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Research paper

Estimates on Initial Coefficients of Certain Subclasses of Bi-Univalent Functions Associated with Quasi-Subordination

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Abstract

In the present investigation we introduce some subclasses of the function class Σ of bi-univalent functions defined in the open unit disk $\mathbb{U}$, which are associated with the quasi-subordination. We obtain the estimates on initial coefficients $|a_2|$ and $|a_3|$ for the functions in these subclasses. Also several related subclasses are considered and connection with some known results are established.

Keywords: *Analytic function; Bi-univalent function; Quasi-subordination; Subordination; Univalent function.*

1. Introduction

Let $\mathscr{A}$ be the class of all analytic functions f which are : (i) normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ and (ii) defined on the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. The Taylor's series expansion of $f \in \mathscr{A}$ is

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

The class of all functions in $\mathscr{A}$ which are univalent in the open unit disk $\mathbb{U}$ is denoted by $\mathscr{S}$. These univalent functions are invertible but their inverse functions may not be defined on the entire unit disk $\mathbb{U}$. The Koebe one-quarter theorem (see [4]) ensures that the image of $\mathbb{U}$ under every function $f \in \mathscr{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathscr{S}$ has an inverse (say g), satisfying $g(f(z)) = z$ for all $z \in \mathbb{U}$ and $f(g(w)) = w$, where $|w| < r_0(f)$, $r_0(f) \geq 1/4$. In fact, it can be easily verified that the inverse function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. The class of all bi-univalent functions defined in $\mathbb{U}$ is denoted by Σ.
Lewin [8] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions in the class Σ. Later, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Also, Netanyahu [11] proved that $max_{f\in\Sigma}|a_2| = 4/3$. Still the coefficient estimate problem is open for each $|a_n|$, $(n = 3, 4, \cdots)$.
Brannan and Taha [3] (see also [17]) introduced certain subclasses of the bi-univalent function class Σ similar to the subclasses $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ of starlike and convex functions of order α $(0 < \alpha \leq 1)$ respectively. Sirvastava et al.[16] introduced and investigated certain subclasses of bi-univalent function class Σ and also obtained the initial coefficient bounds.

Ma and Minda [9] introduced the classes:

$$\mathscr{S}^*(\phi) = \left\{ f \in \mathscr{S}; \left[zf'(z)/f(z) \right] \prec \phi(z) \right\}$$

and

$$\mathscr{K}(\phi) = \left\{ f \in \mathscr{S}; \left[1 + \left(zf''(z)/f'(z) \right) \right] \prec \phi(z) \right\},$$

where ϕ be an analytic function with positive real part in the unit disk $\mathbb{U}$, $\phi(0) = 1$, $\phi'(0) > 0$ and maps $\mathbb{U}$ onto a region which is starlike with respect to 1 and symmetric with respect to the real axis. These classes includes several well known subclasses of starlike and convex functions respectively as special cases.
Robertson [15] introduced the concept of quasi-subordination in 1970. An analytic function f is quasi-subordinate to another analytic function ϕ, written as

$$f(z) \prec_q \phi(z); \quad (z \in \mathbb{U}) \tag{3}$$

if there are the analytic functions ψ and w with $|\psi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \psi(z)\phi(w(z))$. Observe that if $\psi(z) = 1$ then $f(z) = \phi(w(z))$, so that $f(z) \prec \phi(z)$ in $\mathbb{U}$. (See [10] and [14] for work related to quasi-subordination.)
In this investigation we assumed that:

$$\psi(z) = A_0 + A_1 z + A_2 z^2 + \cdots; \quad (|\psi(z)| \leq 1, z \in \mathbb{U}) \tag{4}$$

and $\phi(z)$ is an analytic function in $\mathbb{U}$ with the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots; \quad (B_1 > 0). \tag{5}$$

2. Coefficient Estimates for the Function Class $\mathscr{R}_{\Sigma}^{q}(\lambda, \phi)$

Definition 2.1: *A function $f \in \Sigma$ given by (1) is said to be in the class $\mathscr{R}_{\Sigma}^{q}(\lambda, \phi)$ if the following quasi-subordination holds:*

$$\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) - 1 \right] \prec_q (\phi(z) - 1)$$

and

$$\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)-1\right]\prec_q(\phi(w)-1)$$

where $z,w\in\mathbb{U}$, $\lambda\geq 1$ and the functions g and ϕ are given by (2) and (5) respectively.

