsilon)) = 0$ and $\delta_{\lambda_3}(K_2(\eta, \varepsilon)) = 0$, respectively. Now let

$$K_3(\eta, \varepsilon) = K_1(\eta, \varepsilon) \cap K_1(\eta, \varepsilon).$$

It follows that $\delta_{\lambda_3}(K_3(\eta, \varepsilon)) = 0$ for all $\varepsilon > 0$ which implies $\delta_{\lambda_3}(\mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus K_3(\eta, \varepsilon)) = 1$. If $(j,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus K_3(\eta, \varepsilon)$, we have

$$v_{L_1 - L_2}(\varepsilon) = v_{(L_1 - x_{jkl}) + (x_{jkl} - L_2)}\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) \geq v_{(L_1 - x_{jkl})}\left(\frac{\varepsilon}{2}\right) * v_{(x_{jkl} - L_2)}\left(\frac{\varepsilon}{2}\right) > (1-\eta) * (1-\eta) \geq 1 - t.$$

Since $t > 0$ was arbitrary, we get $v_{L_1 - L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which gives $L_1 = L_2$. Hence $s_{\lambda_3}^{(PN)} - \lim x$ is unique.
■

Theorem 3.5 *Let $(X, v, *)$ be a PN-space. If a sequence $x = (x_{jkl})$ is a triple statistically convergent to L in probabilistic normed space X, then $s_{\lambda_3}^{(PN)} - \lim x = L$ if*

$$\lim \inf_{n,m,p} \frac{\lambda_{nmp}}{nmp} > 0. \tag{1}$$

Proof. Let $\varepsilon > 0$ and $\eta \in (0, 1)$. Then

$$\{j \le n, k \le m, l \le p : \ v_{jkl-L}(\varepsilon) \le 1 - \eta\} \supset \{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) \le 1 - \eta\}.$$

Therefore,

$$\begin{aligned} \frac{1}{nmp} \left|\{j \le n, k \le m, l \le p : \ v_{jkl-L}(\varepsilon) \le 1 - \eta\}\right| &\ge \frac{1}{nmp} \left|\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) \le 1 - \eta\}\right| \\ &\ge \frac{\lambda_{nmp}}{nmp} \frac{1}{nmp} \left|\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) \le 1 - \eta\}\right|. \end{aligned}$$

Taking the limit as $n, m, p \to \infty$ and using (1), we get $s_{\lambda_3}^{(PN)} - \lim x = L$. ■

Theorem 3.6 *Let $(X, v, *)$ be a PN-space. If $v - \lim x = L$ then $s_{\lambda_3}^{(PN)} - \lim x = L$.*

Proof. Let $v - \lim x = L$. Then for every $\eta \in (0, 1)$ and $\varepsilon > 0$, there is a triple $(j_o, k_o, l_o) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $v_{jkl-L}(\varepsilon) > 1 - \eta$ for all $j \ge j_o$, $k \ge k_o$, $l \ge l_o$. Hence the set $\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) \le 1 - \eta\}$ has natural density zero and hence

$$\delta_{\lambda_3}(\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) \le 1 - \eta\}) = 0,$$

that is $s_{\lambda_3}^{(PN)} - \lim x = L$. ■

The following example shows that the converse of this theorem does not hold in general.

Example 3.7 *Let $(\mathbb{R}, |.|)$ denote the space of real numbers with the usual norm. Let $a * b = ab$ and $v_x(\varepsilon) = \frac{\varepsilon}{\varepsilon + |x|}$, where $x \in X$ and $\varepsilon > 0$. In this case, we observe that $(\mathbb{R}, v, *)$ is a PN-space. Now we define a sequence $x = (x_{jkl})$ by*

$$x_{jkl} = \begin{cases} (j, k, l) & , \ \text{for } n - [\sqrt{n} + 1] \le j \le n, \ m - [\sqrt{m} + 1] \le k \le m, \ \text{and } p - \left[\sqrt{p} + 1\right] \le l \le p \\ 0 & , \ \text{otherwise} \end{cases}.$$

It is easy to see that, this sequence is $s_{\lambda_3}^{(PN)}-$convergent to zero in PN-space but $v - \lim x \ne 0$.

