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Research paper

On Properties of meromorphic solutions of difference Painlevé I and II equation

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Abstract

In this paper, we investigate some properties of finite order transcendental meromorphic solutions of difference Painlevé I and II equations, and obtain precise estimations of exponents of convergence of poles of difference $\Delta w(z) = w(z+1) - w(z)$ and divided difference $\frac{\Delta w(z)}{w(z)}$, and of fixed points of $w(z+\eta)$ $(\eta \in \mathbb{C}\setminus\{0\})$.

Keywords: *Difference; Divided Difference; Difference Painlevé Equations; Meromorphic Function*

1. Introduction and Results

In this paper, we assume that the reader is familiar with the basic Nevanlinna's value distribution theory of meromorphic functions (see[2, 10]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote, respectively, the exponent of convergence of zeros and poles of $f(z)$. We also use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$ which is defined as

$$\tau(f) = \limsup_{r\to\infty} \frac{\log N\left(r, \frac{1}{f(z)-z}\right)}{\log r}.$$

We denote by $S(r, f)$ any quantify satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$, possibly outside a set with finite measure.
Recently, a number of papers (including $[3-9, 11-14]$) have focused on complex difference equations and difference analogues of Nevanlinna's theory. As the difference analogues of Nevanlinna's theory are investigated [8,12], many results on the complex difference equations are rapidly obtained.
Halburd and Korhonen [9] used value distribution theory and a reasoning related to the singularity confinement to single out the difference Painlevé I and II equations from difference equation

$$w(z+1) + w(z-1) = R(z, w), \tag{1.1}$$

where R is rational in both of its arguments. They proved that if (1.1) has an admissible meromorphic solutions of finite order, then either w satisfies a difference Riccati equation, or (1.1) may be transformed into some classical difference equations, which include difference Painlevé I equations

$$w(z+1) + w(z-1) = \frac{az+b}{w(z)} + c, \tag{1.2}$$

$$w(z+1) + w(z-1) = \frac{az+b}{w(z)} + \frac{c}{w^2(z)}, \tag{1.3}$$

$$w(z+1) + w(z) + w(z-1) = \frac{az+b}{w(z)} + c, \tag{1.4}$$

and difference Painlevé II equation

$$w(z+1) + w(z-1) = \frac{(az+b)w(z)+c}{1-w^2(z)}, \tag{1.5}$$

where a, b and c are constants.
In 2010, Chen and Shon [13] investigated some properties of meromorphic solutions of difference Painlevé I equation (1.2) and difference Painlevé II equation (1.5), and proved the following results.
Theorem B. (See [13]) *Let a, b, c be constants with $ac \neq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé II equation* (1.5)*, then:*
(i) $w(z)$ has at most one non-zero finite Borel exceptional value;
(ii) $\lambda\left(\frac{1}{w}\right) = \lambda(w) = \sigma(w)$;
(iii) $w(z)$ has infinitely many fixed points and satisfies $\tau(w) = \sigma(w)$.
Theorem C. (See [13]) *Let a, b, c be constants with $a \neq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation* (1.2)*, then*
(i) $w(z)$ has at most one non-zero finite Borel exceptional value;
(ii) $\lambda\left(\frac{1}{w}\right) = \lambda(w) = \sigma(w)$;
(iii) $w(z)$ has infinitely many fixed points and satisfies $\tau(w) = \sigma(w)$.
In 2011, Chen [14] investigated some properties of meromorphic solutions of difference Painlevé I equation (1.3) and obtained the following result.
Theorem D. (See [14]) *Let a, b, c be constants such that $ac \neq 0$. Suppose that $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation* (1.3)*, then*
(i) $w(z)$ has no any Borel exceptional value;
(ii) If $p(z)$ is a non-constant polynomial, then $w(z) - p(z)$ has infinitely many zeros and satisfies $\lambda(w-p) = \sigma(w)$.

In 2012, Chen and Chen [7] investigated some properties of meromorphic solutions of difference Painlevé I equation (1.4) and proved the following result.

Theorem E. (See [7]) *Let a,b,c be constants such that $|a|+|b|\neq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.4), then:*
(i) $\lambda\left(\frac{1}{w}\right)=\lambda(w)=\sigma(w)$;
(ii) If $p(z)$ is a non-constant polynomial, then $w(z)-p(z)$ has infinitely many zeros and satisfies $\lambda(w-p)=\sigma(w)$.
(iii) If $a\neq 0$, then $w(z)$ has no Borel exceptional value;
If $a=0$, then the Borel exceptional value of $w(z)$ can only come from a set $E=\{z|3z^2-cz-b=0\}$.

