Global Journal of Mathematical Analysis, 3 (3) (2015) 134-139
www.sciencepubco.com/index.php/GJMA

doi: 10.14419/gjma.v3i3.5096
Research Paper

Power convexity of a class of Hessian equations in the ball

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Abstract

Power convexities of a class of Hessian equations are considered in this paper. It is proved that some power functions of the smooth admissible solutions to the Hessian equations are strictly convex in the ball. For a special case of the equation, a lower bound principal curvature and Gaussian curvature estimates are given.

Keywords: *Admissible Solution; Hessian Equations; Power Convexity; Strict Convexity.*

1. Introduction

Convexity of solutions to partial differential equations is an interesting issue and has been investigated for a long time. One interesting question in the study of convexity is the following: Is there a monotone real function f, such that $f(u(x))$ is concave or convex. We recall here some results concerning the power convexity. A typical example is that in 1971, Makar-Limanov [1] considered the following elliptic boundary value problem

$$\begin{cases} \triangle u = -1 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega. \end{cases}$$

in a bounded and convex planar domain. By an ingenious argument involving the maximum principle, he proved that the square root $u^{\frac{1}{2}}$ of the solution u is strictly concave. Another well-known example is that in 1976, Brascamp-Lieb [2] used a probabilistic approach to establish the log-concavity of the fundamental solution of diffusion equation with convex potential in a bounded and convex domain in $\mathbb{R}^n$. Consequently, they proved the log-concavity of the first eigenfunction of the Laplacian operator in convex domain. Similar to linear equations, the same kind phenomena appears for the fully nonlinear operators. One typical result is that Ma-Xu [3] considered the smooth admissible solution u of the following Hessian equation

$$\begin{cases} \sigma_2(D^2u) = 1 & in \quad \Omega \subset \mathbb{R}^3, \\ \quad u = 0 & on \quad \partial\Omega, \end{cases}$$

where and in the following, D^2u means the Hessian of u and $\sigma_2(D^2u)$ denotes the Hessian operator which is exactly the second elementary symmetric function of the eigenvalues of D^2u. Under the assumption $\Omega \subset \mathbb{R}^3$ be a bounded and strictly convex domain, they proved that the function $v = -(-u)^{\frac{1}{2}}$ is strictly convex and then gave an example to illustrate the sharpness of the convexity index $\frac{1}{2}$. Another interesting result is that in 2010, Liu-Ma-Xu [4] considered the following eigenvalue problem for Hessian operator in a bounded and strictly convex domain

$$\begin{cases}\sigma_2(D^2u) = \lambda(-u)^2 & in \quad \Omega \subset \mathbb{R}^3,\\ \qquad u = 0 & on \quad \partial\Omega.\end{cases}$$

They obtained the strict logarithmic concavity of the eigenfunction. As an application, they get Brunn-Minkowski inequality for the Hessian eigenvalue and characterize the equality case.

It leaves an open question whether the power convexity holds for the general Hessian operator σ_k and for the domain being in general dimensions. We will answer this question when the domain is the ball $B_R(o) \subset \mathbb{R}^n$.

In this paper, we mainly consider the strict power convexity of the admissible solutions of the following Hessian equations in the ball $B_R(o) \subset \mathbb{R}^n$:

$$\begin{cases}\sigma_k(D^2u) = \lambda(-u)^p & in \quad B_R(o) \subset \mathbb{R}^n,\\ \qquad u < 0 & on \qquad B_R(o),\\ \qquad u = 0 & on \quad \partial B_R(o),\end{cases} \tag{1}$$

where $1 \leq k < n$ and $1 \leq p \leq k$.

For this problem, it has been proved in [8] that for any $p \in [0, 2) \bigcup (2, +\infty)$ and $\lambda > 0$, there exists an admissible solution $u \in C^\infty(\Omega) \bigcap C^{1,1}(\bar{\Omega})$. Furthermore if $p \in [0, 2)$, then comparison theorem holds for the solution and therefore the admissible solution to (1) is unique.

Our main results are stated as follows.