Theorem 3.8 *Let $(X, v, *)$ be a PN-space. Then $s_{\lambda_3}^{(PN)} - \lim x = L$ if and only if there exists a subset $K = \{(j, k, l) : j, k, l = 1, 2, 3, ...\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $\delta_{\lambda_3}(K) = 1$ and $v - \lim_{\substack{j,k,l \to \infty \\ (j,k,l) \in K}} x_{jkl} = L$.*

Proof. We first assume that $s_{\lambda_3}^{(PN)} - \lim x = L$. Now, for any $\varepsilon > 0$ and $r \in \mathbb{N}$, let

$$K(r, \varepsilon) = \left\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) \le 1 - \frac{1}{r}\right\}$$

and

$$M(r, \varepsilon) = \left\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) > 1 - \frac{1}{r}\right\}.$$

Then $\delta_{\lambda_3}(K(r, \varepsilon)) = 0$ and

$$\begin{array}{c} M(1, \varepsilon) \supset M(2, \varepsilon) \supset ... \supset M(i, \varepsilon) \supset M(i + 1, \varepsilon) \supset ... \\ \delta_{\lambda_3}(M(r, \varepsilon)) = 1, \ r = 1, 2, 3, ... \end{array}. \tag{2}$$

Now we have to show that for $(j, k, l) \in M(r, \varepsilon)$, $x = (x_{jkl})$ is $v-$convergent to L. Suppose that $x = (x_{jkl})$ is not $v-$convergent to L. Therefore there is $\eta > 0$ such that $v_{jkl-L}(\varepsilon) \le 1 - \eta$ for finitely many terms. Let

$$M(\eta, \varepsilon) = \{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) \ge 1 - \eta\} \text{ and } \lambda > \frac{1}{\eta}.$$

Then $\delta_{\lambda_3}(M(\eta,\varepsilon)) = 0$ which contradicts (2). Therefore $x = (x_{jkl})$ is $v-$convergent to L.

Conversely, suppose that there exists a subset $K = \{(j,k,l) : j,k,l = 1,2,3,...\} \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $\delta_{\lambda_3}(K) = 1$ and $v - \lim_{\substack{j,k,l\to\infty \\ (j,k,l)\in K}} x_{jkl} = L$, that is there exists $k_o \in \mathbb{N}$ such that for every $\eta \in (0,1)$ and $\varepsilon > 0$ $v_{jkl-L}(\varepsilon) > 1-\eta$, for all $j,k,l \geq k_o$. Now

$$M(\eta,\varepsilon) = \{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}(\varepsilon) \leq 1-\eta\} \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N} - \left\{ \left(j_{k_o+1}, k_{k_o+1}, l_{k_{o+1}}\right), \left(j_{k_o+2}, k_{k_o+2}, l_{k_{o+2}}\right), ...\right\}.$$

Therefore $\delta_{\lambda_3}(M(\eta,\varepsilon)) \leq 1-1 = 0$. Hence, we conclude that $s_{\lambda_3}^{(PN)} - \lim x = L$.

This completes the proof. ■

In the next we now define triple $\lambda-$statistically Cauchy sequence in probabilistic normed space.

Definition 3.9 *Let $(X,v,*)$ be a PN-space. A triple sequence $x = (x_{jkl})$ is said to be $s_{\lambda_3}^{(PN)}-$ Cauchy in PN-space X if for every $\varepsilon > 0$ and $\eta \in (0,1)$, there exist $N = N(\varepsilon)$, $M = M(\varepsilon)$ and $Z = Z(\varepsilon)$ such that for all $j,p \geq N$, $k,q \geq M$ and $l,u \geq Z$*

$$\delta_{\lambda_3}(\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-pqu}(\varepsilon) \leq 1-\eta\}) = 0.$$

Theorem 3.10 *Let $(X,v,*)$ be a PN-space and $x = (x_{jkl}) \in X$. Then $x = (x_{jkl})$ is $s_{\lambda_3}^{(PN)}-$convergent if and only if it is $s_{\lambda_3}^{(PN)}-$Cauchy in X.*

Proof. Let $x = (x_{jkl})$ be $s_{\lambda_3}^{(PN)}-$convergent to L, i.e., $s_{\lambda_3}^{(PN)} - \lim x = L$. Then for a given $\varepsilon > 0$ and $\eta \in (0,1)$, choose $t > 0$ such that $(1-\eta)*(1-\eta) \geq 1-t$. Then, we have

$$\delta_{\lambda_3}(M(\eta,\varepsilon)) = \delta_{\lambda_3}\left(\left\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}\left(\frac{\varepsilon}{2}\right) \leq 1-\eta\right\}\right) = 0 \tag{3}$$

which implies that

$$\delta_{\lambda_3}\left(M^C(\eta,\varepsilon)\right) = \delta_{\lambda_3}\left(\left\{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-L}\left(\frac{\varepsilon}{2}\right) > 1-\eta\right\}\right) = 1.$$

