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Generalized Identities of Companion Fibonacci-Like Sequences

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Abstract

Sequences have been fascinating topic for mathematicians for centuries. Fibonacci sequence, Lucas sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal sequence and Jacobsthal-Lucas sequence are most prominent examples of second order recursive sequences. In this paper, companion Fibonacci-Like sequences are introduced and presented generalized identities of companion Fibonacci-Like sequences with Jacobsthal and Jacobsthal-Lucas sequences.

Keywords: *Fibonacci sequence, Lucas sequence, Companion Fibonacci-Like sequences.*

1 Introduction

It is well known that generalized Fibonacci and Lucas numbers play an important role in many subjects such as algebra, geometry and number theory. Their various elegant properties and wide applications have been studied by many authors.
The generalization of Fibonacci sequence gives new direction for research. Fibonacci sequence and their generalization have many interesting properties and applications in almost every field of science and art [14] below [14]. Authors [1], [6], [13] have been generalized the Fibonacci sequence by preserving the recurrence relation and altering the initial conditions of the sequence, while authors [7],[10],[12] have been generalized the Fibonacci sequence by preserving the initial conditions of the sequence but altering the recurrence relation slightly. Further, authors [2],[8],[9] have been generalized Fibonacci sequence by changing recurrence relation and initial conditions both.

The Fibonacci sequence [14]is defined by the recurrence relation,

$$F_n = F_{n-1} + F_{n-2},\ n \ge 2 \text{ with } F_0 = 0,\ F_1 = 1. \tag{1.1}$$

Lucas sequence [14] is defined by the recurrence relation,

$$L_n = L_{n-1} + L_{n-2},\ n \ge 2 \text{ with } L_0 = 2,\ L_1 = 1. \tag{1.2}$$

The Pell sequence [4] is defined by the recurrence relation,

$$P_n = 2P_{n-1} + P_{n-2},\ n \ge 2 \text{ with } P_0 = 0,\ P_1 = 1. \tag{1.3}$$

The companion Pell or Pell-Lucas sequence [4] is defined by the recurrence relation,

$$Q_n = 2Q_{n-1} + Q_{n-2},\ n \ge 2 \text{ with } Q_0 = 2,\ Q_1 = 2. \tag{1.4}$$

The Jacobsthal sequence [3] is defined by the recurrence relation,

$$J_n = J_{n-1} + 2J_{n-2},\ n \ge 2 \text{ with } J_0 = 0,\ J_1 = 1. \tag{1.5}$$

The Jacobsthal-Lucas sequence [3] is defined by the recurrence relation,

$$j_n = j_{n-1} + 2j_{n-2},\ n \ge 2 \text{ with } j_0 = 2,\ j_1 = 1. \tag{1.6}$$

The Fibonacci- Like sequence [5] is defined by recurrence relation,

$$S_{n+2} = S_{n+1} + S_n \ , \ n \geq 0 \ \text{ with } S_0 = 2, S_1 = 2. \tag{1.7}$$

First few numbers of Fibonacci-Like sequences are 2, 2, 4, 6, 10, 16, and so on.

Two companion Fibonacci-Like sequences $\{A_n\}$ and $\{C_n\}$ are introduced by authors [11]. The companion Fibonacci-Like sequences are not only self-related but also related to Jacobsthal and Jacobsthal-Lucas sequences.

In this paper we state and derive generalized identities of companion Fibonacci- Like sequences with Jacobsthal and Jacobsthal- Lucas sequences.

2 Preliminaries

Companion Fibonacci-Like sequence $\{A_k\}$ [11] is defined by the recurrence relation,

$$A_{k+2} = A_{k+1} + 2A_k \quad \text{for } k \geq 0 \text{ with } \quad A_0 = 2 \ , A_1 = 2. \tag{2.1}$$

The first few terms of sequence $\{A_k\}$ are 2, 2, 6, 10, 22, 42, and so on.

Companion Fibonacci-Like sequence $\{C_k\}$ [11] is defined by the recurrence relation,

$$C_{k+2} = C_{k+1} + 2C_k \quad \text{for } k \geq 0 \text{ with } C_0 = 2 \ , \ C_1 = 0. \tag{2.2}$$

The first few terms of sequence $\{C_k\}$ are 2, 0, 4, 4, 12, 20, and so on.