In this paper, we consider some properties of difference and divided difference of transcendental meromorphic solutions of the difference Painlevé I equations $(1.2)-(1.4)$ and Painlevé II equation (1.5), and obtain the following results.

Theorem 1.1. *Let a,b,c be constants with $|a|+|b|\neq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.2), then*
(i) $\lambda\left(\frac{1}{\Delta w}\right)=\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$;
(ii) For any $\eta\in\mathbb{C}\setminus\{0\}$, $\tau(w(z+\eta))=\sigma(w)$.

Theorem 1.2. *Let a,b,c be constants with $|a|+|b|+|c|\neq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.3), then*
(i) $\lambda\left(\frac{1}{\Delta w}\right)=\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$;
(ii) For any $\eta\in\mathbb{C}\setminus\{0\}$, $\tau(w(z+\eta))=\sigma(w)$.

Theorem 1.3. *Let a,b,c be constants with $|a|+|b|\neq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé I equation (1.4), then $\lambda\left(\frac{1}{\Delta w}\right)=\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$.*

Theorem 1.4. *Let a,b,c be constants with $|a|+|b|+|c|\neq 0$. If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé II equation (1.5), then*
(i) $\lambda\left(\frac{1}{\Delta w}\right)=\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$;
(ii) For any $\eta\in\mathbb{C}\setminus\{0\}$, $\tau(w(z+\eta))=\sigma(w)$.

Remark 1.1. *From the proofs of Theorems $1.1-1.4$, we can also obtain that $\lambda\left(\frac{1}{w}\right)=\sigma(w)$ and $\sigma\left(\frac{\Delta w}{w}\right)=\sigma(\Delta w)=\sigma(w)$.*

Remark 1.2. *Generally, $\tau(w(z+\eta))\neq\tau(w(z))$, where $\eta\in\mathbb{C}\backslash\{0\}$. For example, $w(z)=e^z+z$, $w(z+1)=ee^z+z+1$, $w(z)$ has no any fixed points and $\tau(w(z))=0$, but $w(z+1)$ has infinitely many fixed points and satisfies $\tau(w(z+1))=\sigma(w(z))=1$.*

Example 1.1. *The meromorphic function $w(z)=\frac{e^{i\pi z}-1}{e^{i\pi z}+1}$ satisfies the difference Painlevé I equation*

$$w(z+1)+w(z-1)=\frac{2}{w(z)},$$

with $a=c=0,b=2$ satisfying $|a|+|b|=2(\neq 0)$. We see that

$$\Delta w(z)=\frac{e^{i\pi(z+1)}-1}{e^{i\pi(z+1)}+1}-\frac{e^{i\pi z}-1}{e^{i\pi z}+1}=\frac{4e^{i\pi z}}{e^{i2\pi z}-1},$$

$$\frac{\Delta w(z)}{w(z)}=\frac{4e^{i\pi z}}{e^{i2\pi z}-1}\cdot\frac{e^{i\pi z}+1}{e^{i\pi z}-1}=\frac{4e^{i\pi z}}{(e^{i\pi z}-1)^2},$$

$$w(z+\eta)-z=\frac{e^{i\pi(z+\eta)}-1}{e^{i\pi(z+\eta)}+1}-z=\frac{e^{i\pi(z+\eta)}(1-z)-(z+1)}{e^{i\pi(z+\eta)}+1}.$$

Then, $\lambda\left(\frac{1}{\Delta w}\right)=\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)=1,\quad\lambda(\Delta w)=\lambda\left(\frac{\Delta w}{w}\right)=0$. For any $\eta\in\mathbb{C}\setminus\{0\}$, we have $\tau(w(z+\eta))=\sigma(w)=1$.

