



A new perspective on paranormed Riesz sequence space of non-absolute type

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Abstract

The current article mainly dwells on introducing Riesz sequence space $r^q(\tilde{B}_u^p)$ which generalized the prior studies of Candan & Güneş [28], Candan & Kılınc [30] and consists of all sequences whose $R_u^q \tilde{B}$ -transforms are in the space $\ell(p)$, where $\tilde{B} = B(r_n, s_n)$ stands for double sequential band matrix $(r_n)_{n=0}^\infty$ and $(s_n)_{n=0}^\infty$ are given convergent sequences of positive real numbers. Some topological properties of the new brand sequence space have been investigated as well as α - β - and γ -duals. Additionally, we have also constructed the basis of $r^q(\tilde{B}_u^p)$. Eventually, we characterize a matrix class on the sequence space. These results are more general and more comprehensive than the corresponding results in the literature.

Keywords: Paranormed sequence space, alpha-, beta- and gamma-duals and matrix mappings.

1. Introduction

First of all, we are going to introduce some notations and definitions which are going to be needed in later sections of the article. Let us begin by discussing the concept of a sequence. There are many ways of defining a sequence, each of which is an equivalent way of defining the same thing. A sequence is simply an ordered list $x_1, x_2, \dots, x_n, \dots$ of numbers. Such a sequence is called an infinite sequence. In this article sequences will be infinite thus so from now on when we speak of sequences we will mean infinite sequences. A slightly more sophisticated way of representing the sequence $a_1, a_2, \dots, a_n, \dots$ is with the notation $\{a_i\}_{i=0}^\infty$. We will often abbreviate $\{a_i\}_{i=0}^\infty$ simply with $\{a_i\}$. A sequence $\{a_n\}$ converges with limit a if each neighborhood of a contain almost all terms of the sequence. In this case we say that $\{a_n\}$ converges (or convergent) to a as n goes to ∞ . In a common parlance the words series and sequence are essentially synonymous, however, in mathematics the distinction between two is that a series is the sum of the terms of a sequence. Let $\{a_n\}$ be a sequence and define a new sequence $\{s_n\}$ by the recursion relation $\{s_1\} = \{a_1\}$ and $\{s_{n+1}\} = \{s_n\} + \{a_{n+1}\}$. The sequence $\{s_n\}$ is called the sequence of partial sums of $\{a_n\}$. Let $\{s_n\}$ be sequence of partial sums of $\{a_n\}$. If $\{s_n\}$ converges we say that $\{a_n\}$ is summable. In this case, we denote the $\lim_{k \rightarrow \infty} s_n$ by $\sum_{j=0}^\infty a_j$. The expression $\sum_{j=0}^\infty a_j$ is called an infinite series whether or not the sequence $\{a_n\}$ is summable. When we are given an infinite series $\sum_{j=0}^\infty a_j$ the sequence $\{a_n\}$ is called the sequence of terms. If the sequence of terms is summable, the infinite series is said to be convergent. The set of all convergent sequences in \mathbb{K} are denoted by c . A sequence $\{a_n\}$ in \mathbb{K} is called a null sequence if it converges to zero. The set of all null sequences in \mathbb{K} are denoted by c_0 , where \mathbb{K} denotes either of fields \mathbb{R} and \mathbb{C} . A sequence is bounded if the set of its terms have an upper bound and a lower bound. The set of all bounded sequences is denoted by ℓ_∞ . Any vector subspace

of $\omega = \omega(\mathbb{K}) = \mathbb{K}^{\mathbb{N}}$ is known as a sequence space and $\mathbb{N} := \{0, 1, 2, \dots\}$. It is clear that the sets c , c_0 and ℓ_∞ are the subspaces of the ω . Therefore, c , c_0 and ℓ_∞ , equipped with a vector space structure, forms a sequence space. Also by bs , cs , ℓ_1 and ℓ_p we denote the spaces of all bounded, convergent, absolutely and p -absolutely convergent series, respectively.

Now, let us give the definition of the triangle matrix. Let $T = (t_{nk})$ be a *triangle matrix*, that is $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is clear that $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x . Moreover, a triangle matrix U uniquely has an inverse $U^{-1} = V$ which is also a triangle matrix. Thus, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$.

The concept of matrix domain plays one of the most important role in this article. Therefore, its definition is presented in this paragraph. The *domain* λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A := \{x = (x_k) \in \omega : Ax \in \lambda\}, \tag{1}$$

which is a sequence space.

Even though generally the new sequence space λ_A produced by the limitation matrix A using a sequence space λ is either the expansion or the contraction of the space λ itself, sometimes those spaces may be observed overlap. In fact, it is easy see that the inclusion $\lambda_S \subset \lambda$ strictly holds for $\lambda \in \{\ell_\infty, c, c_0\}$. Because of this property, it can be deduced that the inclusion $\lambda \subset \lambda_{\Delta^{(1)}}$ also strictly holds for $\lambda \in \{\ell_\infty, c, c_0, \ell_p\}$. But, when $\lambda := c_0 \oplus span\{z\}$ is defined with $z = ((-1)^k)$, that is $x \in \lambda$ iff $x := s + \alpha z$ for some $s \in c_0$ and some $\alpha \in \mathbb{C}$, and take the matrix A with the rows A_n described by $A_n := (-1)^n e^{(n)}$ for all $n \in \mathbb{N}$ into consideration, we obtain $Ae = z \in \lambda$ while $Az = e \notin \lambda$ resulting in the sequences $z \in \lambda \setminus \lambda_A$ and $e \in \lambda_A \setminus \lambda$, here $e = (1, 1, 1, \dots)$ and $e^{(n)}$ represents a sequence of which only non-zero element is a 1 on the n^{th} place for each $n \in \mathbb{N}$. In other words, the sequence spaces λ_A and λ overlap while none of them contains the other one. For more detail on the domains of some triangle matrices in certain sequence spaces, the reader may refer to Baar's book entitled "Summability Theory and Its Applications" (see [1, p. 50]).

