Perturbation of zero surfaces

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Abstract

It is proved that if a smooth function $u(x)$, $x \in \mathbb{R}^3$, such that $\inf_{s \in S} |u_N(s)| > 0$, where u_N is the normal derivative of u on S, has a closed smooth surface S of zeros, then the function $u(x) + \varepsilon v(x)$ has also a closed smooth surface S_ε of zeros. Here v is a smooth function and $\varepsilon > 0$ is a sufficiently small number.

Keywords: zero surfaces; perturbation theory.

1. Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain containing inside a connected closed C^3−smooth surface S, which is the set of zeros of a function $u \in C^3(D)$, so that

$$u|_S = 0. \tag{1}$$

Let $N = N_s$ be the unit normal to S, such that $u_N = |\nabla u(s)|$, where u_N is the normal derivative of u on S. Let $u_\varepsilon := u + \varepsilon v$, where $v \in C^3(D)$ and $\varepsilon > 0$ is sufficiently small. Assume that

$$\inf_{s \in S} |\nabla u(s)| \geq 2c_1 > 0, \qquad c_1 = const > 0. \tag{2}$$

The purpose of this paper is to prove Theorem 1.

Theorem 1. *Under the above assumptions there exists a smooth closed surface S_ε such that $u_\varepsilon = 0$ on S_ε.*

In Section 2 Theorem 1 is proved.

Although there are many various results on perturbation theory, see [2], [3], the result formulated in Theorem 1 is new.

2. Proof of Theorem 1

Consider the following equation for t:

$$u(s + tN) + \varepsilon v(s + tN) = 0, \tag{3}$$

where $N = N(s)$ is the normal to S at the point s and t is a parameter. Using the Taylor's formula and relation (1), one gets from (3)

$$t\Big(\nabla u(s) \cdot N + \varepsilon \nabla v(s) \cdot N\Big) + \varepsilon v(s) + t^2 \phi = 0, \tag{4}$$

where $t^2 \phi$ is the Lagrange remainder in the Taylor's formula and

$$\phi := \sum_{i,j=1}^{3} [u_{x_i x_j}(s + \theta tN) + \varepsilon v_{x_i x_j}(s + \theta tN)] N_i N_j, \quad \theta \in (0,1). \tag{5}$$

Since the functions u and v belong to $C^3(D)$, the function $\phi = \phi(t,s,\varepsilon)$ has a bounded derivative with respect to t uniformly with respect to $s \in S$ and $\varepsilon \in (0,1]$.

Consider equation (4) as an equation for $t = t(s)$ in the space $C(S)$. Rewrite (4) as

$$t = -\varepsilon \Big(\nabla u(s) \cdot N + \varepsilon \nabla v(s) \cdot N\Big)^{-1} v(s) - \\ - t^2 \phi \Big(\nabla u(s) \cdot N + \varepsilon \nabla v(s) \cdot N\Big)^{-1} := Bt. \tag{6}$$

Let us check that the operator B satisfies the contraction mapping theorem in the set

$$M := \{t : max_{s \in S} |t(s) - \varepsilon \Big(\nabla u(s) \cdot N + \varepsilon \nabla v(s) \cdot N\Big)^{-1} v(s)| \leq \delta\}, \tag{7}$$

where $\delta > 0$ is a small number, and $M \subset C(S)$.

First, one should check that B maps M into itself. One has

$$max_{s \in S} |Bt(s) - \varepsilon \Big(\nabla u(s) \cdot N + \varepsilon \nabla v(s) \cdot N\Big)^{-1} v(s)| \leq \\ \leq max_{s \in S} \frac{t^2 |\phi|}{\nabla u(s) \cdot N + \varepsilon \nabla v(s) \cdot N}. \tag{8}$$

We have chosen N so that $\nabla u(s) \cdot N = |\nabla u(s)|$. This is possible because equation (1) implies that $\nabla u(s)$ is orthogonal to S at the point $s \in S$. Assumption (2) implies that for sufficiently small ε one has

$$\inf_{s \in S} |\nabla u_\varepsilon(s)| \geq c_1. \tag{9}$$

Since ϕ is continuously differentiable, one has

$$\sup_{s \in S, t \in (0,1)} |\phi(t,s,\varepsilon)| \leq c_2. \tag{10}$$

Therefore, if

$$|t(s)| \leq \delta, \tag{11}$$

then

$$\frac{t^2(s) |\phi(t,s,\varepsilon)|}{|\nabla u(s)| + \varepsilon \nabla v(s) \cdot N} \leq \frac{c_2}{c_1} \delta^2 \leq \delta, \tag{12}$$

provided that

$$\frac{c_2}{c_1}\delta \leq 1. \tag{13}$$

Thus, if (13) holds then B maps M into itself.