Example 1.2. *The meromorphic function $w(z)=\frac{1}{e^{i2\pi z}+z+1}$ satisfies the difference Painlevé II equation*

$$w(z+1)+w(z-1)=\frac{2w(z)}{1-w^2(z)},$$

with $a=c=0,b=2$ satisfying $|a|+|b|+|c|=2(\neq 0)$. We see that

$$\Delta w(z)=\frac{1}{e^{i2\pi z}+z+2}-\frac{1}{e^{i2\pi z}+z+1}=\frac{-1}{(e^{i2\pi z}+z+2)(e^{i2\pi z}+z+1)},$$

$$\frac{\Delta w(z)}{w(z)}=\frac{-1}{(e^{i2\pi z}+z+2)(e^{i2\pi z}+z+1)}\cdot(e^{i2\pi z}+z+1)=\frac{-1}{e^{i2\pi z}+z+2},$$

$$\begin{aligned}w(z+\eta)-z &= \frac{1}{e^{i2\pi(z+\eta)}+z+\eta+1}-z\\ &= \frac{-ze^{i2\pi(z+\eta)}-(z^2+(\eta+1)z-1)}{e^{i2\pi(z+\eta)}+z+\eta+1}.\end{aligned}$$

Then, $\lambda\left(\frac{1}{\Delta w}\right)=\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)=1,\quad\lambda(\Delta w)=\lambda\left(\frac{\Delta w}{w}\right)=0$. For any $\eta\in\mathbb{C}\setminus\{0\}$, we have $\tau(w(z+\eta))=\sigma(w)=1$.

2. Some Lemmas

In order to prove our conclusions, we need the following lemmas.

Lemma 2.1. (See [1], [2, Theorem 2.2.5]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f(z)$,*

$$R(z,f(z))=\frac{\sum_{i=0}^{m}a_i(z)f(z)^i}{\sum_{j=0}^{n}b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z),b_j(z)(a_m(z)b_n(z)\not\equiv 0)$ being small with respect to $f(z)$, the characteristic function of $R(z,f(z))$ satisfies

$$T(r,R(z,f(z)))=\max\{m,n\}T(r,f)+S(r,f).$$

Lemma 2.2. (See [3, Theorem 2.4], [8]) *Let f be a transcendental meromorphic solution of finite order σ of the difference equation*

$$P(z,f)=0,$$

where $P(z,f)$ is a difference polynomial in $f(z)$ and its shifts. If $P(z,a)\not\equiv 0$ for a slowly moving target meromorphic function a, that is, $T(r,a)=S(r,f)$, then

$$m\left(r,\frac{1}{f-a}\right)=O(r^{\sigma-1+\varepsilon})+S(r,f),$$

outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.3. (See [3, Theorem 2.3], [8]) *Let f be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$U(z,f)P(z,f)=Q(z,f),$$

where $U(z,f),P(z,f)$ and $Q(z,f)$ are difference polynomials such that the total degree $\deg_f U(z,f)=n$ in $f(z)$ and its shifts, and $\deg_f Q(z,f)\leq n$. Moreover, we assume $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then for each $\varepsilon>0$,

$$m(r,P(z,f))=O(r^{\sigma-1+\varepsilon})+S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.4. (See [12, Corollary 2.5]) *Let $f(z)$ be a meromorphic function of finite order σ and let η be a non-zero complex number. Then for each $\varepsilon>0$, we have*

$$m\left(r,\frac{f(z+\eta)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+\eta)}\right)=O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.5. (See [12, Theorem 2.1]) *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f), \sigma < +\infty$, and let η be a fixed non-zero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 2.6. (See [12, Theorem 2.2]) *Let f be a meromorphic function with exponent of convergence of poles $\lambda\left(\frac{1}{f}\right) = \lambda < \infty, \eta \neq 0$ be fixed, then for each $\varepsilon > 0$,*

$$N(r, f(z+\eta)) = N(r, f(z)) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

3. Proof of Theorems

Proof of Theorem 1.1
(i) Firstly, we will prove $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right) = \sigma(w)$. By equation (1.2), Lemma 2.1, Lemma 2.5 and $|a| + |b| \neq 0$, we have

$$\begin{aligned} 2T(r, w(z)) &= T\left(r, \frac{az+b+cw(z)}{w^2(z)}\right) + O(\log r) \\ &= T\left(r, \frac{w(z+1)+w(z-1)}{w(z)}\right) + O(\log r) \\ &\leq T\left(r, \frac{w(z+1)}{w(z)}\right) + T\left(r, \frac{w(z)}{w(z-1)}\right) + O(\log r) \\ &= 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S\left(r, \frac{w(z+1)}{w(z)}\right) + O(\log r) \\ &\leq 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S(r, w) \\ &= 2T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w), \end{aligned}$$

that is,

$$T(r, w) \leq T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w). \tag{3.1}$$