Another essential definition is the paranorm definition which is going to be needed in later. Now, let us give this concept. The function g on a X satisfies the properties of a paranorm $g(\theta) = 0$, $g(x) = g(-x)$, $g(x+y) = g(x)+g(y)$, $|\alpha_n - \alpha| \rightarrow 0$ and $g(x_n - x) \rightarrow 0$ imply $g(\alpha_n x_n - \alpha x) \rightarrow 0$ for all $\alpha \in \mathbb{R}$ and all $x \in X$, where θ is the zero vector in the linear space X . Recall that a linear topological space X over the real field \mathbb{R} with a paranorm obeying these rules is called a paranormed space.

From now on, let us assume that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max \{1, H\}$ and $1/p_k + 1/p'_k = 1$ provided $1 < \inf p_k \leq H < \infty$. Then, the linear spaces $\ell_\infty(p)$ and $\ell(p)$ were defined by Maddox in [2] and [3] (see also Simons [4] and Nakano [5]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\}$$

and

$$\ell_\infty(p) = \left\{ x = (x_k) \in w : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty \right\},$$

which are the complete spaces paranormed by

$$g_1(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M} \quad \text{and} \quad g_2(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/M} \quad \text{iff} \quad \inf p_k > 0$$

, respectively.

For the sake of simplicity, here and in what follows, it will be assumed that the summation without limits runs from 0 to ∞ .

Recently, the approach to construct a new sequence space by means of the matrix domain of a particular triangle has been used by some of the writers in many research articles. They defined and examined the sequence spaces $X_p = (\ell_p)_{C_1}$ in [6], $r^t(p) = (\ell(p))_{R^t}$ in [7], $e_p^r = (\ell_p)_{E^r}$ and $e^r(p) = (\ell(p))_{E^r}$ in [8, 9, 10]. $Z(u, v, \ell_p) = (\ell_p)_{G(u, v)}$ and $\ell(u, v, p) = (\ell(p))_{G(u, v)}$ in [11, 12], $a^r(p) = (\ell_p)_{A^r}$ and $a^r(u, p) = (\ell(p))_{A_u^r}$ in [13, 14], $bv_p = (\ell_p)_\Delta$ and $bv(u, p) = (\ell(p))_{A_u}$ in [15, 16, 17], $\overline{\ell(p)} = (\ell(p))_S$ in [18], $\ell_p^\lambda = (\ell_p)_\Lambda$ in [19], $\lambda_{B(r, s)}$ in [20] $\lambda_{B(\bar{r}, \bar{s})}$ in [21], $f_0(B)$ and $f(B)$ in [22], $f_0(\tilde{B})$ and $f(\tilde{B})$ in [23] and etc., where $C_1 = \{c_{nk}\}$, $R^t = \{r_{nk}^t\}$, $E^r = \{e_{nk}^r\}$, $S = \{s_{nk}\}$, $\Delta = \{\delta_{nk}\}$, $G(u, v) = \{g_{nk}\}$, $\Delta^{(m)} = \{\Delta_{nk}^{(m)}\}$, $A^r = \{a_{nk}^r\}$, $A_u^r = \{a_{nk}^r(r)\}$, $A^u = \{a_{nk}^u\}$, $B(r, s) =$

$\{b_{nk}(r, s)\}$, $B(\tilde{r}, \tilde{s}) = \{b_{nk}(\tilde{r}, \tilde{s})\}$, $\Lambda = \{\lambda_{nk}\}_{n,k=0}^{\infty}$ and $A(\lambda) = \{a_{nk}(\lambda)\}$ denote the Cesro, Riesz, Euler, generalized weighted means or factorable matrix, summation matrix, difference matrix, generalized difference matrix and sequential band matrix, respectively. Moreover, the ones who are more interested in the subject are advised to read [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]. We should note here, there are many different ways to construct new sequence spaces from old ones. To get more detailed information, one can look at the articles [53, 54, 55, 56].

In this paragraph, we shall introduce the notion of a matrix transformation from X to Y . Let X, Y be any two sequence spaces. Given any infinite matrix $A = (a_{nk})$ of real numbers a_{nk} , where $n, k \in \mathbb{N}$, any sequence x , we write $Ax = ((Ax)_n)$, the A -transform of x , if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}$. If $x \in X$ implies that $Ax \in Y$ then we say that A defines a matrix mapping from X into Y and denote it by $A : X \rightarrow Y$. By $(X : Y)$, we mean the class of all infinite matrices such that $A : X \rightarrow Y$.

In the present article, we introduce the new sequence spaces derived by Riesz mean (R, q_n) and generalized difference matrix $\tilde{B}(r, s)$.

The layout of the rest of the present article is organized as follows:

Section 2 is devoted to the spaces of difference sequences and some historical developments related to this matter are given. Additionally, the concept of generalized difference matrix is introduced. In section 3, the paranormed sequence space $r^q(\tilde{B}_u^p)$ of non-absolute type which is the set of all sequences whose $R_u^q \tilde{B}$ -transforms are in the spaces $\ell(p)$ and then their alpha-, beta- and gamma-duals are computed. In addition to this, the basis of the space $r^q(\tilde{B}_u^p)$ is obtained. In the final section of the article, we characterize a matrix class on the sequence space.