Theorem 1.1 *Let $B_R(o)$ be the ball in $\mathbb{R}^n$ with radius $R > 0$ centering at o and $0 \leq p < k$. Let $u \in C^\infty(B_R) \bigcap C^{1,1}(\overline{B_R})$ be the admissible solution of equation (1) in $B_R(o)$. Then the function $v = -(-u)^{\frac{k-p}{2k}}$ is strictly convex in $B_R(o)$.*

For the special case $p = 0$ of equation (1), we also confirm the strict power convexity by computing the principal curvatures of the following problem.

$$\begin{cases}\sigma_k(D^2u) = \lambda & in \quad B_R(o) \subset \mathbb{R}^n,\\ \qquad u < 0 & on \qquad B_R(o),\\ \qquad u = 0 & on \quad \partial B_R(o),\end{cases} \tag{2}$$

For this problem (2), we have the following theorem concerning the quantitative convexity estimates.

Theorem 1.2 *Let u be the smooth admissible solution of problem (2). Then for the graph $v = -(-u)^{\frac{1}{2}}$, we denote $\kappa_{\min}$ and K_G denote the smallest principal curvature and the Gaussian curvatures of the graph v respectively. Then*

(1) If $\lambda \leq 2^k C_n^k$, we have the lower bound estimates for the principal curvatures $\kappa_{\min} \geq (\frac{\lambda}{2^k C_n^k})^{\frac{1}{2k}} R^{-1}$ and lower bound estimates for the Gaussian curvature $K_G \geq (\frac{\lambda}{2^k C_n^k})^{\frac{n}{2k}} R^{-n}$ respectively.

(2) If $\lambda > 2^k C_n^k$, we have $\kappa_{\min} \geq (\frac{2^k C_n^k}{\lambda})^{\frac{1}{k}} R^{-1}$ and $K_G \geq (\frac{2^k C_n^k}{\lambda})^{\frac{n}{k}} R^{-n}$ respectively.

Hence the graph v is strictly convex.

The main step of the proof of Theorem 1.2 is to calculate the principal curvatures and the Gaussian curvatures of the graph v.

We will use the definitions of elementary symmetric function and curvature formulas for the surfaces during the proving process. These definitions and formulas are standard, the readers can consult them on other reference books, such as [5], [6] and [7], etc..

The paper is organized as follows. In section 2, we use a contradiction argument to prove that the solutions to a class of Hessian equations (1) in the ball is strictly power convex. In section 3, we calculate the principal curvatures and the Gaussian curvature of the solution surface $v = -(-u)^{\frac{1}{2}}$ w.r.t. equation (2) and confirm the strict power convexity of (2) in a quantitative viewpoint.

2. Proof of theorem 1.1

In this section, we use a contradiction argument to prove Theorem 1.1. We show that the smooth admissible solution to a class of Hessian equations (1) in the ball $B_R(o)$ are all strictly power convex.

When the domain of the problem (1) is the ball $B_R(o)$ with radius R and center o, Chou-Wang [8] or Tso [9] have proved that, if $p \in (0, k)$, the equation (1) has a unique negative admissible solution $u \in C^\infty(B_R) \bigcap C^{1,1}(\overline{B_R})$ which is radially symmetric.

Therefore we may assume the solution has the form

$$u(x) = \phi(|x|) = \phi(r), \qquad \text{for} \quad |x| = r,$$

where $r \in [0, R]$ and we see that $\phi(r) < 0$ for $r \in [0, R]$. Then ϕ is an increasing function in $(0, R)$ and satisfies $\phi'(0) = \phi(R) = 0$.

For $1 \le i, j \le n$, a calculation shows that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r},$$
$$\frac{\partial^2 r}{\partial x_i \partial x_j} = -r^{-3} x_i x_j + r^{-1} \delta_{ij}.$$

Then it follows that

$$u_{ij} = (\phi^{''} r^{-2} - \phi^{'} r^{-3}) x_i x_j + \phi' r^{-1} \delta_{ij}.$$

Using the fact that the determinant of the matrix $\{A x_i x_j + B \delta_{ij}\}_{1 \le i,j \le k}$ is $AB^{k-1}|x|^2 + B^k$, we have the radial form of the σ_k operator

$$\sigma_k(D^2 u) = C_{n-1}^{k-1} \phi'^{k-1} \phi^{''} r^{-k+1} + C_{n-1}^{k} (\phi')^k r^{-k}, \tag{3}$$

where C_{n-1}^{k} and C_{n-1}^{k-1} are the combinatorial constants.