Theorem 2.2: *Let $f(z)$ given by (1) be in the class $\mathscr{R}^q_\Sigma(\lambda,\phi)$. Then,*

$$|a_2|\leq min\left\{\frac{|A_0|B_1}{1+\lambda},\sqrt{\frac{|A_0|(B_1+|B_2-B_1|)}{1+2\lambda}}\right\} \tag{6}$$

and

$$|a_3|\leq min\left\{\frac{(|A_0|+|A_1|)B_1}{1+2\lambda}+\frac{A_0^2B_1^2}{(1+\lambda)^2},\ \frac{|A_1|B_1+|A_0|(B_1+|B_2-B_1|)}{1+2\lambda}\right\}. \tag{7}$$

Proof: Since $f\in\mathscr{R}^q_\Sigma(\lambda,\phi)$, there exist two analytic functions $u,v:\mathbb{U}\to\mathbb{U}$ with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$ and a function ψ defined by (4) satisfies:

$$\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)-1\right]=\psi(z)\left[\phi(u(z))-1\right] \tag{8}$$

and

$$\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)-1\right]=\psi(w)\left[\phi(v(w))-1\right]. \tag{9}$$

Define the functions p and q such that:

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+\cdots$$

and

$$q(w)=\frac{1+v(w)}{1-v(w)}=1+d_1w+d_2w^2+\cdots$$

equivalently,

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right] \tag{10}$$

and

$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left[d_1w+\left(d_2-\frac{d_1^2}{2}\right)w^2+\cdots\right]. \tag{11}$$

Clearly p and q are analytic in $\mathbb{U}$ with $p(0)=q(0)=1$ and have their positive real part in $\mathbb{U}$. Hence $|c_i|\leq 2$ and $|d_i|\leq 2$ (see [12]). Using (10) and (11) together with (4) and (5) in the RHS of (8) and (9), we get

$$\psi(z)\left[\phi(u(z))-1\right]=\frac{1}{2}A_0B_1c_1z+\left\{\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2\right\}z^2+\cdots \tag{12}$$

and

$$\psi(w)\left[\phi(v(w))-1\right]=\frac{1}{2}A_0B_1d_1w+\left\{\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2\right\}w^2+\cdots. \tag{13}$$

Since the function f and its inverse g are given by (1) and (2) respectively, we have

$$\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)-1\right]=(1+\lambda)a_2z+(1+2\lambda)a_3z^2+\cdots \tag{14}$$

and

$$\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)-1\right]=-(1+\lambda)a_2w+(1+2\lambda)(2a_2^2-a_3)w^2+\cdots. \tag{15}$$

Using (12) to (15) in (8) and (9) and then comparing the coefficients of z,z^2,w and w^2; we get

$$(1+\lambda)a_2=\frac{1}{2}A_0B_1c_1, \tag{16}$$

$$(1+2\lambda)a_3=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{A_0B_2}{4}c_1^2, \tag{17}$$

$$-(1+\lambda)a_2=\frac{1}{2}A_0B_1d_1 \tag{18}$$

and

$$(1+2\lambda)(2a_2^2-a_3)=\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2. \tag{19}$$