It follows from (3.1) and Lemma 2.4 that

$$\begin{aligned} N\left(r, \frac{\Delta w(z)}{w(z)}\right) &= T\left(r, \frac{\Delta w(z)}{w(z)}\right) - m\left(r, \frac{\Delta w(z)}{w(z)}\right) \\ &\geq T(r, w(z)) + S(r, w). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right) \geq \sigma(w)$, that is $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right) = \sigma(w)$.
Next, we prove $\lambda\left(\frac{1}{\Delta w}\right) = \sigma(w)$. By equation (1.2),

$$\begin{aligned} \Delta w(z) - \Delta w(z-1) &= w(z+1) + w(z-1) - 2w(z) \\ &= \frac{az+b}{w(z)} + c - 2w(z) \\ &= \frac{az+b+cw(z)-2w^2(z)}{w(z)}. \end{aligned} \tag{3.2}$$

From (3.2), Lemma 2.1, Lemma 2.5 and $|a| + |b| \neq 0$, we have

$$\begin{aligned} 2T(r, w(z)) &= T\left(r, \frac{az+b+cw(z)-2w^2(z)}{w(z)}\right) + O(\log r) \\ &= T(r, \Delta w(z) - \Delta w(z-1)) + O(\log r) \\ &\leq T(r, \Delta w(z)) + T(r, \Delta w(z-1)) + O(\log r) \\ &= 2T(r, \Delta w(z)) + S(r, \Delta w(z)) + O(\log r) \\ &\leq 2T(r, \Delta w(z)) + S(r, w), \end{aligned}$$

that is,

$$T(r, w(z)) \leq T(r, \Delta w(z)) + S(r, w). \tag{3.3}$$

It follows from Lemma 2.5 that

$$\begin{aligned} T(r, \Delta w(z)) &\leq T(r, w(z+1)) + T(r, w(z)) + O(1) \\ &= 2T(r, w(z)) + S(r, w) \end{aligned} \tag{3.4}$$

By equation (1.2), we obtain

$$w(z)(w(z+1) + w(z-1)) = az + b + cw(z). \tag{3.5}$$

From (3.5) and Lemma 2.3, we see that for each $\varepsilon > 0$, there is a subset $E_1 \subset (1, \infty)$ having finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup E_1$,

$$m(r, w(z+1) + w(z-1)) = O(r^{\sigma(w)-1+\varepsilon}) + S(r, w). \tag{3.6}$$

It follows from equation (1.2), Lemma 2.1 and $|a| + |b| \neq 0$ that

$$T(r, w(z+1) + w(z-1)) = T\left(\frac{az+b}{w(z)} + c\right) = T(r, w) + S(r, w). \tag{3.7}$$

From (3.6), (3.7) and Lemma 2.6, we obtain

$$\begin{aligned} &T(r, w(z)) + S(r, w) = N(r, w(z+1) + w(z-1)) \\ &\leq N(r, w(z+1)) + N(r, w(z-1)) = 2N(r, w(z)) + S(r, w). \end{aligned} \tag{3.8}$$

It follows from Lemma 2.4 that

$$m(r, \Delta w(z)) \leq m\left(r, \frac{\Delta w(z)}{w(z)}\right) + m(r, w(z)) = m(r, w(z)) + S(r, w). \tag{3.9}$$

From (3.3) − (3.4) and (3.8) − (3.9), we see

$$\begin{aligned} N(r, \Delta w(z)) &= T(r, \Delta w(z)) - m(r, \Delta w(z)) \\ &\geq T(r, \Delta w(z)) - (T(r, w(z)) \\ &\quad - \frac{1}{4}T(r, \Delta w(z))) + S(r, w) \\ &= \frac{5}{4}T(r, \Delta w(z)) - T(r, w(z)) + S(r, w) \\ &\geq \frac{1}{4}T(r, w(z)) + S(r, w). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\Delta w}\right) \geq \sigma(w)$, that is $\lambda\left(\frac{1}{\Delta w}\right) = \sigma(w)$.
(ii) For any $\eta \in \mathbb{C} \setminus \{0\}$, substituting $z + \eta$ into equation (1.2), we obtain

$$w(z+\eta+1) + w(z+\eta-1) = \frac{a(z+\eta)+b}{w(z+\eta)} + c. \tag{3.10}$$

Set $g(z) = w(z+\eta)$. Rewriting equation (3.10) as

$$g(z)(g(z+1) + g(z-1)) = cg(z) + (a(z+\eta) + b).$$

Denote

$$P_1(z, g) := g(z)(g(z+1) + g(z-1)) - cg(z) - (a(z+\eta) + b) = 0.$$

Then, we have

$$\begin{aligned} P_1(z, z) &= z(z+1+z-1) - cz - (a(z+\eta) + b) \\ &= 2z^2 - (a+c)z - (a\eta + b) \not\equiv 0. \end{aligned}$$