Let $(p,q,u) \in M^C(\eta,\varepsilon)$. Then $v_{pqu-L}\left(\frac{\varepsilon}{2}\right) > 1-\eta$. Now let

$$K(t,\varepsilon) = \{j \in I_n, k \in J_m, l \in K_p : \ v_{jkl-pqu}(\varepsilon) \leq 1-t\}.$$

We need to show that $K(\eta,\varepsilon) \subset M(\eta,\varepsilon)$. Let $(j,k,l) \in K(\eta,\varepsilon) - M(\eta,\varepsilon)$. Then we have $v_{jkl-pqu}(\varepsilon) \leq 1-t$ and $v_{jkl-L}\left(\frac{\varepsilon}{2}\right) > 1-\eta$, in particular $v_{pqu-L}\left(\frac{\varepsilon}{2}\right) > 1-\eta$. Then

$$1-t \geq v_{jkl-pqu}(\varepsilon) \geq v_{jkl-L}\left(\frac{\varepsilon}{2}\right) * v_{pqu-L}\left(\frac{\varepsilon}{2}\right) > (1-\eta)*(1-\eta) \geq 1-t$$

which is not possible. Hence $K(\eta,\varepsilon) \subset M(\eta,\varepsilon)$. Therefore, by (3), $\delta_{\lambda_3}(K(\eta,\varepsilon)) = 0$. Hence $x = (x_{jkl})$ is $s_{\lambda_3}^{(PN)}-$Cauchy in X.

Conversely, $x = (x_{jkl})$ is $s_{\lambda_3}^{(PN)}-$Cauchy in X but it is not $s_{\lambda_3}^{(PN)}-$convergent. Now

$$v_{jkl-pqu}(\varepsilon) \geq v_{jkl-L}\left(\frac{\varepsilon}{2}\right) * v_{pqu-L}\left(\frac{\varepsilon}{2}\right) > (1-\eta)*(1-\eta) \geq 1-t,$$

since $x = (x_{jkl})$ is not $s_{\lambda_3}^{(PN)}-$convergent. Therefore $\delta_{\lambda_3}\left(K^C(t,\varepsilon)\right) = 0$ and so $\delta_{\lambda_3}(K(t,\varepsilon)) = 1$, which is contradiction, since $x = (x_{jkl})$ was $s_{\lambda_3}^{(PN)}-$Cauchy in X. Hence $x = (x_{jkl})$ must be $s_{\lambda_3}^{(PN)}-$convergent in X . This completes the proof. ■

References

[1] E. Savaş and A. Esi, Statistical convergence of triple sequences on probabilistic normed space, Annals of the University of Craiova, Mathematics and Computer Science Series, **39(2)**(2012), 226-236.

[2] A. Esi and E. Savaş, On lacunary statistically convergent triple sequences in probabilistic normed space, submitted.

[3] E. Savaş and S. A. Mohiuddine, $\overline{\lambda}$-statistically convergent double sequences in probabilistic normed spaces, Math. Slovaca, **62(1)**(2012), 99-108.

[4] C. Alsina, B. Schweizer, A. Sklar, Continuity properties of probabilistic norms, J. Math. Anal. Appl. 208 (1997), 446–452.

[5] H. Çakalli, A study on statistical convergence, Funct. Anal. Approx. Comput., 1(2)(2009), 19-24, MR2662887.

[6] J. Connor, M.A. Swardson, Measures and ideals of $C^*(X)$, Ann. N.Y. Acad.Sci.704(1993), 80-91.

[7] A. Esi, M. K. Özdemir, "Generalized Δ^m-Statistical convergence in probabilistic normed space, Jour. Comput. Anal. Appl., 13(5)(2011), 923-932.

[8] H. Fast, Sur la convergence statistique, Colloq. Math. 2(1951) 241-244.

[9] J. A. Fridy, On statistical convergence, Analysis, 5(1985) 301-313.