From (16) and (18), it follows that

$$c_1=-d_1 \tag{20}$$

and

$$8(1+\lambda)^2a_2^2=A_0^2B_1^2(c_1^2+d_1^2). \tag{21}$$

Also by adding (17) in (19) in light of (20), we get

$$8(1+2\lambda)a_2^2=2A_0B_1(c_2+d_2)+A_0(B_2-B_1)(c_1^2+d_1^2). \tag{22}$$

Applying $|c_i|\leq 2$ and $|d_i|\leq 2$ in (21) and (22), we get the desired result (6).
Next, for the bound on $|a_3|$, by subtracting (19) from (17), we obtain

$$a_3=a_2^2+\frac{2A_1B_1c_1+A_0B_1(c_2-d_2)}{4(1+2\lambda)}. \tag{23}$$

Using (21) with $|c_i|\leq 2$ and $|d_i|\leq 2$ in (23), we get

$$|a_3|\leq\frac{(|A_0|+|A_1|)B_1}{(1+2\lambda)}+\frac{A_0^2B_1^2}{(1+\lambda)^2}. \tag{24}$$

Also, using (22) with $|c_i|\leq 2$ and $|d_i|\leq 2$ in (23), we get

$$|a_3|\leq\frac{|A_1|B_1+|A_0|(B_1+|B_2-B_1|)}{1+2\lambda}. \tag{25}$$

From (24) and (25), we get the desired result (7).
This completes the proof of Theorem 2.2.
Observe that, if we set $\psi(z)=1$ in Definition 2.1, then the quasi-subordination reduces to subordination and the subclass $\mathscr{R}^q_\Sigma(\lambda,\phi)$ reduces to $\mathscr{R}_\Sigma(\lambda,\phi)$. Hence we get the following corollary:

Corollary 2.3: *Let the function $f(z)$ given by (1) be in the class $\mathscr{R}_\Sigma(\lambda,\phi)$. Then,*

$$|a_2|\leq min\left\{\frac{B_1}{1+\lambda},\sqrt{\frac{B_1+|B_2-B_1|}{1+2\lambda}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{B_1}{1+2\lambda}+\frac{B_1^2}{(1+\lambda)^2}, \frac{B_1+|B_2-B_1|}{1+2\lambda}\right\}.$$

If we set $\psi(z)=1$ and $\lambda=1$ in Theorem 2.2, then we get the following corollary:
Corollary 2.4: *Let the function $f(z)$ given by (1) be in the class $\mathscr{R}_\Sigma(\phi)$. Then,*

$$|a_2 \leq min\left\{\frac{B_1}{2}, \sqrt{\frac{B_1+|B_2-B_1|}{3}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{B_1}{3}+\frac{B_1^2}{4}, \frac{B_1+|B_2-B_1|}{3}\right\}.$$

Remark 2.5: Corollaries (2.3) and (2.4) are the improvements of the estimates obtained in Theorem 2.1 given by Kumar et al. [7] and Theorem 2.1 given by Ali et al. [1], respectively.
Remark 2.6: If we set

$$\begin{aligned}\phi(z) &= \frac{1+(1-2\beta)z}{1-z}\\ &= 1+2(1-\beta)z+2(1-\beta)z^2+\cdots; \quad (0\leq\beta<1)\end{aligned}$$

in Corollaries (2.3) and (2.4) then we get the improvements of the estimates obtained in Theorem 3.2 given by Frasin and Aouf [5] and Theorem 2 given by Srivastava et al. [16], respectively.

3. Coefficient Estimates for the Function Class $\mathscr{S}_\Sigma^{*,q}(\phi)$

Definition 3.1: *A function $f\in\Sigma$ given by (1) is said to be in the class $\mathscr{S}_\Sigma^{*,q}(\phi)$ if the following quasi-subordination holds:*

$$\left[\frac{zf'(z)}{f(z)}-1\right] \prec_q (\phi(z)-1)$$

and

$$\left[\frac{wg'(w)}{g(w)}-1\right] \prec_q (\phi(w)-1)$$

where $z,w\in\mathbb{U}$ and the functions g and ϕ are given by (2) and (5) respectively.
Theorem 3.2: *Let $f(z)$ given by (1) be in the class $\mathscr{S}_\Sigma^{*,q}(\phi)$. Then,*

$$|a_2| \leq min\{L,M,N\} \tag{26}$$

where,

$$L=\sqrt{|A_0|(B_1+|B_2-B_1|)},\ M=\sqrt{\frac{A_0^2B_1^2+|A_0|(B_1+|B_2-B_1|)}{2}},$$
$$N=\frac{|A_0|B_1\sqrt{|A_0|B_1}}{\sqrt{A_0^2B_1^2+|A_0||B_1-B_2|}}$$