For $v = -(-\phi)^{\frac{k-p}{2k}}$, we have $\phi = -(-v)^{\frac{2k}{k-p}}$ and

$$\begin{aligned} \phi' &= \frac{2k}{k-p}(-v)^{\frac{k+p}{k-p}} v', \\ \phi^{''} &= \frac{2k}{k-p}(-v)^{\frac{k+p}{k-p}} v^{''} - \frac{2k}{k-p}\frac{k+p}{k-p}(-v)^{\frac{2p}{k-p}} v'^2. \end{aligned} \tag{4}$$

Substituting (4) into (3), the equation (1) can be rewritten as

$$r(-v)^k v'^{k-1} v^{''} - \frac{k+p}{k-p} r(-v)^{k-1} v'^{k+1} + \frac{n-k}{k}(-v)^k v'^k = \frac{\lambda}{C_{n-1}^{k-1}} (\frac{k-p}{2k})^k r^k, \quad \text{for} \quad 0 < r < R \tag{5}$$

with conditions

$$v^{'}(0) = 0 \quad \text{and} \quad v^{'}(r) > 0 \qquad \text{for} \quad 0 < r < R.$$

From the above, we know that

$$v^{''}(0) \ge 0.$$

To get a qualitative value of $v^{''}(0)$, we first obtain from the L'Hospital rule that

$$v^{''}(0) = \lim_{r \to 0^+} v^{''}(r) = \lim_{r \to 0^+} [\frac{v^{'}(r)}{r}].$$

Dividing r^k on the both sides of equation (5) and letting $r \to 0$, we obtain

$$(-v(0))^k v''(0)^k + \frac{n-k}{k}(-v(0))^k v''(0)^k = \frac{\lambda}{C_{n-1}^{k-1}} (\frac{k-p}{2k})^k,$$

and we obtain

$$v''(0) = \sqrt[k]{\frac{k\lambda}{n C_{n-1}^{k-1}}} \frac{k-p}{2k}[-v(0)] = \sqrt[k]{\frac{\lambda}{C_n^k}} \frac{k-p}{2k}[-v(0)] > 0.$$

We use contradiction argument to show that $v(r)$ is strictly convex for all $r \in [0, R]$. If not, there exists a smallest positive r_0 such that $v^{''}(r_0) = 0$. We know $v^{'}(r) > 0$ for $0 < r \le R$ and $v^{''}(r) > 0$ for $0 < r < r_0$ and therefore

$$v^{'''}(r_0) \le 0. \tag{6}$$

On the other hand, we will prove in the following that $v'''(r_0) > 0$ and a contradiction follows.

To this aim, we differentiate the equation (5) with respect to r and evaluate at $r = r_0$ to obtain

$$\begin{aligned} & r_0[-v(r_0)]^k v'(r_0)^{k-1} v'''(r_0) - \big(n - k + \frac{k+p}{k-p}\big)[-v(r_0)]^{k-1} v'(r_0)^{k+1} \\ & + (k-1)\frac{k+p}{k-p} r_0[-v(r_0)]^{k-2} v'(r_0)^{k+2} = \frac{\lambda}{C_{n-1}^{k-1}}(\frac{k-p}{2k})^k k r_0^{k-1}, \end{aligned}$$

i.e.,

$$\begin{aligned} & r_0[-v(r_0)]^k v'(r_0)^{k-1} v'''(r_0) = \frac{\lambda}{C_{n-1}^{k-1}}(\frac{k-p}{2k})^k k r_0^{k-1} \\ & + \big(n - k + \frac{k+p}{k-p}\big)[-v(r_0)]^{k-1} v'(r_0)^{k+1} - (k-1)\frac{k+p}{k-p} r_0[-v(r_0)]^{k-2} v'(r_0)^{k+2}, \end{aligned} \tag{7}$$

where we have used the assumption $v''(r_0) = 0$.

To prove that the sign of $v'''(r_0)$ is positive, we still need to estimate the sign of the right hand side of (7).