From $P_1(z, z) \not\equiv 0$ and Lemma 2.2, we see

$$m\left(r, \frac{1}{g(z) - z}\right) = S(r, g).$$

Thus, by Lemma 2.5, we have

$$\begin{aligned} N\left(r, \frac{1}{w(z+\eta) - z}\right) &= N\left(r, \frac{1}{g(z) - z}\right) = T(r, g) + S(r, g) \\ &= T(r, w(z+\eta)) + S(r, w(z+\eta)) \\ &= T(r, w(z)) + S(r, w). \end{aligned}$$

Hence, for any $\eta \in \mathbb{C} \setminus \{0\}$, $\tau(w(z+\eta)) = \sigma(w)$.
This completes the proof of Theorem 1.1.
Proof of Theorem 1.2
If $c = 0$, equation (1.3) is a special case of equation (1.2). In what

follows, we assume $c\neq 0$. (i) Firstly, we will prove $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$. By equation (1.3), Lemma 2.1, Lemma 2.5 and $c\neq 0$, we have

$$\begin{aligned} 3T(r,w(z)) &= T\left(r,\frac{(az+b)w(z)+c}{w^3(z)}\right)+O(\log r)\\ &= T\left(r,\frac{w(z+1)+w(z-1)}{w(z)}\right)+O(\log r)\\ &\leq 2T\left(r,\frac{\Delta w(z)}{w(z)}\right)+S(r,w), \end{aligned}$$

that is,

$$\frac{3}{2}T(r,w(z))\leq T\left(r,\frac{\Delta w(z)}{w(z)}\right)+S(r,w). \tag{3.11}$$

It follows from (3.11) and Lemma 2.4 that

$$\begin{aligned} N\left(r,\frac{\Delta w(z)}{w(z)}\right) &= T\left(r,\frac{\Delta w(z)}{w(z)}\right)-m\left(r,\frac{\Delta w(z)}{w(z)}\right)\\ &\geq \frac{3}{2}T(r,w(z))+S(r,w). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)\geq\sigma(w)$, that is $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$.
Next, we prove $\lambda\left(\frac{1}{\Delta w}\right)=\sigma(w)$. By equation (1.3),

$$\Delta w(z)-\Delta w(z-1)=\frac{c+(az+b)w(z)-2w^3(z)}{w^2(z)}. \tag{3.12}$$

From (3.12), Lemma 2.1, Lemma 2.5 and $c\neq 0$, we have

$$\begin{aligned} 3T(r,w(z)) &= T\left(r,\frac{c+(az+b)w(z)-2w^3(z)}{w^2(z)}\right)+O(\log r)\\ &= T(r,\Delta w(z)-\Delta w(z-1))+O(\log r)\\ &\leq 2T(r,\Delta w(z))+S(r,w), \end{aligned}$$

that is,

$$\frac{3}{2}T(r,w(z))\leq T(r,\Delta w(z))+S(r,w). \tag{3.13}$$

By Lemma 2.4,

$$m(r,\Delta w(z))=m\left(r,\frac{\Delta w(z)}{w(z)}\right)+m(r,w(z))\leq T(r,w(z))+S(r,w). \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\begin{aligned} N(r,\Delta w(z)) &= T(r,\Delta w(z))-m(r,\Delta w(z))\\ &\geq \frac{3}{2}T(r,w(z))-T(r,w(z))+S(r,w)\\ &= \frac{1}{2}T(r,w(z))+S(r,w). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\Delta w}\right)\geq\sigma(w)$, that is $\lambda\left(\frac{1}{\Delta w}\right)=\sigma(w)$.
(ii) For any $\eta\in\mathbb{C}\setminus\{0\}$, substituting $z+\eta$ into equation (1.3), we obtain

$$w(z+\eta+1)+w(z+\eta-1)=\frac{a(z+\eta)+b}{w(z+\eta)}+\frac{c}{w(z+\eta)^2}. \tag{3.15}$$