and

$$|a_3| \leq min\{P,Q,R\} \tag{27}$$

where,

$$P=\frac{|A_1|B_1}{2}+|A_0|(B_1+|B_2-B_1|),$$
$$Q=\frac{A_0^2B_1^2+|A_0|(B_1+|B_2-B_1|)-2|A_1|B_1}{2},$$
$$R=\frac{1}{4}\left[(|A_0|+2|A_1|)B_1+3|A_0|B_1\cdot max\left\{1,\left|\frac{B_1-4B_2}{3B_1}\right|\right\}\right].$$

Proof: Since $f\in\mathscr{S}_\Sigma^{*,q}(\phi)$, there exist two analytic functions $u,v:\mathbb{U}\to\mathbb{U}$ with $u(0)=v(0)=0$, $|u(z)|<1$, $|v(w)|<1$ and a function ψ defined by (4) satisfies:

$$\left[\frac{zf'(z)}{f(z)}-1\right]=\psi(z)\left[\phi(u(z))-1\right] \tag{28}$$

and

$$\left[\frac{wg'(w)}{g(w)}-1\right]=\psi(w)\left[\phi(v(w))-1\right]. \tag{29}$$

Define the functions p and q analytic in $\mathbb{U}$ as in (10) and (11) and then proceed similarly up to (13). Also on expanding LHS of (28) and (29) using (1) and (2), we get

$$\left[\frac{zf'(z)}{f(z)}-1\right]=a_2z+(2a_3-a_2^2)z^2+\cdots \tag{30}$$

and

$$\left[\frac{wg'(w)}{g(w)}-1\right]=-a_2w+(3a_2^2-2a_3)w^2+\cdots. \tag{31}$$

Using (12), (13), (30) and (31) in (28) and (29) and then equating the coefficients of z,z^2,w,w^2; we get

$$a_2=\frac{1}{2}A_0B_1c_1, \tag{32}$$

$$2a_3=\frac{1}{2}A_0B_1c_1a_2+\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}A_0B_2c_1^2, \tag{33}$$

$$-a_2=\frac{1}{2}A_0B_1d_1 \tag{34}$$

and

$$\begin{aligned}4a_2^2-2a_3 = &-\frac{1}{2}A_0B_1d_1a_2+\frac{1}{2}A_1B_1d_1+\\ &\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}A_0B_2d_1^2\end{aligned} \tag{35}$$

Using (32) and (34), we get

$$c_1=-d_1, \tag{36}$$

$$8a_2^2=(c_1^2+d_1^2)A_0^2B_1^2 \tag{37}$$

and

$$4a_2=(c_1-d_1)A_0B_1. \tag{38}$$

Adding (33) and (35) and then using (38), we get

$$8a_2^2=A_0\left[2(c_2+d_2)B_1+(c_1^2+d_1^2)(B_2-B_1)\right]. \tag{39}$$

Adding (33) and (35) and then using (32) and (36), we get

$$16a_2^2=2A_0^2B_1^2d_1^2+2(c_2+d_2)A_0B_1+A_0(c_1^2+d_1^2)(B_2-B_1). \tag{40}$$

Adding (33) and (35) and then using (37) and (38), we get

$$4(A_0^2B_1^2+A_0B_1-A_0B_2)a_2^2=(c_2+d_2)A_0^3B_1^3. \tag{41}$$

Now, (39), (40) and (41) along with $|c_i|\leq 2$ and $|d_i|\leq 2$, gives the desired estimate on a_2 as asserted in (26).