Let us check that B is a contraction mapping on M. One has

$$|Bt_1 - Bt_2| \leq c_1^{-1}|t_1^2\phi(t_1,s,\varepsilon) - t_2^2\phi(t_2,s,\varepsilon)| \leq c_3|t_1 - t_2|, \tag{14}$$

where $c_3 \in (0,1)$ if δ is sufficiently small. Indeed,

$$c_3 = max_{s\in S, t\leq\delta}\left(2t|\phi(t,s,\varepsilon)| + t^2|\frac{\partial\phi}{\partial t}|\right) \leq c_4\delta < 1, \tag{15}$$

if δ is sufficiently small. Here c_4 is a constant.

Thus, B is a contraction on M. By the contraction mapping principle, equation (6) is uniquely solvable for t. Its solution $t = t(s)$ allows one to construct the zero surface S_ε of the function u_ε by the equation $r = s + t(s)N$, where $r = r(s)$ is the radius vector of the points on S_ε. Theorem 1 is proved.   □

Remark 1. Condition (2) is a sufficient condition for the validity of Theorem 1. Although this condition is not necessary, if it does not hold one can construct counterexamples to the conclusion of Theorem 1. For example, assume that $u(x) \geq 0$ in $\mathbb{R}^3$ and $u(x) = 0$ on S, let $v > 0$ in $\mathbb{R}^3$ and $\varepsilon > 0$. Then the function $u_\varepsilon = u + \varepsilon v$ does not have zeros in $\mathbb{R}^3$.

Remark 2. In scattering theory the following question is of interest: assume that $u(x)$ is an entire function of exponential type, $u(x) = \int_{S^2} e^{ik\beta\cdot x}f(\beta)d\beta$, where $f \in L^2(S^2)$, S^2 is the unit sphere in $\mathbb{R}^3$. Assume that $u = 0$ on S, where S is a bounded closed smooth connected surface in $\mathbb{R}^3$.

The question of interest is:

Is there a bounded closed smooth connected surface of zeros of an entire function u_ε of exponential type, $u_\varepsilon = \int_{S^2} e^{ik\beta\cdot x}[f(\beta) + \varepsilon g(\beta)]d\beta$, where $g \in L^2(S^2)$ and $\varepsilon > 0$ is a small parameter?

We will not use Theorem 1 since assumption (2) may not hold, but sketch an argument, based on the fact that S in the above question is the intersection of an analytic set with $\mathbb{R}^3$, see, for example, [1] for the definition and properties of analytic sets. The functions u and u_ε in Remark 2 solve the differential equation

$$\nabla^2 u + k^2 u = 0 \qquad in \quad \mathbb{R}^3, \quad k^2 = const > 0. \tag{16}$$

The function u_N may vanish on S at most on the closed set $\sigma \subset S$ which is of the surface measure zero (by the uniqueness of the solution to the Cauchy problem for equation (16)). For every point $s \in S \setminus \sigma$ the argument given in the proof of Theorem 1 yields the existence of $t(s)$, the unique solution to (6). Since S is real analytic, the set $\tilde{S}_\varepsilon$, defined in the proof of Theorem 1, is analytic and $\tilde{S}_\varepsilon$ is a part of the analytic set defined by the equation $u_\varepsilon = 0$. In our problem S is a bounded closed real analytic surface. We guess that the set $\tilde{S}_\varepsilon$ can be continued analytically to an analytic set which intersects the real space $\mathbb{R}^3$ over a real analytic surface S_ε. It is still an *open problem* to prove (or disprove) that the analytic continuation of the set $\tilde{S}_\varepsilon$ intersects $\mathbb{R}^3$ over a bounded closed real analytic surface $S_\varepsilon \in \mathbb{R}^3$.

References

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