2. Difference sequence spaces

In this section, we are going to give some knowledge about literature concerning the spaces of difference sequence.

The difference sequence spaces have been studied by several authors in different ways. At first, Kızmaz introduced the difference sequence space in 1981. Now, let us briefly explain further. If $\lambda \in \{\ell_\infty, c, c_0\}$ then, $\lambda(\Delta)$ consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k+1}) \in \lambda$ is called as the *difference sequence spaces* which were introduced by Kızmaz [57]. In recent years, the difference spaces bv_p consisting of the sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_p$ have been studied in the case $0 < p < 1$ by Altay and Başar [58], and in the case $1 \leq p < \infty$ by Başar and Altay [59], and Çolak, Et and Malkowsky [60].

The concept of difference sequences was generalized by Çolak and Et [61]. They defined and examined the sequence spaces

$$\Delta^m \lambda = \left\{ x = (x_k) \in \omega : \Delta^m x \in \lambda \right\},$$

where $\Delta^1 x = (x_k - x_{k+1})$ and $\Delta^m x = \Delta(\Delta^{m-1} x)$ for $m \in \{1, 2, 3, \dots\}$. In [62], Malkowsky and Parashar introduced the sequence spaces as follows

$$\Delta^{(m)} \lambda = \left\{ x = (x_k) \in \omega : \Delta^{(m)} x \in \lambda \right\},$$

where $m \in \mathbb{N}$, $\Delta^{(1)} x = (x_k - x_{k-1})$ and $\Delta^{(m)} x = \Delta^{(1)}(\Delta^{(m-1)} x)$. More recently, in [63], Polat and Başar introduced the spaces $e_0^r(\Delta^{(m)})$, $e_c^r(\Delta^{(m)})$ and $e_\infty^r(\Delta^{(m)})$ consisting of all sequences whose m^{th} order differences are in the Euler spaces e_0^r , e_c^r and e_∞^r , respectively. Finally, Altay [64] studied the space $\ell_p(\Delta^{(m)})$ consisting of all sequences whose m^{th} order differences are p -absolutely summable which is a generalization of the spaces bv_p defined by Başar and Altay [59], and Çolak, Et and Malkowsky [60].

Let r_n and s_n be non-zero real numbers for all $n \in \mathbb{N}$, and define the double sequential band matrix $\tilde{B} = B(\tilde{r}, \tilde{s}) = \{b_{nk}(\tilde{r}, \tilde{s})\}$ by

$$b_{nk}(\tilde{r}, \tilde{s}) := \begin{cases} r_n & , \quad (k = n), \\ s_n & , \quad (k = n - 1), \\ 0 & , \quad (0 \leq k < n - 1 \text{ or } k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$. Let us note here that the matrix $B(\tilde{r}, \tilde{s})$ can be reduced to the generalized difference matrix $B(r, s)$ in the case $r_n = r$, $s_n = s$ for all $n \in \mathbb{N}$. Thus, the results related to the domain of the matrix $B(\tilde{r}, \tilde{s})$ are the generalization of the corresponding consequences of the matrix domain of $B(r, s)$.

Generalized difference matrix $B(r, s)$ has been used by some of the writers in many research articles. Purely for the development of this approach, the articles Kirişçi and Başar [20, 22] are recommended.

3. The Riesz Sequence Space $r^q(\tilde{B}_u^p)$ of Non-absolute Type

In this section, we will focus on the new paranormed sequence space $r^q(\tilde{B}_u^p)$, using the Riesz mean and double sequential band matrix.

Before going into the details, we first introduce some notations and definitions. We begin by talking about the Riesz mean.

The transformation given by

$$t_n = \frac{q_0x_0 + q_1x_1 + \dots + q_nx_n}{Q_n}$$

is called the Riesz mean (T, q_n) or simply the (T, q_n) means of a sequence (x_n) , where (q_k) is a sequence of positive numbers and $Q_n = q_0 + q_1 + \dots + q_n$.

The matrix of the (T, q_n) method is given by

$$t_{nk}^q := \begin{cases} \frac{q_k}{Q_n} & , \quad (0 \leq k \leq n), \\ 0 & , \quad (k > n). \end{cases}$$

In order to avoid any confusion throughout the article the symbol (R, q_n) will be used instead of Riesz mean (T, q_n) . Now, we are ready to establish the set $r^q(\tilde{B}_u^p)$, using the Riesz mean and double sequential band matrix.

For $0 < p_k \leq H < \infty$, let us define the set $r^q(\tilde{B}_u^p)$ as the set of all sequences whose $R_u^q \tilde{B}$ -transforms is in the sequence space $\ell(p)$, that is

$$r^q(\tilde{B}_u^p) = \left\{ x = (x_k) \in w : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^k u_j q_j \tilde{B}x_j \right|^{p_k} < \infty \right\},$$

where $u = (u_k)$ is an arbitrary fixed sequence.