Next, for estimate on $|a_3|$ subtracting (33) from (35) and then using (36), we get

$$-4a_3 = -4a_2^2 + A_1B_1c_1 + \frac{1}{2}(d_2 - c_2)A_0B_1. \tag{42}$$

Using (40) in (42), we get

$$16a_3 = 2A_0^2B_1^2d_1^2 + 4A_0B_1c_2 + A_0(c_1^2 + d_1^2)(B_2 - B_1) - 4A_1B_1c_1. \tag{43}$$

Subtracting (35) from (33) and then using (39), we get

$$4a_3 = \frac{1}{2}(3c_2 + d_2)A_0B_1 + c_1^2A_0(B_2 - B_1) + A_1B_1c_1 \tag{44}$$

or

$$4a_3 = \frac{1}{2}A_0B_1d_2 + \frac{3A_0B_1}{2}\left[c_2 - \frac{2(B_1 - B_2)}{3B_1}c_1^2\right] + A_1B_1c_1. \tag{45}$$

On applying the result given by Keogh and Merkes [6] (see also [13]), that is for any complex number z, $|c_2 - zc_1^2| \leq 2 \cdot max\{1, |2z-1|\}$, along with $|d_2| \leq 2$ in (45), we obtain

$$4|a_3| \leq |A_0|B_1 + 2|A_1|B_1 + 3|A_0|B_1 \cdot max\left\{1, \left|\frac{B_1 - 4B_2}{3B_1}\right|\right\}. \tag{46}$$

Equations (43), (44) and (46) along with $|c_i| \leq 2$ and $|d_i| \leq 2$, gives the desired estimate on a_3 as asserted in (27) .
This completes the proof of Theorem 3.2.
Remark 3.3: If we set $\psi(z) = 1$ and $\phi(z) = [1 + (1-2\beta)z]/(1-z); (0 \leq \beta < 1)$ in Theorem 3.2, then we have $B_1 = B_2 = 2(1-\beta)$ and the class $\mathscr{S}_{\Sigma}^{*,q}(\phi)$ reduce to the class $\mathscr{S}_{\Sigma}^{*}(\beta)$ studied by Brannan and Taha [3]. Note that in the estimate of a_2 for the class $\mathscr{S}_{\Sigma}^{*}(\beta)$ we get an improvement in Theorem 3.1 given by Brannan and Taha [3].
Remark 3.4: If we set $\psi(z) = 1$ and $\phi(z) = [(1+z)/(1-z)]^{\alpha}; (0 < \alpha \leq 1)$ in Theorem 3.2, then we have $B_1 = 2\alpha$, $B_2 = 2\alpha^2$ and the class $\mathscr{S}_{\Sigma}^{*,q}(\phi)$ reduce to the class $\mathscr{S}_{\Sigma,\alpha}^{*}$ studied by Brannan and Taha [3]. Note that for the class $\mathscr{S}_{\Sigma,\alpha}^{*}$ we get the same estimate $|a_2| \leq 2\alpha/\sqrt{1+\alpha}$ as in Theorem 2.1 given by Brannan and Taha [3].

4. Coefficient Estimates for the Function Class $\mathscr{K}_{\Sigma}^{q}(\phi)$

Definition 4.1: *A function $f \in \Sigma$ given by (1) is said to be in the class $\mathscr{K}_{\Sigma}^{q}(\phi)$ if the following quasi-subordination holds:*

$$\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right] \prec_q (\phi(z) - 1)$$

and

$$\left[\left(1 + \frac{wg''(w)}{g'(w)}\right) - 1\right] \prec_q (\phi(w) - 1)$$

where $z, w \in \mathbb{U}$ and the functions g and ϕ are given by (2) and (5) respectively.
Theorem 4.2: *Let $f(z)$ given by (1) be in the class $\mathscr{K}_{\Sigma}^{q}(\phi)$. Then,*

$$|a_2| \leq min\left\{\sqrt{\frac{A_0^2B_1^2 + |A_0|(B_1 + |B_2 - B_1|)}{6}}, \frac{|A_0|B_1}{2}\right\} \tag{47}$$

and

$$|a_3| \leq min\left\{\frac{A_0^2B_1^2 + |A_0|(B_1 + |B_2 - B_1|) - |A_1|B_1}{6}, \frac{3A_0^2B_1^2 + 2(|A_0| + |A_1|)B_1}{12}\right\}. \tag{48}$$