Set $g(z)=w(z+\eta)$. Then (3.15) can be rewritten as

$$g^2(z)(g(z+1)+g(z-1))=g(z)(a(z+\eta)+b)+c.$$

Denote

$$P_2(z,g):=g^2(z)(g(z+1)+g(z-1))-g(z)(a(z+\eta)+b)-c=0.$$

Then, we have

$$P_2(z,z)=z^2(z+1+z-1)-z(a(z+\eta)+b)-c\not\equiv 0.$$

From $P_2(z,z)\not\equiv 0$ and Lemma 2.2, we see

$$m\left(r,\frac{1}{g(z)-z}\right)=S(r,g).$$

Thus, by Lemma 2.5, we have

$$\begin{aligned} N\left(r,\frac{1}{w(z+\eta)-z}\right) &= N\left(r,\frac{1}{g(z)-z}\right)=T(r,g)+S(r,g)\\ &= T(r,w(z+\eta))+S(r,w(z+\eta))\\ &= T(r,w)+S(r,w). \end{aligned}$$

Hence, for any $\eta\in\mathbb{C}\setminus\{0\}$, $\tau(w(z+\eta))=\sigma(w)$.
This completes the proof of Theorem 1.2.

Proof of Theorem 1.3
Using the same method as the proof of Theorem 1.1, we can easily obtain $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\lambda(\Delta w)=\sigma(w)$.

Proof of Theorem 1.4
(i) In what follows, we consider three cases: Case 1, $c=0$; Case 2, $c\neq 0$, either $a=0,b-c=0$, or $a=0,b+c=0$; Case 3, $c\neq 0$, either $a\neq 0$, or $b-c\neq 0$, or $b+c\neq 0$.

Case 1, $c=0$. Firstly, we prove we prove $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$. By equation (1.5), Lemma 2.1, Lemma 2.5 and $|a|+|b|\neq 0$, we have

$$\begin{aligned} 2T(r,w(z)) &= T\left(r,\frac{az+b}{1-w^2(z)}\right)+O(\log r)\\ &= T\left(r,\frac{w(z+1)+w(z-1)}{w(z)}\right)+O(\log r)\\ &\leq 2T\left(r,\frac{\Delta w(z)}{w(z)}\right)+S(r,w), \end{aligned}$$

that is,

$$T(r,w(z))\leq T\left(r,\frac{\Delta w(z)}{w(z)}\right)+S(r,w). \tag{3.16}$$

It follows from (3.16) and Lemma 2.4 that

$$\begin{aligned} N\left(r,\frac{\Delta w(z)}{w(z)}\right) &= T\left(r,\frac{\Delta w(z)}{w(z)}\right)-m\left(r,\frac{\Delta w(z)}{w(z)}\right)\\ &\geq T(r,w(z))+S(r,w). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)\geq\sigma(w)$, that is $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$.
Next, we prove $\lambda\left(\frac{1}{\Delta w}\right)=\sigma(w)$. By equation (1.5),

$$\Delta w(z)-\Delta w(z-1)=\frac{2w^3(z)+(az+b-2)w(z)}{1-w^2(z)}. \tag{3.17}$$

From (3.17), Lemma 2.1, Lemma 2.5 and $|a|+|b|\neq 0$, we have

$$\begin{aligned} 3T(r,w(z)) &= T\left(r,\frac{2w^3(z)+(az+b-2)w(z)}{1-w^2(z)}\right)+O(\log r)\\ &= T(r,\Delta w(z)-\Delta w(z-1))+O(\log r)\\ &\leq 2T(r,\Delta w(z))+S(r,w), \end{aligned}$$

that is,

$$\frac{3}{2}T(r,w(z))\leq T(r,\Delta w(z))+S(r,w). \tag{3.18}$$

From (3.14) and (3.18), we have

$$\begin{aligned} N(r,\Delta w(z)) &= T(r,\Delta w(z))-m(r,\Delta w)\\ &\geq \frac{3}{2}T(r,w(z))-T(r,w(z))+S(r,w)\\ &= \frac{1}{2}T(r,w(z))+S(r,w). \end{aligned}$$

Thus, $\lambda\left(\frac{1}{\Delta w}\right)\geq\sigma(w)$, that is $\lambda\left(\frac{1}{\Delta w}\right)=\sigma(w)$.
Case 2, $c\neq 0$, either $a=0,b-c=0$, or $a=0,b+c=0$. We divide

this proof into the following two subcases.