With the help of the notation of (1), we can rewrite the set $r^q(\tilde{B}_u^p)$ by

$$r^q(\tilde{B}_u^p) = \{\ell_p\}_{R_u^q \tilde{B}}$$

where $R_u^q \tilde{B} = (r_{nk}^{q_u})$ matrix defined as follows:

$$r_{nk}^{q_u} = \begin{cases} \frac{1}{Q_n} (r_k u_k q_k + s_k u_{k+1} q_{k+1}) & , \quad 0 \leq k \leq n-1, \\ \frac{r_n q_n u_n}{Q_n} & , \quad k = n, \\ 0 & , \quad k > n. \end{cases}$$

Ahead of frequently used sequence $y = (y_k)$ by the $R_u^q \tilde{B}$ - transform of any given sequence $x = (x_k)$, i.e.,

$$y_k = \frac{1}{Q_k} \sum_{j=0}^k u_j q_j \tilde{B}x_j. \tag{2}$$

From now on when we speak of the sequences $x = (x_k)$ and $y = (y_k)$, we will mean that they are connected with the relation (2).

Now, it is time to give the following theorem.

Theorem 3.1 *The set $r^q(\tilde{B}_u^p)$ is a linear space together with coordinatewise addition and scalar multiplication, in other words, $r^q(\tilde{B}_u^p)$ represents the sequence space.*

Proof: Since the proof of this theorem can be obtained by using elementary linear algebra, we omit the details.

Let us return to explaining our main subject. More recently, the Riesz sequence spaces $r^q(u, p)$ and $r^q(\Delta_u^p)$ of non-absolute type have been introduced and studied by Ganie and Sheikh [65, 66]. After then, some new Riesz sequence spaces have been introduced and examined Candan & Güneş [28] and Candan & Kılınç [30]. When compared to the corresponding results in the literature; it is seen that the results of the present study are more general and more inclusive.

Now, let us start with one of the main results which is going to be used in later sections.

Theorem 3.2 Let $0 < p_k \leq H < \infty$. Then, $r^q(\tilde{B}_u^p)$ is the complete linear metric space paranormed by g , described via the following equality

$$g_{\tilde{B}}(x) = \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} (r_j u_j q_j + s_j u_{j+1} q_{j+1}) x_j + \frac{r_k u_k q_k}{Q_k} x_k \right|^{p_k} \right]^{\frac{1}{M}}$$

Proof: Clearly, in order to prove the theorem, it is sufficient to show that the conditions of the paranorm are satisfied. To do this, firstly we show that the linearity of $r^q(\tilde{B}_u^p)$ with respect to the coordinate wise addition and scalar multiplication is true. Let us assume that $z, x \in r^q(\tilde{B}_u^p)$. Therefore, the linearity of $r^q(\tilde{B}_u^p)$ is obtained from the following rudimentary calculations

$$\begin{aligned} g_{\tilde{B}}(x+z) &= \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} (r_j u_j q_j + s_j u_{j+1} q_{j+1})(x_j + z_j) + \frac{r_k u_k q_k}{Q_k} (x_k + z_k) \right|^{p_k} \right]^{\frac{1}{M}} \\ &\leq \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} (r_j u_j q_j + s_j u_{j+1} q_{j+1}) x_j + \frac{r_k u_k q_k}{Q_k} x_k \right|^{p_k} \right]^{\frac{1}{M}} \\ &\quad + \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} (r_j u_j q_j + s_j u_{j+1} q_{j+1}) z_j + \frac{r_k u_k q_k}{Q_k} z_k \right|^{p_k} \right]^{\frac{1}{M}} \\ &= g_{\tilde{B}}(x) + g_{\tilde{B}}(z) \end{aligned} \tag{3}$$

for any $\mu \in \mathbb{R}$ (see [2, p. 30])

$$|\mu|^{p_k} \leq \max\{1, |\mu|^M\}. \tag{4}$$

It is fairly easy to get the next two conditions $g_{\tilde{B}}(\theta) = 0$ and $g_{\tilde{B}}(x) = g_{\tilde{B}}(-x)$ which are valid for all $x \in r^q(\tilde{B}_u^p)$. One more time, the inequalities (3) and (4) result in the subadditivity of g and the following inequality

$$g_{\tilde{B}}(\mu x) \leq \max\{1, |\mu|^M\} g_{\tilde{B}}(x)$$

obviously holds.

Now let us observe that the scalar multiplication is continuous. To do this, let us assume that (x^n) be arbitrary sequence of the points lying in $r^q(\tilde{B}_u^p)$ such that $g_{\tilde{B}}(x^n - x) \rightarrow 0$ and (μ_n) also be arbitrary sequence of scalars such that $\mu_n \rightarrow \mu$. We can get

$$\begin{aligned} g_{\tilde{B}}(\mu_n x^n - \mu x) &= \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} (r_j u_j q_j + s_j u_{j+1} q_{j+1})(\mu_n x_j^n - \mu x_j) \right|^{p_k} \right]^{\frac{1}{M}} \\ &\leq |\mu_n - \mu|^{\frac{1}{M}} g_{\tilde{B}}(x^n) + |\mu|^{\frac{1}{M}} g_{\tilde{B}}(x^n - x) \end{aligned}$$

tending to be zero if we take $n \rightarrow \infty$ since $\{g_{\tilde{B}}(x^n)\}$ is bounded due to the inequality

$$g_{\tilde{B}}(x^n) \leq g_{\tilde{B}}(x) + g_{\tilde{B}}(x^n - x)$$

which is valid because of subadditive of $g_{\tilde{B}}$. This shows that the scalar multiplication is continuous. Namely, from now on, we say that $g_{\tilde{B}}$ is paranorm on the space $r^q(\tilde{B}_u^p)$.