Proof: Since $f \in \mathscr{K}_{\Sigma}^{q}(\phi)$, there exist two analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$ and a function ψ defined by (4) satisfies:

$$\left[\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right] = \psi(z)\left[\phi(u(z)) - 1\right] \tag{49}$$

and

$$\left[\left(1 + \frac{wg''(w)}{g'(w)}\right) - 1\right] = \psi(w)\left[\phi(v(w)) - 1\right]. \tag{50}$$

Proceeding similarly as in Theorem 2.2 and Theorem 3.2, we get

$$2a_2z + (6a_3 - 4a_2^2)z^2 + \cdots = \frac{1}{2}A_0B_1c_1z + \left\{\frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2\right\}z^2 + \cdots \tag{51}$$

and

$$-2a_2w + (8a_2^2 - 6a_3)w^2 + \cdots = \frac{1}{2}A_0B_1d_1w + \left\{\frac{1}{2}A_1B_1d_1 + \frac{1}{2}A_0B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{A_0B_2}{4}d_1^2\right\}w^2 + \cdots. \tag{52}$$

Equating the coefficients of z, z^2 in (51) and w, w^2 in (52), we get

$$2a_2 = \frac{1}{2}A_0B_1c_1, \tag{53}$$

$$6a_3 = A_0B_1c_1a_2 + \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2, \tag{54}$$

$$-2a_2 = \frac{1}{2}A_0B_1d_1 \tag{55}$$

and

$$(12a_2^2 - 6a_3) = -A_0B_1d_1a_2 + \frac{1}{2}A_1B_1d_1 + \frac{1}{2}A_0B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{A_0B_2}{4}d_1^2 \tag{56}$$

From (53) and (55), we get

$$c_1 = -d_1, \tag{57}$$

$$8a_2 = (c_1 - d_1)A_0B_1 \tag{58}$$

and

$$32a_2^2 = (c_1^2 + d_1^2)A_0^2B_1^2. \tag{59}$$

Adding (54) in (56) and then using (58) and (59), we get

$$48a_2^2 = 2A_0^2B_1^2c_1^2 + 2(c_2 + d_2)A_0B_1 + A_0(c_1^2 + d_1^2)(B_2 - B_1). \tag{60}$$

Clearly, (58), (59) and (60) along with $|c_i| \leq 2$ and $|d_i| \leq 2$, yields the desired result (47).
Next, subtracting (54) from (56) and then using (57), we get

$$-12a_3 = -12a_2^2 + \frac{1}{2}(d_1 - c_1)A_1B_1 + \frac{1}{2}(d_2 - c_2)A_0B_1. \tag{61}$$

Using (60) and (61), we get

$$48a_3 = 2A_0^2B_1^2c_1^2 + 4A_0B_1c_2 + A_0(c_1^2 + d_1^2)(B_2 - B_1) - 2(d_1 - c_1)A_1B_1. \tag{62}$$

Using (58) in (61), we get

$$-12a_3 = \frac{1}{2}(d_2-c_2)A_0B_1 + \frac{1}{2}(d_1-c_1)A_1B_1 - \frac{3(c_1-d_1)^2A_0^2B_1^2}{16}. \tag{63}$$

Clearly, (62) and (63) along with $|c_i| \leq 2$ and $|d_i| \leq 2$, yields the desired result (48).

This completes the proof of Theorem 4.2.

Remark 4.3: If we set $\psi(z) = 1$ and $\phi(z) = [1+(1-2\beta)z]/(1-z); (0 \leq \beta < 1)$ in Theorem 4.2, then we have $B_1 = B_2 = 2(1-\beta)$ and the class $\mathscr{K}_\Sigma^q(\phi)$ reduce to the class $\mathscr{K}_\Sigma(\beta)$ studied by Brannan and Taha [3]. Note that we get $|a_2| \leq 1-\beta$ and $|a_3| \leq (1-\beta)(3-2\beta)/3$ for the functions in the class $\mathscr{K}_\Sigma(\beta)$, which is an improvement in Theorem 4.1 given by Brannan and Taha [3].

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