[10] S. Gähler, 2-metrische Raume and ihre topologische Struktur, Math. Nachr. 26(1963) 115-148.

[11] I . Goleţ , On probabilistic 2-normed spaces, Novi Sad J. Math. 35 (2006), 95–102.

[12] M. Gürdal and S. Pehlivan, Statistical convergence in 2-normed spaces, South. Asian Bull. Math.33, (2009), 257-264.

[13] M. Gürdal, S. Pehlivan, The statistical convergence in 2-Banach spaces, Thai J. Math. 2 (1), (2004), 107–113.

[14] S. Karakus, Statistical convergence on probabilistic normed spaces, Math. Commun. 12 (2007), 11–23.

[15] S. Karakus, K. Demirci and O. Duman, Statistical convergence on intuitionistic fuzzy normed spaces, Chaos, Solitons and Fractals, 35 (2008), 763-769.

[16] V. Kumar and M. Mursaleen, On (λ, μ)-statistical convergence of double sequences on intuitionistic fuzzy normed spaces, Filomat, 25(2) (2011) 109-120.

[17] L. Leindler, Uber die de la Vallee-Pousinsche Summierbarkeit allgenmeiner Othogonalreihen, Acta Math. Acad. Sci. Hungar, 16(1965), 375-387.

[18] I. J. Maddox, Statistical convergence in a locally convex space, Math. Proc. Cambridge Philos. Soc., 104(1)(1988), 141-145.

[19] G. D. Maio, L. D. R. Kocinac, Statistical convergence in topology, Topology Appl. 156, (2008), 28-45.

[20] K. Menger, Statistical metrics, Proc. Natl. Acad. Sci. USA 28 (1942), 535–537.

[21] H. I. Miller, A measure theoretical subsequence characterization of statistical convergence, Trans. Amer. Math. Soc., 347(5)(1995), 1811-1819.

[22] M. Mursaleen, Statistical convergence in random 2-normed spaces, Acta Sci.Math.(Szeged), 76(1-2)(2010), 101-109.

[23] M. Mursaleen, $\lambda-$statistical convergence, Mathematica Slovaca, 50(1)(2000), 111-115.

[24] M. Mursaleen and A. Alotaibi, Statistical summability and approximation by de la Vallee-Pousin mean, Applied Math. Letters, 24 (2011) 320-324.

[25] M. Mursaleen, C. Çakan, S. A. Mohiuddine and E. Savaş, Generalized statistical convergence and statitical core of double sequences, Acta Math. Sinica, 26(11)(2010) 2131-2144.

[26] M. Mursaleen and Osama H. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl., 288(2003) 223-231.

[27] M. Mursaleen and Osama H. H. Edely, Generalized statistical convergence, Information Sciences , 162(2004), 287-294.

[28] M. Mursaleen and Osama H. H. Edely, On the invariant mean and statistical convergence, Appl. Math. Letters,22(2009) 1700-1704.

[29] S. A. Mohiuddine and Q. M. Danish Lohani, On generalized statistical convergence in intuitionistic fuzzy normed space, Chaos, Solitons and Fractals, 42 (2009) 1731-1737.

[30] M. Mursaleen and S. A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, Jour. Comput. Appl. Math., 233(2)(2009) 142-149.

[31] T. Šalát, On statistical convergence of real numbers, Math. Slovaca, 30(1980), 139-150.

[32] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66(1959) 361-375.

[33] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 313–334.

[34] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North Holland, New York- Amsterdam-Oxford, 1983.

[35] C. Sempi, A short and partial history of probabilistic normed spaces, Mediterr. J. Math. 3(2006), 283-300.

[36] C. Sencimen and S. Pehlivan, Statistical convergence in fuzzy normed linear spaces, Fuzzy Sets and Systems, 159 (2008), 361-370.

[37] A. N. Serstnev, On the notion of a random normed space, Dokl. Akad. Nauk SSSR 149(1963), 280-283.

[38] R. F. Patterson and E. Savas, Lacunary statistical convergence of double sequences, *Math. Commun.*, **10**(2005), no. 1, 55–61.

[39] A. Alotaibi, Generalized statistical convergence in probabilistic normed spaces, Open Math.J., **1**(2008), 82-88.

[40] A. Asadollah and K. Nourouzi, Convex sets in probabilistic normed spaces, Chaos, Solitons and Fractals, 36(2)(2008), 322-328.