Here, if we prove the completeness of the space $r^q(\tilde{B}_u^p)$ then the proof ends. Let us suppose that $\{x^i\}$ be an arbitrary Cauchy sequence in the space $r^q(\tilde{B}_u^p)$, where $x^i = \{x_0^i, x_1^i, \dots\}$. In that case, there exists a positive integer $n_0(\epsilon)$

$$g_{\tilde{B}}(x^i - x^j) < \infty \tag{5}$$

for all $i, j \geq n_0(\epsilon)$ for a given $\epsilon > 0$. We get by using definition of $g_{\tilde{B}}$, for each fixed $k \in \mathbb{N}$

$$\left| (R_u^q \tilde{B} x^i)_k - (R_u^q \tilde{B} x^j)_k \right| \leq \left[\sum_k \left| (R_u^q \tilde{B} x^i)_k - (R_u^q \tilde{B} x^j)_k \right|^{p_k} \right]^{\frac{1}{M}} < \infty$$

for $i, j \geq n_0(\epsilon)$. This result in the fact that $\left\{ (R_u^q \tilde{B}x^0)_k, (R_u^q \tilde{B}x^1)_k, \dots \right\}$ is a Cauchy sequence of real number for every fixed $k \in \mathbb{N}$. Since we already know that \mathbb{R} is complete, it converge, say $(R_u^q \tilde{B}x^i)_k \rightarrow (R_u^q \tilde{B}x)_k$ as $i \rightarrow \infty$. Using these infinitely many limits $(R_u^q \tilde{B}x)_0, (R_u^q \tilde{B}x)_1, \dots$ we describe the sequence $\left\{ (R_u^q \tilde{B}x)_0, (R_u^q \tilde{B}x)_1, \dots \right\}$. From (5) for each $m \in \mathbb{N}$ and $i, j \geq n_0(\epsilon)$, we have

$$\sum_{k=0}^m \left| (R_u^q \tilde{B}x^i)_k - (R_u^q \tilde{B}x^j)_k \right|^{p_k} \leq g_{\tilde{B}}(x^i - x^j)^M < \epsilon^M. \tag{6}$$

Let us pass to limit first as $j \rightarrow \infty$ and next as $m \rightarrow \infty$ in (6), then we obtain

$$g_{\tilde{B}}(x^i - x) \leq \infty.$$

Finally, if we take $\epsilon = 1$ in (6) and $i \geq n_0(1)$, then using Minkowsky's inequality for each $m \in \mathbb{N}$, we have

$$\left[\sum_{k=0}^m \left| (R_u^q \tilde{B}x)_k \right|^{p_k} \right]^{\frac{1}{M}} \leq g_{\tilde{B}}(x^i - x) + g_{\tilde{B}}(x^i) \leq 1 + g_{\tilde{B}}(x^i).$$

This imply that $x \in r^q(\tilde{B}_u^p)$. Because of the fact that $g_{\tilde{B}}(x^i - x) \leq \infty$ for all $i \geq n_0(\epsilon)$, it is reached the end of that $x^i \rightarrow x$ as $i \rightarrow \infty$, thus we show that $r^q(\tilde{B}_u^p)$ is complete and proof of the theorem is completed.

Note that, it can easily be seen that the absolute property is invalid on the space $r^q(\tilde{B}_u^p)$, that is $g_{\tilde{B}}(x) \neq g_{\tilde{B}}(|x|)$ for at least one sequence in the space $r^q(\tilde{B}_u^p)$ and this says us that $r^q(\tilde{B}_u^p)$ is a sequence space of non-absolute type.

Theorem 3.3 *Let $0 < p_k \leq H < \infty$. Then the sequence space $r^q(\tilde{B}_u^p)$ is linearly isomorphic to the space $\ell(p)$.*

Proof: The first step in proving the theorem is to show the existence of a linear bijection between the spaces $r^q(\tilde{B}_u^p)$ and $\ell(p)$, where $0 < p_k \leq H < \infty$. For this, we use the notation of (3), consider the transformation T that defined from $r^q(\tilde{B}_u^p)$ to $\ell(p)$ by $x \rightarrow y = Tx$. Indeed, the linearity of T is fairly easy. Also it is clear that $x = \theta$ whenever $Tx = \theta$. Hence T is injective.

Let $y \in \ell(p)$ and define the sequence $x = (x_k)$ as follows:

$$x_k = \sum_{n=0}^{k-1} \prod_{j=n+1}^k \left(\frac{-s_j}{r_{j+1}} \right) \left(\frac{1}{r_n u_n q_n} + \frac{1}{s_n u_{n+1} q_{n+1}} \right) Q_n y_n + \frac{Q_k}{r_k u_k q_k} y_k$$

for $k \in \mathbb{N}$. We eventually obtain,

$$\begin{aligned} g_{\tilde{B}}(x) &= \left[\sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k-1} (r_j u_j q_j + s_j u_{j+1} q_{j+1}) x_j + \frac{r_k u_k q_k}{Q_k} x_k \right|^{p_k} \right]^{\frac{1}{M}} \\ &= \left[\sum_k \left| \sum_{j=0}^k \delta_{kj} y_j \right|^{p_k} \right]^{\frac{1}{M}} \\ &= \left[\sum_k |y_k|^{p_k} \right]^{\frac{1}{M}} \\ &= g_1(y) < \infty \end{aligned}$$

where

$$\delta_{kj} = \begin{cases} 1 & , \quad k = j, \\ 0 & , \quad k \neq j. \end{cases}$$

Consequently, we obtain that $x \in r^q(\tilde{B}_u^p)$ i.e., T is surjective and its paranorm is preserving. Hence T is a linear bijection and the proof is completed.

