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Research Paper

The common value and uniqueness theorems of algebroid functions

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Abstract

In this paper, we investigate the common values and growth relationship between two algebroid functions and prove several growth theorems involving common values. Moreover, on this basis, we study uniqueness theory of algebroid function and prove several elegant uniqueness theorems.

Keywords: *Algebroid functions, Common values, Growth, Uniqueness theorems*

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1. Introduction and main results

Let D be a simply connected domain in the complex plane $\mathbf{C}$. An algebroid function of k-valued in D is defined as

$$W(z) = \{w \in \mathbf{C}; w^k + A_1(z)w^{k-1} + \cdots + A_k(z) = 0\}, \quad z \in D, \tag{1}$$

where $A_1(z), \cdots, A_k(z)$ are meromorphic in D. Because the set of branch points of an algebroid function $W(z)$ is discrete, we can cut D by a polyline that pass through all branch points to obtain a simply connected domain $\widetilde{D}$. Then we get k single-valued branch functions of $W(z)$:

$$w_1(z),\ w_2(z),\ \cdots,\ w_k(z).$$

Thus, we can write $W(z)$ simply as

$$W(z) = \{w_j(z)\}_{j=1}^{k}. \tag{2}$$

and

$$\begin{aligned}\Psi(z,w) &= w^k + A_1(z)w^{k-1} + \cdots + A_k(z)\\ &= \big(w - w_1(z)\big)\big(w - w_2(z)\big)\cdots\big(w - w_k(z)\big) = 0,\ z \in \widetilde{D}.\end{aligned}$$

Let $W(z)$ and $M(z)$ be two k-valued and s-valued non-constant algebroid functions defined by (1). Suppose that a is an arbitrary constant or ∞. We denote by $\overline{E}_{t)}(a, W)$ the set of zeros of $W(z) - a$, whose multiplicities are not greater than t; and denote by $\overline{n}_{t)}(r, \frac{1}{W-a})$ the number of the distinct points of the set $\overline{E}_{t)}(a, W)\bigcap\{|z| \le r\}$.

If $\overline{E}_{t)}(a, W) = \overline{E}_{t)}(a, W)$, then a is called a t-common value of $W(z)$ and $M(z)$. If $t = \infty$, then a is called a IM common value of $W(z)$ and $M(z)$.

Algebroid functions are extremely important multiple-valued functions. There are lots of complex differential equations that possess global algebroid solutions. For example, the 2-valued algebroid function $w^2 - \tan z = 0$ solves the simple first order non-linear differential equation $2ww' - w^4 = 1$. However, there are still many difficulties with the study of algebroid functions because of multivalue and branch points. For instance, we can not plus two algebroid functions like we plus two meromorphic functions.

In [6] and [7], Sun and Gao defined the addition $(W \oplus M)(z)$ of any two algebroid functions $W(z)$ and $M(z)$ as follows:

Definition 1.1 [6,7] *Let $W(z)$ and $M(z)$ be two k-valued and s-valued algebroid functions. Then, the sum $W(z) \oplus M(z)$ of them is defined as*

$$(W \oplus M)(z) = \{w \in \mathbf{C}; \prod_{i=1,j=1}^{i=k,j=s} [w - (w_i(z) + m_j(z))] = 0\}, \quad z \in \widetilde{D'},$$

where $\widetilde{D'}$ is a simply connected domain obtained by cutting D by the polyline that pass through all branch points of $W(z)$ and $M(z)$; $w_i(z)$ and $m_j(z)$ are single-valued branches of $W(z)$ and $M(z)$, respectively.

In [6] and [7], Sun and Gao proved that the sum $W(z) \oplus M(z)$ of a k-valued algebroid function and a s-valued algebroid function is a ks-valued algebroid function. By using this new operation, one can study the common values of algebroid functions in depth. For example, the authors of this paper [2, 6, 7], Cao & Yi [1], Xuan & Gao [9], Zhang & Sun [12], Liu & Sun [5] and other researchers studied the common values by applying this new operation and obtained numbers of uniqueness theorems for algebroid functions. For instance, In [6], Sun and Gao proved the following theorem:

Theorem 1.2 [6] *Let $W(z)$ and $M(z)$ be two k-valued non-constant algebroid functions defined by (1). Suppose that $a_1, a_2, \ldots, a_{4k+1}$ are $4k + 1$ distinct complex numbers. If*

$$\overline{E}_{2k+1)}(a_j, W) = \overline{E}_{2k+1)}(a_j, M) \quad (j = 1, 2, \ldots, 2k + 1)$$

and

$$\overline{E}_{2k)}(a_j, W) = \overline{E}_{2k)}(a_j, M) \quad (j = 2k + 2, \ldots, 4k + 1),$$

then $W(z) \equiv M(z)$.