By taking $r = r_0$ in equation (5) and noting $v''(r_0) = 0$, we have

$$\frac{n-k}{k}[-v(r_0)]^k v'(r_0)^k - \frac{k+p}{k-p} r_0[-v(r_0)]^{k-1} v'(r_0)^{k+1} = \frac{\lambda}{C_{n-1}^{k-1}}(\frac{k-p}{2k})^k r_0^k. \tag{8}$$

We can rewrite the above equation (8) into

$$[-v(r_0)]^{k-1} v'(r_0)^{k+1} = \frac{k\lambda}{(n-k)C_{n-1}^{k-1}}(\frac{k-p}{2k})^k r_0^k \frac{v'(r_0)}{[-v(r_0)]} + \frac{k(k+p)}{(n-k)(k-p)} r_0[-v(r_0)]^{k-2} v'(r_0)^{k+2}. \tag{9}$$

Substituting (9) into (7), we obtain that

$$\begin{aligned} & r_0[-v(r_0)]^k v'(r_0)^{k-1} v'''(r_0) = \big(n - k + \frac{k+p}{k-p}\big)\frac{k\lambda}{(n-k)C_{n-1}^{k-1}}(\frac{k-p}{2k})^k r_0^k \frac{v'(r_0)}{[-v(r_0)]} \\ & + \frac{\lambda}{C_{n-1}^{k-1}}(\frac{k-p}{2k})^k k r_0^{k-1} + (1 + \frac{k(k+p)}{(n-k)(k-p)})\frac{k+p}{k-p} r_0[-v(r_0)]^{k-2} v'(r_0)^{k+2} > 0. \end{aligned} \tag{10}$$

Noting that $0 \le p < k$, $-v(r_0) > 0$ and $v'(r_0) > 0$ by our assumption, therefore from (10), we know that $v'''(r_0) > 0$, which is a contradiction with (6). The contradiction tell us that $v''(r) > 0$ in $[0, R)$ and hence $v = -(-u)^{\frac{k-p}{2k}}$ is strictly convex in the ball $B_R(o)$.

3. Proof of theorem 1.2

In this section, we confirm the strict power convexity in Theorem 1.2 in a quantitative viewpoint.

For equation (2), by solving the corresponding ordinary differential equation, we derive that $u = \sqrt[k]{\frac{\lambda}{C_n^k}}\frac{(|x|^2 - R^2)}{2}$ is the unique admissible solution. It is easy to see that $v = -\sqrt{-u} = -\sqrt{\sqrt[k]{\frac{\lambda}{C_n^k}}\frac{(R^2 - |x|^2)}{2}}$ is strictly convex. In the following, we will give quantitative convexity estimates of the solution by computing the principal curvatures of the solution surface. For convenience of the calculation, we assume $v = A\sqrt{R^2 - |x|^2}$, where we denote $A = \sqrt[2k]{\frac{\lambda}{2^k C_n^k}}$. To calculate the principal curvatures, we take the first and second derivatives of v to get

$$\begin{aligned} v_i &= \frac{Ax_i}{\sqrt{R^2 - |x|^2}}, \\ v_{ij} &= \frac{A}{\sqrt{R^2 - |x|^2}}(\delta_{ij} + \frac{x_i x_j}{R^2 - |x|^2}). \end{aligned}$$

Since the position vector of the graph v can be written as $X = (x, v(x))$, $x \in \mathbb{R}^n$. Then the tangent vector and the unit normal vector of the graph are given by

$$X_i = (0, \cdots, 0, 1, 0, \cdots, 0, \frac{Ax_i}{\sqrt{R^2 - |x|^2}}), \text{ the } i\text{-th component being } 1, \text{ for } \quad 1 \le i \le n$$

and

$$\overrightarrow{n}=(-\frac{Ax_1}{\sqrt{R^2+(A^2-1)|x|^2}},\cdots,-\frac{Ax_n}{\sqrt{R^2+(A^2-1)|x|^2}},\frac{\sqrt{R^2-|x|^2}}{R^2+(A^2-1)|x|^2})$$

respectively.

Therefore the induced metric and the second fundamental form of the graph are given by

$$g_{ij}=<X_i,X_j>=\delta_{ij}+\frac{A^2}{R^2-|x|^2}$$

and

$$h_{ij}=<X_{ij},\overrightarrow{n}>=\frac{A}{\sqrt{R^2+(A^2-1)|x|^2}}(\delta_{ij}+\frac{x_ix_j}{R^2-|x|^2})$$

respectively.