The characteristic equation for both is $x^2 - x - 2 = 0.$, which has two real roots $\mu_1 = 2$ and $\mu_2 = -1.$
Also $\mu_1 + \mu_2 = 1,\ \mu_1 - \mu_2 = 3,\ \mu_1\mu_2 = -2,\ \mu_1^2 = 4,\ \mu_2^2 = 1$ and $\mu_1 - 1 = -\mu_2,\ \mu_2 - 1 = -\mu_1.$

The generating function for companion Fibonacci-Like sequence $\{A_k\}$ [11] is given by

$$\sum_{k=0}^{\infty} A_k x^k = \frac{2}{1 - x - 2x^2}. \tag{2.3}$$

The generating function for companion Fibonacci-Like sequence $\{C_k\}$ [11] is given by

$$\sum_{k=0}^{\infty} C_k x^k = \frac{2(1-x)}{1 - x - 2x^2}. \tag{2.4}$$

The Binet's formula for companion Fibonacci-Like sequence $\{A_k\}$ [11] is given by

$$A_k = 2\,\frac{\mu_1^{k+1} - \mu_2^{k+1}}{\mu_1 - \mu_2}. \tag{2.5}$$

The Binet's formula for companion Fibonacci-Like sequence $\{C_k\}$ [11] is given by

$$C_k = 4\,\frac{\mu_1^{k-1} - \mu_2^{k-1}}{\mu_1 - \mu_2} \tag{2.6}$$

The Binet's formula for Jacobsthal sequence [3] is given by

$$J_k = \frac{\mu_1^k - \mu_2^k}{\mu_1 - \mu_2} \tag{2.7}$$

The Binet's formula for Jacobsthal-Lucas sequence [3] is given by

$$j_k = \mu_1^k + \mu_2^k. \tag{2.8}$$

3 Generalized Identities

Now we present generalized identities of companion Fibonacci- Like sequences with Jacobsthal and Jacobsthal- Lucas sequences.

Theorem 3.1: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p} j_{2k+p} = A_{4k+2p} + 2(-2)^{2k+p},$

(ii) $C_{2k+p} j_{2k+p} = C_{4k+2p} + 2(-2)^{2k+p}.$

Proof (i): By Binet's formula (2.5) and (2.8), we have

$$A_{2k+p} j_{2k+p} = 2\left(\frac{\mu_1^{2k+p+1} - \mu_2^{2k+p+1}}{\mu_1 - \mu_2} \right)\left(\mu_1^{2k+p} + \mu_2^{2k+p}\right)$$

$$= \frac{2}{\mu_1 - \mu_2}\left(\mu_1^{4k+2p+1} - \mu_2^{4k+2p+1} \right) + \frac{2}{\mu_1 - \mu_2}\left(\mu_1\mu_2\right)^{2n+p}\left(\mu_1 - \mu_2\right)$$

$$A_{2k+p} j_{2k+p} = A_{4k+2p} + 2(-2)^{2k+p}.$$

Proof (ii): By Binet's formula (2.6) and (2.8), we have

$$C_{2k+p} j_{2k+p} = 4\left(\frac{\mu_1^{2k+p-1} - \mu_2^{2k+p-1}}{\mu_1 - \mu_2} \right)\left(\mu_1^{2k+p} + \mu_2^{2k+p}\right)$$

$$= \frac{4}{\mu_1 - \mu_2}\left(\mu_1^{4k+2p-1} - \mu_2^{4k+2p-1} \right) + \frac{4}{\mu_1 - \mu_2}\left(\mu_1\mu_2\right)^{2k+p}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)$$

$$C_{2k+p} j_{2k+p} = C_{4k+2p} + 2(-2)^{2k+p}.$$

Corollary 3.2: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p} j_{2k+p} = 2J_{4k+2p+1} + 2(-2)^{2k+p},$