In this paper, we firstly investigate the relation of characteristic functions between any two k-valued and s-valued algebroid functions that have $2k + 2s$ IM common values and we obtain

Theorem 1.3 *Let $W(z)$ and $M(z)$ be two k-valued and s-valued non-constant algebroid functions defined by (1). If they have $2k + 2s$ IM common values $a_1, a_2, \ldots, a_{2k+2s}$, then*

$$T(r, W) = T(r, M) + S(r, M),$$

$$T(r, M) = T(r, W) + S(r, W).$$

If $k = s$, then we can get the following conclusion from Theorem 1.3 immediately.

Corollary 1.4 *Let $W(z)$ and $M(z)$ be two k-valued non-constant algebroid functions defined by (1). If they have $4k$ IM common values $a_1, a_2, \ldots, a_{4k}$, then*

$$T(r, W) = T(r, M) + S(r, M),$$

$$T(r, M) = T(r, W) + S(r, W).$$

We also investigate the relation of characteristic functions between two k-valued algebroid functions that have $2k+q$ $(q \geq 1)$ IM common values and obtain

Theorem 1.5 *Let $W(z)$ and $M(z)$ be two k-valued non-constant algebroid functions defined by (1). If they have $3k + 1$ IM common values $a_1, a_2, \ldots, a_{3k+1}$, then*

$$\frac{1}{\sigma_k} T(r, W) + S(r, W) \leq T(r, M) \leq \sigma_k T(r, W) + S(r, W),$$

where $\sigma_k = \min\{k,\ 3 - 1/(k + 1)\}$.

And so, we can obtain the following corollary:

Corollary 1.6 *Let $W(z)$ and $M(z)$ be two k-valued non-constant algebroid functions defined by (1). If they have $2k+q$ $(q\geq 1)$ IM common values $a_1, a_2, \ldots, a_{2k+q}$, then*

$$\rho(W)=\rho(M).$$

where $\rho(W)$ denotes the order of $W(z)$.

Furthermore, by applying these conclusions and considering the common values involved multiplicity of any two algebroid functions, we prove an uniqueness theorem of algebroid functions as follows.

Theorem 1.7 *Let $W(z)$ and $M(z)$ be two k-valued and s-valued non-constant algebroid functions defined by (1), and suppose that $a_1, a_2, \ldots, a_{2k+2s+1}$ are $2k+2s+1$ distinct complex numbers. If*

$$\overline{E}_{k+s+1)}(a_j, W)=\overline{E}_{k+s+1)}(a_j, M) \quad (j=1,2,\ldots,k+s+1)$$

and

$$\overline{E}_{k+s)}(a_j, W)=\overline{E}_{k+s)}(a_j, M) \quad (j=k+s+2,\ldots,2k+2s+1),$$

then $W(z)\equiv M(z)$.

It is clear that this theorem improves Theorem 1.2, and the following He's uniqueness theorem [4, 8] is a corollary of Theorem 1.7.

Corollary 1.8 [4,8] *Let $W(z)$ and $M(z)$ be two k-valued and s-valued non-constant algebroid functions defined by (1). If they have $2k+2s+1$ IM common values, then $W(z)\equiv M(z)$.*

In addition, the reader who is not familiar with the other notations and terms used in this paper can refer to [3, 4, 8, 10, 11].

2. Some lemmas

Definition 2.1 *Suppose that $W(z)$ is a non-constant k-valued algebroid function in $|z|<R$. Denote by $S(r,W)$ an arbitrary function X defined on $\{0\leq r<R\}$ and such that:*

(1) If $R=+\infty$ and $W(z)$ has finite order, then

$$X=O\{\log T(r,W)\}+O\{\log r\}=O\{\log r\}$$

as $r\to+\infty$.

(2) If $R=+\infty$ and $W(z)$ has infinite order, then

$$X=O\{\log T(r,W)\}+O\{\log r\}=O\{\log[rT(r,W)]\}$$

as $r\to+\infty$ outside a set E_0 with finite linear measure otherwise.

(3) If $R\in(0,+\infty)$, then

$$X=O\{\log^+ T(r,W)+\log\frac{1}{R-r}\}$$

as $r\to R$ outside a set E_0 with

$$\int_{E_0}\frac{dr}{R_0-r}\leq 2,$$

and there must exit a point r outside E_0 for which $r\in(\rho,\rho')$ provided that $\rho\in(0,R_0)$ and $\rho'\in(R_0-\frac{R_0-\rho}{e^2},R_0)$.