4. Basis and $\alpha-$, $\beta-$ and $\gamma-$ duals of the space $r^q(\tilde{B}_u^p)$

In this section, after we define $M(\lambda, \mu)$ multiplier space of any sequence spaces λ and μ , we will research basis and $\alpha-$, $\beta-$ and $\gamma-$ duals of the space $r^q(\tilde{B}_u^p)$.

If $\lambda, \mu \subset w$ and z arbitrary sequence, we can write

$$z^{-1} * \lambda = \{x = (x_k) \in w : xz = (x_k z_k) \in \lambda\}$$

and

$$M(\lambda, \mu) = \cap_{x \in \lambda} x^{-1} * \mu.$$

If we choose $\mu = \ell_1, cs$ and bs , then we obtain the $\alpha-$, $\beta-$ and $\gamma-$ duals of the space λ , respectively as

$$\lambda^\alpha = M(\lambda, \ell_1) = \{a = (a_k) \in w : ax = (a_k x_k) \in \ell_1 \text{ for all } x \in \lambda\},$$

$$\lambda^\beta = M(\lambda, cs) = \{a = (a_k) \in w : ax = (a_k x_k) \in cs \text{ for all } x \in \lambda\},$$

$$\lambda^\gamma = M(\lambda, bs) = \{a = (a_k) \in w : ax = (a_k x_k) \in bs \text{ for all } x \in \lambda\}.$$

Let us now state the following lemmas. In this way, the fundamental results will be used in proofs of our theorems.

Lemma 4.1 [67]

(i) Let $1 < p_k \leq H < \infty$. Then $A \in (\ell(p) : \ell_1)$ if and only if there exists an integer $B > 1$ such that

$$\sup_{K \in F} \sum_k \left| \sum_{n \in K} a_{nk} B^{-1} \right|^{p'_k} < \infty.$$

(ii) Let $0 < p_k \leq 1$. Then $A \in (\ell(p) : \ell_1)$ if and only if

$$\sup_{K \in F} \sup_k \left| \sum_{n \in K} a_{nk} \right|^{p_k} < \infty.$$

Lemma 4.2 [68]

(i) Let $1 < p_k \leq H < \infty$. Then $A \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $B > 1$ such that

$$\sup_n \sum_k |a_{nk} B^{-1}|^{p'_k} < \infty. \tag{7}$$

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (\ell(p) : \ell_\infty)$ if and only if

$$\sup_{n,k} |a_{nk}|^{p_k} < \infty. \tag{8}$$

Lemma 4.3 [68] $A \in (\ell(p) : c)$ if and only if there exists an integer $B > 1$ provided that (7) and (8) hold,

$$\lim_n a_{nk} = \beta_k \text{ for } k \in \mathbb{N} \tag{9}$$

also holds. Where $0 < p_k \leq H < \infty$ for every given $k \in \mathbb{N}$.

Theorem 4.4 Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$ and the sets $D_1(u, p), D_2(u, p)$ and $D_3(u, p)$ are defined by following equations

$$D_1(u, p) = \bigcup_{B > 1} \left\{ a = (a_k) \in w : \sup_{K \in F} \sum_k \left| \sum_{n \in K} \left[\prod_{j=n}^{k-1} \left(\frac{-s_j}{r_{j+1}} \right) A_k a_n Q_k + \frac{a_n}{r_n u_n q_n} Q_n \right] B^{-1} \right|^{p'_k} < \infty \right\},$$

$$D_2(u, p) = \bigcup_{B>1} \left\{ a = (a_k) \in w : \sup_n \sum_k \left| \left[\left(\frac{a_k}{r_k u_k q_k} + A_k \sum_{i=k+1}^n a_i \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \right) Q_k \right] B^{-1} \right|^{p_k} < \infty \right\},$$

where $A_k = A(r_k, s_k, u_k, q_k) = \frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}}$, and

$$D_3(u, p) = \left\{ a = (a_k) \in w : \lim_{n \rightarrow \infty} \sum_{i=k+1}^n a_i \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \text{ exists} \right\}.$$

In this case,

$$\left[r^q(\tilde{B}_u^p) \right]^\alpha = D_1(u, p), \quad \left[r^q(\tilde{B}_u^p) \right]^\beta = D_2(u, p) \cap D_3(u, p), \quad \left[r^q(\tilde{B}_u^p) \right]^\gamma = D_2(u, p).$$

Proof: Firstly, let research that α - dual of space $r^q(\tilde{B}_u^p)$. Hence we will consider definition of α - dual. Let us suppose that any $a = (a_k) \in w$. Then we easily obtain by means of that

$$\begin{aligned} a_n x_n &= \sum_{k=0}^{n-1} \prod_{j=k}^{n-1} \left(\frac{-s_{j-1}}{r_j} \right) \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) a_n Q_k y_k + \frac{a_n}{r_n u_n q_n} Q_n y_n \\ &= (Dy)_n \end{aligned} \tag{10}$$

where the matrix $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} \prod_{j=n}^k \left(\frac{-s_{j-1}}{r_j} \right) \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) a_n Q_n & , \quad 0 \leq k \leq n-1, \\ \frac{a_n}{r_n u_n q_n} Q_n & , \quad k = n, \\ 0 & , \quad k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$. Thus we observe from (10) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_n) \in r^q(\tilde{B}_u^p)$ if and only if $Dy \in \ell_1$ whenever $y \in \ell(p)$. This means that $D \in (\ell(p), \ell_1)$. Thus we obtain from Lemma 4.1 (ii) that $\left[r^q(\tilde{B}_u^p) \right]^\alpha = D_1(u, p)$.