(ii) $C_{2k+p} j_{2k+p} = 4J_{4k+2p-1} + 2(-2)^{2k+p}.$

Proof (i): By Binet's formula (2.5) and (2.8), we have

$$A_{2k+p} j_{2k+p} = 2\left(\frac{\mu_1^{2k+p+1} - \mu_2^{2k+p+1}}{\mu_1 - \mu_2} \right)\left(\mu_1^{2k+p} + \mu_2^{2k+p}\right)$$

$$= \frac{2}{\mu_1 - \mu_2}\left(\mu_1^{4k+2p+1} - \mu_2^{4k+2p+1} \right) + \frac{2}{\mu_1 - \mu_2}\left(\mu_1\mu_2\right)^{2k+p}\left(\mu_1 - \mu_2\right).$$

From (2.7), it follows that

$$A_{2k+p} j_{2k+p} = 2J_{4k+2p+1} + 2(-2)^{2k+p}.$$

Proof (ii): By Binet's formula (2.6) and (2.8), it follows that

$$C_{2k+p} j_{2k+p} = 4\left(\frac{\mu_1^{2k+p-1} - \mu_2^{2k+p-1}}{\mu_1 - \mu_2} \right)\left(\mu_1^{2k+p} + \mu_2^{2k+p}\right)$$

$$= \frac{4}{\mu_1 - \mu_2}\left(\mu_1^{4k+2p-1} - \mu_2^{4k+2p-1} \right) + \frac{4}{\mu_1 - \mu_2}\left(\mu_1\mu_2\right)^{2k+p}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right).$$

From (2.7), it follows that

$$C_{2k+p} j_{2k+p} = 4J_{4k+2p-1} + 2(-2)^{2k+p}.$$

Following identities can be derived same as theorem (3.1) and corollary (3.2).

Theorem 3.3: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k-p} j_{2k-p} = A_{4k-2p} + 2(-2)^{2k-p},$

(ii) $C_{2k-p} j_{2k-p} = C_{4k-2p} - 4(-2)^{2k-p-1}.$

Corollary 3.4: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k-p} j_{2k-p} = 2J_{4k-2p+1} + 2(-2)^{2k-p},$

(ii) $C_{2k-p} j_{2k-p} = 4J_{4k-2p-1} - 4(-2)^{2k-p-1}.$

Theorem 3.5: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p} j_{2k-p} = A_{4k} + (-2)^{2k-p} A_{2p},$

(ii) $C_{2k+p} j_{2k-p} = C_{4k} + (-2)^{2k-p} C_{2p}.$

Corollary 3.6: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p} j_{2k-p} = 2\left[J_{4k+1} + (-2)^{2k-p} J_{2p+1}\right],$

(ii) $C_{2k+p} j_{2k-p} = 4\left[J_{4k-1} + (-2)^{2k-p} J_{2p-1}\right].$

Theorem 3.7: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k-p} j_{2k+p} = A_{4k} - (-2)^{2k-p+1} A_{2p-2},$

(ii) $C_{2k-p} j_{2k+p} = C_{4k} - (-2)^{2k-p-1} C_{2p+2}.$

Corollary 3.8: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k-p} j_{2k+p} = 2\left[J_{4k+1} - (-2)^{2k-p+1} J_{2p-1}\right],$

(ii) $C_{2k-p} j_{2k+p} = 4\left[J_{4k-1} - (-2)^{2k-p-1} J_{2p+1}\right].$

Theorem 3.9: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k} j_{2k+p} = A_{4k+p} - 2(-2)^{2k+1} A_{p-2},$

(ii) $C_{2k} j_{2k+p} = C_{4k+p} - (-2)^{2k-1} C_{p+2}.$

Corollary 3.10: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k} j_{2k+p} = 2\left[J_{4k+p+1} - (-2)^{2k+1} J_{p-1}\right],$

(ii) $C_{2k} j_{2k+p} = 4\left[J_{4k+p-1} - (-2)^{2k-1} J_{p+1}\right].$

Theorem 3.11: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p} j_{2k} = A_{4k+p} + 4^k A_p,$

(ii) $C_{2k+p} j_{2k} = C_{4k+p} + 4^k C_p.$

Corollary 3.12: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p} j_{2k} = 2J_{4k+p+1} + 2^{2k+1} J_{p+1},$

(ii) $C_{2k+p} j_{2k} = 4J_{4k+p-1} + 4^{k+1} J_{p-1}.$

Theorem 3.13: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k} j_{2k+p+1} = A_{4k+p+1} - (-2)^{2k+1} A_{p-1},$