We denote the inverse matrix of $\{g_{ij}\}$ by $\{g^{ij}\}$. By direct computation, we know that $g^{ij}=\delta_{ij}-\frac{A^2x_ix_j}{R^2+(A^2-1)|x|^2}$ and hence the shape operator of the graph v is

$$h^i_j=\sum_{k=1}^n g^{ik}h_{kj}=\delta_{ij}-\frac{(A^2-1)x_ix_j}{R^2+(A^2-1)|x|^2}.$$

By calculation, we obatain that the principal curvatures of the graph v which are the eigenvalues of the shape operator are $\frac{A}{\sqrt{R^2+(A^2-1)|x|^2}}$ with n-1 multiplicities and $\frac{AR^2}{(R^2+(A^2-1)|x|^2)^{\frac{3}{2}}}$. We see from the positiveness of the principal curvatures that the graph v is strictly convex.

We further compute the Gaussian curvature of the graph v by multiplying all the principal curvatures and obtain

$$K_G=A^nR^2\big[R^2+(A^2-1)|x|^2\big]^{-\frac{n+2}{2}}=(\frac{\lambda}{2^kC_n^k})^{\frac{n}{2k}}R^2\big\{R^2+\big[(\frac{\lambda}{2^kC_n^k})^{\frac{1}{k}}-1\big]|x|^2\big\}^{-\frac{n+2}{2}}.$$

Therefore, if $\lambda\le 2^kC_n^k$, then we have lower bound estimates for the smallest principal curvature $\kappa_{\min}$ and the Gaussian curvature K_G, i.e.,

$$\kappa_{\min}=\frac{A}{\sqrt{R^2+(A^2-1)|x|^2}}\ge AR^{-1}\ge(\frac{\lambda}{2^kC_n^k})^{\frac{1}{2k}}R^{-1}$$

and

$$K_G\ge(\frac{\lambda}{2^kC_n^k})^{\frac{n}{2k}}R^{-n}.$$

respectively.

On the other hand, if $\lambda>2^kC_n^k$, then the smallest principal curvature and the Gaussian curvature attains their minimums on the boundary of the domain, which are

$$\kappa_{\min}=\frac{AR^2}{(R^2+(A^2-1)|x|^2)^{\frac{3}{2}}}\ge\frac{1}{A^2R}=(\frac{2^kC_n^k}{\lambda})^{\frac{1}{k}}R^{-1}$$

and

$$K_G\ge(\frac{2^kC_n^k}{\lambda})^{\frac{n}{k}}R^{-n}$$

respectively. Therefore we have completed the proof of Theorem 1.2.

4. Conclusion

In this paper, we consider power convexity of a Hessian equation in the n-dimensional ball and obtain lower bound estimates for the principal curvatures and the Gaussian curvature. However, people are interesting in the upper and lower bound curvature estimates for the solutions of fully-nonlinear partial differential equations. For the corresponding results of general convex domains, we meet some technical difficulties. In our further studies, we will focus on the study of convexity and curvature estimates for general Hessian equations and general convex domain. These kind of convexities and convexity estimates are of great interest in the study of fully-nonlinear elliptic equations. Finding their various applications of the convexities will help us understanding of the geometry of the solution surfaces.

References

[1] L.G. Makar-Limanov, *Solution of Dirichlet's problem for the equation* $\Delta u = -1$ *on a convex region*, Math. Notes Acad. Sci. USSR **9**, (1971), 52–53.

[2] H.J. Brascamp and E.H. Lieb, *On extensions of the Brunn-Minkowski and Prekopa-Leindler theorems, including inequalities for log-concave functions, with an application to the diffusion equation*, J. Func.Anal., **22** (1976), 366–389.

[3] X.N. Ma and X. Lu, *The convexity of solution of a class Hessian equation in bounded convex domain in* $\mathbb{R}^3$, J. Funct. Anal. **255** (2008), no. 7, 1713–1723.

[4] P.Liu, X.N.Ma and L.Xu, *A Brunn-Minkowski inequatlity for the Hessian eigenvalue in three-dimensional convex domain*, Adv.Math., **225** (2010), 1616–1633.

[5] G.Huisken, C.Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math.,**183**(1),(1999), 45–70.

[6] R. C. Reilly, *On the Hessian of a function and the curvatures of its graph*, Michigan Math. J., **20**,(1973), 373–383.

[7] M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1976.

[8] K.S. Chou and X.J.Wang *A variational theory of the Hessian equations*, Comm. Pure Appl.Math., **54** (2001), 1029–1064.

[9] K.S. Tso, *Remarks on critical exponents for Hessian operators*, Ann. Inst. H.Poincaré Anal. Non Linéaire **7**(1990), No.2, 113–122.