Now, let us research that β - dual of space $r^q(\tilde{B}_u^p)$, using the definition of β - dual. We consider following equation,

$$\begin{aligned} \sum_{k=0}^n a_k x_k &= \sum_{k=0}^n \left[\left(\frac{a_k}{r_k u_k q_k} + \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^n a_i \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \right) Q_k \right] y_k \\ &= (Ey)_n \end{aligned} \tag{11}$$

where, $E = (e_{nk})$ is defined as following

$$e_{nk} = \begin{cases} \left(\frac{a_k}{r_k u_k q_k} + \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^n a_i \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \right) Q_k & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n. \end{cases}$$

From (11), $ax = (a_k x_k) \in cs$ whenever $x \in r^q(\tilde{B}_u^p)$ if and only if $Ey \in c$ whenever $y \in \ell(p)$. In other words $E \in (\ell(p), c)$. We obtain $\left[r^q(\tilde{B}_u^p) \right]^\beta = D_2(u, p) \cap D_3(u, p)$, using Lemma 4.3.

Finally, let find out that γ - dual of space $r^q(\tilde{B}_u^p)$, using the definition of γ - dual. Using (11) $ax = (a_k x_k) \in bs$ whenever $x \in r^q(\tilde{B}_u^p)$ if and only if $Ey \in \ell_\infty$ whenever $y \in \ell(p)$. In other words, $a = (a_k) \in [r^q(\tilde{B}_u^p)]^\gamma$ iff $E \in (\ell(p), \ell_\infty)$. Then from Lemma 4.2 (ii) obtain $\left[r^q(\tilde{B}_u^p) \right]^\gamma = D_2(u, p)$. Hence, the proof is completed.

Theorem 4.5 Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$ and define the sets $D_4(u, p)$ and $D_5(u, p)$ with the following equations

$$D_4(u, p) = \left\{ a = (a_k) \in w : \sup_{K \in F} \sup_k \left| \sum_{n \in K} \left[\prod_{j=n+1}^k \left(\frac{-s_{j-1}}{r_j} \right) \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) a_n Q_k + \frac{a_n}{r_n u_n q_n} Q_n \right] \right|^{p_k} < \infty \right\},$$

$$D_5(u, p) = \left\{ a = (a_k) \in w : \sup_k \left| \left[\left(\frac{a_k}{r_k u_k q_k} + \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^n a_i \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \right) Q_k \right] \right|^{p_k} < \infty \right\}.$$

Then,

$$\left[r^q(\tilde{B}_u^p) \right]^\alpha = D_4(u, p), \quad \left[r^q(\tilde{B}_u^p) \right]^\beta = D_3(u, p) \cap D_5(u, p), \quad \left[r^q(\tilde{B}_u^p) \right]^\gamma = D_5(u, p).$$

Proof: The proof of theorem is obtained alike the proof of Theorem 4.4.

Theorem 4.6 Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}$ of the elements of the space $r^q(\tilde{B}_u^p)$ for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)}(q) = \begin{cases} \frac{Q_k}{r_k u_k q_k}, & n = k, \\ \prod_{j=k+1}^n \left(\frac{-s_{j-1}}{r_j} \right) \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) Q_k, & n > k, \\ 0, & n < k. \end{cases}$$

Then, the sequence $b^{(k)}(q)$ is a basis for the space $r^q(\tilde{B}_u^p)$ and any $x \in r^q(\tilde{B}_u^p)$ has a unique representation of the form

$$x = \sum_k \lambda_k(q) b^k(q) \tag{12}$$

where, $\lambda_k(q) = (R_u^q \tilde{B}x)_k$ for all $k \in \mathbb{N}$.

Proof: Let $0 < p_k \leq H < \infty$. Then it is not difficult to verify of the relation $\{b^{(k)}(q)\} \subset r^q(\tilde{B}_u^p)$. Really, for $k \in \mathbb{N}$

$$R_u^q \tilde{B}b^{(k)}(q) = e^{(k)} \in \ell(p) \tag{13}$$

where $e^{(k)}$ is a sequence that 1 in k^{th} term for each $k \in \mathbb{N}$ another term 0.

Moreover, let $x \in r^q(\tilde{B}_u^p)$. For all non-negative integer m , we put

$$x^{[m]} = \sum_{k=0}^m \lambda_k(q) b^{(k)}(q). \tag{14}$$

In this case, we have that $R_u^q \tilde{B}$ to (14) that for $i, m \in \mathbb{N}$

$$R_u^q \tilde{B}x^{[m]} = \sum_{k=0}^m \lambda_k(q) R_u^q \tilde{B}b^{(k)}(q) = \sum_{k=0}^m (R_u^q \tilde{B}x)_k e^{(k)}$$

and hence

$$\left(R_u^q \tilde{B}x - x^{[m]} \right)_i = \begin{cases} 0 & , \quad 0 \leq i \leq m, \\ (R_u^q \tilde{B}x)_i & , \quad i > m. \end{cases}$$

Also, for any given $\epsilon > 0$, there exists an integer m_0 such that for every $m \geq m_0$

$$\left(\sum_{i=m_0}^{\infty} \left| (R_u^q \tilde{B}x)_i \right|^{p_k} \right)^{\frac{1}{M}} < \frac{\epsilon}{2}.$$

Hence, we obtain for all $m \geq m_0$ that

$$\begin{aligned} g_{\tilde{B}}(x - x^m) &= \left(\sum_{i=m}^{\infty} \left| (R_u^q \tilde{B}x)_i \right|^{p_k} \right)^{\frac{1}{M}} \\ &\leq \left(\sum_{i=m_0}^{\infty} \left| (R_u^q \tilde{B}x)_i \right|^{p_k} \right)^{\frac{1}{M}} \\ &< \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

A little rewriting and the use of limit properties give $\lim_{m \rightarrow \infty} g_{\tilde{B}}(x - x^m) = 0$ that is x is represented as (12).