(ii) $C_{2k} j_{2k+p+1} = C_{4k+p+1} - (-2)^{2k-1} C_{p+3}.$

Corollary 3.14: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k} j_{2k+p+1} = 2\left[J_{4k+p+2} - (-2)^{2k+1} J_p\right],$

(ii) $C_{2k} j_{2k+p+1} = 4\left[J_{4k+p} - (-2)^{2k-1} J_{p+2}\right].$

Theorem 3.15: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p+1} j_{2k} = A_{4k+p+1} + 4^k A_{p+1},$

(ii) $C_{2k+p+1} j_{2k} = C_{4k+p+1} + 4^k C_{p+1}.$

Corollary 3.16: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p+1} j_{2k} = 2\left[J_{4k+p+2} + 4^k J_{p+2}\right],$

(ii) $C_{2k+p+1} j_{2k} = 4\left[J_{4k+p+1} + 4^k J_p\right].$

Theorem 3.17 For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k} j_{2k+p-1} = A_{4k+p-1} - (-2)^{2k+1} A_{p-3},$

(ii) $C_{2k} j_{2k+p-1} = C_{4k+p-1} - (-2)^{2k-1} C_{p+1}.$

Corollary 3.18: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k} j_{2k+p-1} = 2\left[J_{4k+p} - (-2)^{2k+1} J_{p-2}\right],$

(ii) $C_{2k} j_{2k+p-1} = 4\left[J_{4k+p-2} - (-2)^{2k-1} J_p\right].$

Theorem 3.19: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p-1} j_{2k} = A_{4k+p-1} + 4^k A_{p-1},$

(ii) $C_{2k+p-1} j_{2k} = C_{4k+p-1} + 4^k C_{p-1}.$

Theorem 3.20: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p-1} j_{2k} = 2\left[J_{4k+p} + 4^k J_p\right],$

(ii) $C_{2k+p-1} j_{2k} = 4\left[J_{4k+p-2} + 4^k J_{p-2}\right].$

Theorem 3.21 For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p} j_{2k+p+1} = A_{4k+2p+1},$

(ii) $C_{2k+p} j_{2k+p+1} = C_{4k+2p+1} - 4(-2)^{2k+p-1}.$

Corollary 3.22: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p} j_{2k+p+1} = 2J_{4k+2p+2},$

(ii) $C_{2k+p} j_{2k+p+1} = 4J_{4k+2p} - 4(-2)^{2k+p-1}.$

Theorem 3.23: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p+1} j_{2k+p} = A_{4k+2p+1} + 2(-2)^{2k+p},$

(ii) $C_{2k+p+1} j_{2k+p} = C_{4k+2p+1}.$

Corollary 3.24: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p+1} j_{2k+p} = 2J_{4k+2p+2} + 2(-2)^{2k+p},$

(ii) $C_{2k+p+1} j_{2k+p} = 4J_{4k+2p}.$

Theorem 3.25: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p}\, j_{2k+p-1} = A_{4k+2p-1} + 2(-2)^{2k+p-1},$

(ii) $C_{2k+p}\, j_{2k+p-1} = C_{4k+2p-1}.$

Corollary 3.26: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p}\, j_{2k+p-1} = 2J_{4k+2p} + 2(-2)^{2k+p-1},$

(ii) $C_{2k+p}\, j_{2k+p-1} = 4J_{4k+2p-2}.$

Theorem 3.27: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p-1}\, j_{2k+p} = A_{4k+2p-1}.$

(ii) $C_{2k+p-1}\, j_{2k+p} = C_{4k+2p-1} - 4(-2)^{4k+p-2}.$

Corollary 3.28: For positive integer $k \geq 1,\ p \geq 0,$ prove that

(i) $A_{2k+p-1}\, j_{2k+p} = 2J_{4k+2p},$

(ii) $C_{2k+p-1}\, j_{2k+p} = 4J_{4k+2p-2} - 4(-2)^{4k+p-2}.$

4 Conclusion

In this paper we have stated and derived generalized identities of companion Fibonacci-Like sequences with Jacobsthal and Jacobsthal-Lucas sequences. Mainly Binet's formulae are used for derivation of all the identities.

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