Then we say that the function X satisfies remainder conditions for $W(z)$ and write $X=S(r,W)$.

It is easy to see that

Lemma 2.2 *Suppose that $M(z)$ and $W(z)$ are two non-constant algebroid functions and $X=S(r,M)$. If there is a constant $c>0$ such that $T(r,W)>cT(r,M)$ for all r in the common domain of definitions of $M(z)$ and $W(z)$, then X also satisfies the remainder conditions for $W(z)$, that is, $X=S(r,W)$.*

Lemma 2.3 [7] *Let $W(z)$ and $M(z)$ be two k-valued and s-valued non-constant algebroid functions defined by (1). Then, the sum $W(z) \oplus M(z)$ and difference $W(z) \ominus M(z)$ of them are all ks-valued generalized algebroid functions, and*

$$T(r, W \oplus M) \le T(r, M) + T(r, M) + \log 2,$$

$$T(r, M \ominus M) \le T(r, W) + T(r, M) + \log 2.$$

Lemma 2.4 *Let $W(z)$ and $M(z)$ be two k-valued non-constant algebroid functions defined by (1). If they have $2k+q$ $(q \ge 1)$ IM common values $a_1, a_2, \ldots, a_{2k+q}$, then*

$$(1 - \frac{2k}{2k+q})T(r, M) + S(r, M) \le T(r, W) \le (1 + \frac{2k}{q})T(r, M) + S(r, M).$$

Proof: These relation clearly hold when $W(z) \equiv M(z)$. Thus, in the following verification, we assume that $W(z) \not\equiv M(z)$. We first know from the second fundamental theorem of algebroid functions that

$$\begin{aligned}(2k+q-2k)T(r,W) &\le \sum_{j=1}^{2k+q} \overline{N}(r, \frac{1}{M - a_j}) + S(r, W) \\ &= \sum_{j=1}^{2k+q} \overline{N}(r, \frac{1}{M - a_j}) + S(r, W) \\ &\le (2k+q)T(r, \frac{1}{M - a_j}) + S(r, W)\end{aligned}$$

By the first fundamental theorem of algebroid functions, we have

$$qT(r, W) \le (2k+q)T(r, M) + S(r, W).$$

That is,

$$T(r, W) \le (1 + \frac{2k}{q})T(r, M) + S(r, W).$$

In a similar way, we can obtain

$$(1 - \frac{2k}{2k+q})T(r, M) \le T(r, W) + S(r, M).$$

Moreover, we get from Lemma 2.2 that

$$S(r, M) = S(r, W).$$

Combining above three relations, we obtain the Lemma 2.4.

Lemma 2.5 [7] *Suppose that $W(z)$ is a k-valued non-constant algebroid function defined by (1). Let $a_1, a_2, \ldots, a_p$ be p distinct complex numbers, and suppose that $t_1, t_2, \ldots, t_p$ are p positive integers, then*

$$(p - 2k - \sum_{j=1}^{p} \frac{1}{t_j + 1})T(r, W) < \sum_{t=1}^{p} \frac{t_j}{t_j + 1} \overline{N}_{t_j)}(r, \frac{1}{W - a_t}) + S(r, W).$$

3. Proofs of theorems

Proof of Theorem 1.3: These relations clearly hold when $W(z) \equiv M(z)$. Thus, in the verification, we may assume that $W(z) \neq M(z)$. We first know from the second fundamental theorem of algebroid functions that

$$
\begin{aligned}
&(2k+2s-2k)T(r,W) \leq \sum_{j=1}^{2k+2s} \overline{N}(r,\frac{1}{W-a_j}) + S(r,W) \\
= \;& \frac{1}{k}\sum_{j=1}^{2k+2s}\Big[\int_0^r (\overline{n}(t,\frac{1}{W-a_j}) - \overline{n}(0,\frac{1}{W-a_j}))\frac{1}{t}dt + \overline{n}(0,\frac{1}{W-a_j})\log r\Big] + S(r,W) \\
\leq \;& \frac{1}{k}\Big[\int_0^r (\overline{n}(t,\frac{1}{W\ominus M}) - \overline{n}(0,\frac{1}{W\ominus M}))\frac{1}{t}dt + \overline{n}(0,\frac{1}{W\ominus M})\log r\Big] + S(r,W) \\
= \;& s\frac{1}{ks}\Big[\int_0^r (\overline{n}(t,\frac{1}{W\ominus M}) - \overline{n}(0,\frac{1}{W\ominus M}))\frac{1}{t}dt + \overline{n}(0,\frac{1}{W\ominus M})\log r\Big] + S(r,W) \\
= \;& s\overline{N}(r,\frac{1}{W\ominus M}) + S(r,W) \\
\leq \;& sT(r,\frac{1}{W\ominus M}) + S(r,W)
\end{aligned}
$$