Now, if we prove the uniqueness of the representation (12) of $x \in r^q(\tilde{B}_u^p)$ then the proof ends. For this, suppose the contrary. That is, it have two representation like $x = \sum_k \mu_k(q)b^{(k)}$ and $x = \sum_k \lambda_k(q)b^{(k)}$.

Since we know that the linear transformation from $r^q(\tilde{B}_u^p)$ to $\ell(p)$ is continuous, we get

$$\begin{aligned} (R_u^q \tilde{B}x)_n &= \sum_k \mu_k(q) \left(R_u^q \tilde{B}b^{(k)}(q) \right)_n \\ &= \sum_k \mu_k(q) e_n^{(k)} = \mu_n(q) \end{aligned}$$

for $n \in \mathbb{N}$. We acknowledge that $(R_u^q \tilde{B}x)_n = \lambda_n$ for all $n \in \mathbb{N}$. Hence $\lambda_n(q) = \mu_n(q)$ and so unique of representation (12) is obtained. This ends the proof of the last part of the theorem.

5. Matrix Mapping on the Space $r^q(\tilde{B}_u^p)$

One of the most important ideas is matrix transformation in this article. So, we focus on this concept in the present section. Now, we emphasize the characterization of $(r^q(\tilde{B}_u^p), \ell_\infty)$.

Theorem 5.1 (i) $A \in (r^q(\tilde{B}_u^p), \ell_\infty)$ if and only if there exists an integer $B > 0$ such that

$$C(B) = \sup_n \sum_k \left| \left[\left(\frac{a_{nk}}{r_k u_k q_k} \right) + \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^n a_{ni} \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \right] Q_k B^{-1} \right|^{p'_k} < \infty \tag{15}$$

and

$$\{a_{nk}\}_{k \in \mathbb{N}} \in cs \quad (n \in \mathbb{N})$$

where $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$.

(ii) $A \in (r^q(\tilde{B}_u^p), \ell_\infty)$ if and only if

$$\sup_{n,k} \left| \left[\left(\frac{a_{nk}}{r_k u_k q_k} \right) + \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^n a_{ni} \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \right] Q_k \right|^{p_k} < \infty \tag{16}$$

and

$$\{a_{nk}\}_{k \in \mathbb{N}} \in cs \quad (n \in \mathbb{N})$$

where $0 < p_k \leq 1 < \infty$ for every $k \in \mathbb{N}$.

Proof: (i) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$ and $A \in (r^q(\tilde{B}_u^p), \ell_\infty)$. Then Ax exists for $x \in r^q(\tilde{B}_u^p)$, $\{a_{nk}\}_{k \in \mathbb{N}} \in [r^q(\tilde{B}_u^p)]^\beta$ for each $n \in \mathbb{N}$. Also consider the following equality obtained by using the relation (10) that

$$\sum_{k=0}^m a_{nk} x_k = \sum_{k=0}^m \left[\left(\frac{a_{nk}}{r_k u_k q_k} + \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^n a_{ni} \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \right) Q_k \right] y_k. \tag{17}$$

From Lemma (4.1) and (17), we obtain (15) expression.

Conversely, supposed that is provided (15) expression and $\{a_{nk}\}_{k \in \mathbb{N}} \in cs$ for each $n \in \mathbb{N}$ $x \in r^q(\tilde{B}_u^p)$. Since $\{a_{nk}\}_{k \in \mathbb{N}} \in [r^q(\tilde{B}_u^p)]^\beta$ for every fixed $n \in \mathbb{N}$, A-transform of x exists. We easily can derive from (17) as $m \rightarrow \infty$ that

$$\sum_{k=0}^\infty a_{nk} x_k = \sum_{k=0}^\infty \left[\left(\frac{a_{nk}}{r_k u_k q_k} + \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^\infty a_{ni} \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \right) Q_k \right] y_k. \tag{18}$$

Now, by combining (18) and inequality holding for an arbitrary $Z > 0$ and complex numbers a, b

$$|ab| \leq Z \left\{ |aZ^{-1}|^{p'} + |b|^p \right\}$$

where $p > 1$ and $1/p + 1/p' = 1$. We can easily obtain

$$\begin{aligned} \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right| &\leq \sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} \left[\left(\frac{a_{nk}}{r_k u_k q_k} + \left(\frac{1}{r_k u_k q_k} + \frac{1}{s_k u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^{\infty} a_{ik} \prod_{j=k+1}^i \left(\frac{-s_{j-1}}{r_j} \right) \right) Q_k \right] |y_k| \\ &\leq Z[C(B) + g_1^M(y)] < \infty. \end{aligned}$$

This mean that $Ax \in \ell_{\infty}$ whenever $x \in r^q(\tilde{B}_u^p)$. Hence, the proof is completed. (ii) The proof of (ii) be done alike (i).

6. Note

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