Case 2.1, $c\neq 0$, $a=0, b-c=0$. Firstly, we prove $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$. By equation (1.5), Lemma 2.1, Lemma 2.5 and $b=c(\neq 0)$, we have

$$\begin{aligned}2T(r,w(z)) &= T\left(r,\frac{b}{w(z)(1-w(z))}\right)+O(1)\\ &= T\left(r,\frac{w(z+1)+w(z-1)}{w(z)}\right)+O(1)\\ &\leq 2T\left(r,\frac{\Delta w(z)}{w(z)}\right)+S(r,w),\end{aligned}$$

hence,

$$T(r,w(z))\leq T\left(r,\frac{\Delta w(z)}{w(z)}\right)+S(r,w). \qquad (3.19)$$

From (3.19) and Lemma 2.4, we see

$$\begin{aligned}N\left(r,\frac{\Delta w(z)}{w(z)}\right) &= T\left(r,\frac{\Delta w(z)}{w(z)}\right)-m\left(r,\frac{\Delta w(z)}{w(z)}\right)\\ &\geq T(r,w(z))+S(r,w).\end{aligned}$$

Thus, $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)\geq\sigma(w)$, that is $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$.

Next, we prove $\lambda\left(\frac{1}{\Delta w}\right)=\sigma(w)$. By equation (1.5),

$$\Delta w(z)-\Delta w(z-1)=\frac{2w^2(z)-2w(z)+b}{1-w(z)}. \qquad (3.20)$$

From (3.20), Lemma 2.1, Lemma 2.5 and $b=c(\neq 0)$, we have

$$\begin{aligned}2T(r,w(z)) &= T\left(r,\frac{2w^2(z)-2w(z)+b}{1-w(z)}\right)+O(1)\\ &= T(r,\Delta w(z)-\Delta w(z-1))+O(1)\\ &\leq 2T(r,\Delta w(z))+S(r,w),\end{aligned}$$

that is,

$$T(r,w(z))\leq T(r,\Delta w(z))+S(r,w). \qquad (3.21)$$

By equation (1.5), we obtain

$$w(z)(w(z+1)+w(z-1))=w(z+1)+w(z-1)-b. \qquad (3.22)$$

From (3.22) and Lemma 2.3, we see that for each $\varepsilon>0$, there is a subset $E_2\subset(1,\infty)$ having finite logarithmic measure such that for $|z|=r\notin[0,1]\cup E_2$,

$$m(r,w(z+1)+w(z-1))=O(r^{\sigma(w)-1+\varepsilon})+S(r,w). \qquad (3.23)$$

By equation (1.5), Lemma 2.1 and $b=c(\neq 0)$, we see

$$T(r,w(z+1)+w(z-1))=T\left(\frac{b}{1-w(z)}\right)=T(r,w)+S(r,w). \qquad (3.24)$$

From (3.23), (3.24) and Lemma 2.6, we obtain

$$\begin{aligned}T(r,w(z))+S(r,w) &= N(r,w(z+1)+w(z-1))\\ &\leq 2N(r,w(z))+S(r,w)\end{aligned} \qquad (3.25)$$

From (3.4), (3.9), (3.21) and (3.25), we see

$$\begin{aligned}N(r,\Delta w(z)) &= T(r,\Delta w(z))-m(r,\Delta w(z))\\ &\geq T(r,\Delta w(z))-(T(r,w(z))\\ &\quad -\frac{1}{4}T(r,\Delta w(z)))+S(r,w)\\ &= \frac{5}{4}T(r,\Delta w(z))-T(r,w(z))+S(r,w)\\ &\geq \frac{1}{4}T(r,w(z))+S(r,w).\end{aligned}$$

Thus, $\lambda\left(\frac{1}{\Delta w}\right)\geq\sigma(w)$, that is $\lambda\left(\frac{1}{\Delta w}\right)=\sigma(w)$.