By the first fundamental theorem of algebroid functions and Lemma 2.3, we have

$$2sT(r,W) \leq sT(r,W\ominus M) + S(r,W) \leq s[T(r,W)+T(r,M)] + S(r,W).$$

That is,

$$T(r,W) \leq T(r,M) + S(r,W).$$

Then, the same procedure may be easily adapted to obtain

$$T(r,M) \leq T(r,W) + S(r,M).$$

Combining above two inequalities and applying Lemma 2.2, we can obtain the Theorem 1.3 immediately.

Proof of Theorem 1.5: If $W(z) \equiv M(z)$, it reaches the conclusion immediately. If not, by making the substitution $q = k+1$ in Lemma 2.4 firstly, we obtain that

$$(1-\frac{2k}{3k+1})T(r,M) + S(r,M) \leq T(r,W) \leq (3-\frac{2}{k+1})T(r,M) + S(r,M). \tag{3}$$

Then, applying a similar approach that taken in the proof of the Theorem 1.3, we have

$$
\begin{aligned}
(3k+1-2k)T(r,W) &\leq \sum_{j=1}^{3k+1} \overline{N}(r,\frac{1}{W-a_j}) + S(r,W) \\
&\leq kT(r,W\ominus M) + S(r,W).
\end{aligned}
$$

Thus

$$T(r,W) \leq kT(r,M) + S(r,W). \tag{4}$$

Also, we can obtain

$$T(r,M) \leq kT(r,W) + S(r,M) \tag{5}$$

in a similar way. Finally, we get the Theorem 1.5 from the three inequalities (3), (4), and (5) and Lemma 2.2.

Proof of Theorem 1.7: If $W(z) \not\equiv M(z)$, then Lemma 2.5 tells us

$$\begin{aligned}
&(2k+2s+1-2k-\frac{k+s+1}{k+s+2}-\frac{k+s}{k+s+1})T(r,W)\\
\leq\ & \sum_{j=1}^{k+s+1}\frac{k+s+1}{k+s+2}\overline{N}_{k+s+1)}(r,\frac{1}{W-a_j})+\sum_{j=k+s+2}^{2k+2s+1}\frac{k+s}{k+s+1}\overline{N}_{k+s)}(r,\frac{1}{W-a_j})+S(r,W)\\
\leq\ & \frac{k+s+1}{k+s+2}[\sum_{j=1}^{k+s+1}\overline{N}_{k+s+1)}(r,\frac{1}{W-a_j})+\sum_{j=k+s+2}^{2k+2s+1}\overline{N}_{k+s)}(r,\frac{1}{W-a_j})]+S(r,W)\\
\leq\ & \frac{k+s+1}{k+s+2}\cdot s\overline{N}(r,\frac{1}{W\ominus M})+S(r,W)\\
\leq\ & sT(r,\frac{1}{W\ominus M})+S(r,W)
\end{aligned}$$

By applying the first fundamental theorem of algebroid functions and Lemma 2.3, we can obtain

$$\begin{aligned}
(2s-1-\frac{1}{k+s+2}-\frac{1}{k+s+1})T(r,W) &\leq s(1-\frac{1}{k+s+2})T(r,W\ominus M)+S(r,W)\\
&\leq s(1-\frac{1}{k+s+2})[T(r,W)+T(r,M)]+S(r,W)
\end{aligned}$$

Then, we have from Theorem 1.3 that

$$(2s-1-\frac{1}{k+s+2}-\frac{1}{k+s+1})T(r,W)\leq 2s(1-\frac{1}{k+s+2})T(r,W)+S(r,W). \tag{6}$$

By following the same procedure, we get

$$(2k-1-\frac{1}{k+s+2}-\frac{1}{k+s+1})T(r,W)=(2k-1-\frac{1}{k+s+2}-\frac{1}{k+s+1})T(r,M)+S(r,W)$$

$$\leq 2k(1-\frac{1}{k+s+2})T(r,W)+S(r,W). \tag{7}$$

By adding corresponding sides of the relations (8) and (9), we find that

$$2(\frac{1}{k+s+1}-\frac{1}{k+s+2})T(r,W)\leq S(r,W).$$

Since $1/(k+s+1)-1/(k+s+2)>0$, we arrive at a contradiction, and the proof of Theorem 1.7 is complete.

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