Case 2.2, $c\neq 0$, $a=0, b+c=0$. Using the same method as the proof of subcase 2.1, we can also obtain $\lambda\left(\frac{1}{\Delta w}\right)=\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$;

Case 3, $c\neq 0$, either $a\neq 0$, or $b-c\neq 0$, or $b+c\neq 0$. Firstly, we prove $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$. By equation (1.5), Lemma 2.1, Lemma 2.5 and $c\neq 0$, either $a\neq 0$, or $b-c\neq 0$, or $b+c\neq 0$, we have

$$\begin{aligned}3T(r,w(z)) &= T\left(r,\frac{(az+b)w(z)+c}{w(z)(1-w^2(z))}\right)+O(\log r)\\ &= T\left(r,\frac{w(z+1)+w(z-1)}{w(z)}\right)+O(\log r)\\ &\leq 2T\left(r,\frac{\Delta w(z)}{w(z)}\right)+S(r,w),\end{aligned}$$

that is,

$$\frac{3}{2}T(r,w(z))\leq T\left(r,\frac{\Delta w(z)}{w(z)}\right)+S(r,w). \qquad (3.26)$$

From (3.26) and Lemma 2.4, we see

$$\begin{aligned}N\left(r,\frac{\Delta w(z)}{w(z)}\right) &= T\left(r,\frac{\Delta w(z)}{w(z)}\right)-m\left(r,\frac{\Delta w(z)}{w(z)}\right)\\ &\geq \frac{3}{2}T(r,w(z))+S(r,w).\end{aligned}$$

Thus, $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)\geq\sigma(w)$, that is $\lambda\left(\frac{1}{\frac{\Delta w}{w}}\right)=\sigma(w)$.

Next, we prove $\lambda\left(\frac{1}{\Delta w}\right)=\sigma(w)$. By equation (1.5),

$$\Delta w(z)-\Delta w(z-1)=\frac{2w^3(z)+(az+b-2)w(z)+c}{1-w^2(z)}. \qquad (3.27)$$

From (3.27), Lemma 2.1, Lemma 2.5 and $c\neq 0$, either $a\neq 0$, or $b-c\neq 0$, or $b+c\neq 0$, we have

$$\begin{aligned}3T(r,w) &= T\left(r,\frac{2w^3(z)+(az+b-2)w(z)+c}{1-w^2(z)}\right)+O(\log r)\\ &= T(r,\Delta w(z)-\Delta w(z-1))+O(\log r)\\ &\leq 2T(r,\Delta w(z))+S(r,w),\end{aligned}$$

that is,

$$\frac{3}{2}T(r,w(z))\leq T(r,\Delta w(z))+S(r,w). \qquad (3.28)$$

From (3.14) and (3.28), we have

$$\begin{aligned}N(r,\Delta w(z)) &= T(r,\Delta w(z))-m(r,\Delta w(z))\\ &\geq \frac{3}{2}T(r,w(z))-T(r,w(z))+S(r,w)\\ &= \frac{1}{2}T(r,w(z))+S(r,w).\end{aligned}$$

Thus, $\lambda\left(\frac{1}{\Delta w}\right)\geq\sigma(w)$, that is $\lambda\left(\frac{1}{\Delta w}\right)=\sigma(w)$.

(ii) For any $\eta\in\mathbb{C}\setminus\{0\}$, substituting $z+\eta$ into equation (1.5), we obtain

$$w(z+\eta+1)+w(z+\eta-1)=\frac{(a(z+\eta)+b)w(z+\eta)+c}{1-w(z+\eta)^2}, \qquad (3.29)$$

Set $g(z)=w(z+\eta)$. Then (3.29) can be rewritten as

$$(1-g^2(z))(g(z+1)+g(z-1))=g(z)(a(z+\eta)+b)+c.$$

Denote

$$\begin{aligned}P_3(z,g) &:= (1-g^2(z))(g(z+1)+g(z-1))\\ &\quad -g(z)(a(z+\eta)+b)-c=0.\end{aligned}$$

Then, we have

$$P_3(z,z)=(1-z^2)(z+1+z-1)-z(a(z+\eta)+b)-c\not\equiv 0.$$

From $P_3(z,z) \not\equiv 0$ and Lemma 2.2, we see that

$$m\left(r, \frac{1}{g(z)-z}\right) = S(r,g).$$

Thus, by Lemma 2.5, we have

$$\begin{aligned} N\left(r, \frac{1}{w(z+\eta)-z}\right) &= N\left(r, \frac{1}{g(z)-z}\right) = T(r,g)+S(r,g) \\ &= T(r,w(z+\eta))+S(r,w(z+\eta)) \\ &= T(r,w(z))+S(r,w). \end{aligned}$$

Hence, for any $\eta \in \mathbb{C}\setminus\{0\}$, $\tau(w(z+\eta)) = \sigma(w)$.
This completes the proof of Theorem 1.4.

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