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Research Paper

$\mathcal{I}_2$-Cauchy double sequences in 2-normed spaces

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Abstract

The concept $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences were studied by Gürdal and Açık in [*On $\mathcal{I}$-Cauchy sequences in 2-normed spaces*, Math. Inequal. Appl. **11** (2) (2008), 349–354]. In this paper, we introduce the notions of $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy double sequences, and study their some properties with the property (AP2) in 2-normed spaces.

Keywords: *Ideal; Double Sequences; $\mathcal{I}_2$-Convergence; $\mathcal{I}_2$-Cauchy; 2-normed spaces.*

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [5] and Schoenberg [25]. This concept was extended to the double sequences by Mursaleen and Edely [16].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [14] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers [5, 6]. Nuray and Ruckle [20] indepedently introduced the same with another name generalized statistical convergence. Das et al. [2] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Dündar and Altay [4] studied the concepts of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy for double sequences and they gave the relation between $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of double sequences of functions defined between linear metric spaces. A lot of development have been made in this area after the works of [3, 15, 17–19, 24, 26–28].

The concept of 2-normed spaces was initially introduced by Gähler [7, 8] in the 1960's. Since then, this concept has been studied by many authors, see for instance [9–11, 13]. Şahiner et al. [26] and Gürdal [13] studied $\mathcal{I}$-convergence in 2-normed spaces. Gürdal and Açık [12] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan et al. [22, 23] investigated $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences in 2-normed spaces. They also examined the concepts $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points in 2-normed spaces.

In this paper, we introduce the notions of $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy double sequence, and study their some properties with the property (AP2) in 2-normed spaces.

2. Definitions and notations

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ denotes the set of all real numbers.

Now, we recall the concept of 2-normed space, ideal, ideal convergence of the sequences, double sequences and some fundamental definitions and notations (See [1, 2, 7, 10, 12, 14, 21–23]).

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case, we write $\lim_{m,n\to\infty} x_{mn} = L$.

A double sequence $x = (x_{mn})$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is,

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,

(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,

(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,

(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,

(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1 ([14]) *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X\backslash A)\}$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$, for each $x \in X$.

Throughout the paper, we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$, for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

In this section, we consider the $\mathcal{I}_2$ and $\mathcal{I}_2^*$-convergence of double sequences in the more general structure of a metric space (X, ρ). Unless otherwise mentioned we shall denote the metric space (X, ρ) by X only.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we say that x is $\mathcal{I}_2$-convergent and we write $\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L$.

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies $\mathcal{I}_2$-convergence.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N}\backslash M \in \mathcal{I}_2$) such that $\lim_{m,n\to\infty} x_{mn} = L$, for $(m, n) \in M$ and we write $\mathcal{I}_2^* - \lim_{m,n\to\infty} x_{mn} = L$.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2$-Cauchy, if for every $\varepsilon > 0$ there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon\} \in \mathcal{I}_2.$$

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2^*$-Cauchy sequence if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that for every $\varepsilon > 0$ and for $(m,n),(s,t) \in M$, $m,n,s,t > k_0 = k_0(\varepsilon)$, $\rho(x_{mn}, x_{st}) < \varepsilon$. In this case, we write $\lim_{m,n,s,t\to\infty} \rho(x_{mn}, x_{st}) = 0$.

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property $(AP2)$ if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.

(ii) $\|x, y\| = \|y, x\|$.

(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.

(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

Now, we give definitions of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence of sequences and double sequences and $\mathcal{I}$-Cauchy sequence, $\mathcal{I}^*$-Cauchy sequence in 2-normed space.

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal. The sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}$-convergence to $x \in X$, if for each $\varepsilon > 0$ and nonzero $z \in X$,

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|x_n - x, z\| \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write

$$\mathcal{I} - \lim_{n\to\infty} \|x_n - x, z\| = 0 \;\; or \;\; \mathcal{I} - \lim_{n\to\infty} \|x_n, z\| = \|x, z\|.$$

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal. The sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}^*$-convergence to $L \in X$, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, $M \in F(\mathcal{I})$ such that $\lim_{k\to\infty} \|x_{m_k} - L, z\| = 0$, for each nonzero $z \in X$.

Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. The sequence (x_n) is said to be $\mathcal{I}$-Cauchy sequence in X, if for each $\varepsilon > 0$ and nonzero $z \in X$ there exists a number $N = N(\varepsilon, z)$ such that

$$\{n \in \mathbb{N} : \|x_n - x_N, z\| \geq \varepsilon\} \in \mathcal{I}.$$

Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. The sequence (x_n) is said to be $\mathcal{I}^*$-Cauchy sequence in X, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, $M \in F(\mathcal{I})$ such that $\lim_{k,p\to\infty} \|x_{m_k} - x_{m_p}, z\| = 0$, for each nonzero $z \in X$.

Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}_2$-convergence to $L \in X$, if for each $\varepsilon > 0$ and nonzero $z \in X$,

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - L, z\| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we write $\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L$.

Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}_2^*$-convergence to $L \in X$, if there exists a set $M \in F(\mathcal{I}_2)$ (i.e. $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that $\lim_{m,n\to\infty} \|x_{mn} - L, z\| = 0$, for $(m,n) \in M$ and for each nonzero $z \in X$. In this case, we write $\mathcal{I}_2^* - \lim_{m,n\to\infty} x_{mn} = L$.

Lemma 2.2 ([4],Theorem 3.3) *Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}\times\mathbb{N}$ such that $P_i \in F(\mathcal{I}_2)$ for each i, where $\mathcal{F}(\mathcal{I}_2)$ is a filter associate with a strongly admissible ideal $\mathcal{I}_2$ with the property $(AP2)$. Then, there exists a set $P \subset \mathbb{N}\times\mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and the set $P\backslash P_i$ is finite for all i.*

3. $\mathcal{I}_2$-Cauchy double sequences in 2-normed spaces

Now, we introduce the notions of $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy double sequence in 2-normed space.

Definition 3.1 *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-Cauchy if for each $\varepsilon > 0$ and nonzero z in X there exist $s = s(\varepsilon, z)$, $t = t(\varepsilon, z) \in \mathbb{N}$ such that*

$$A(\varepsilon, z) := \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - x_{st}, z\| \geq \varepsilon\} \in \mathcal{I}_2.$$

Theorem 3.2 *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If $x = (x_{mn})$ in X is $\mathcal{I}_2$-convergent then $x = (x_{mn})$ is an $\mathcal{I}_2$-Cauchy double sequence in 2-normed space $(X, \|\cdot,\cdot\|)$.*

Proof. Suppose that $x = (x_{mn})$ is $\mathcal{I}_2$-convergent to L in X. Then, for each $\varepsilon > 0$ and nonzero $z \in X$,

$$A(\frac{\varepsilon}{2}, z) = \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - L, z\| \geq \frac{\varepsilon}{2}\Big\} \in \mathcal{I}_2.$$

This implies that the set

$$A^c(\frac{\varepsilon}{2}, z) = \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - L, z\| < \frac{\varepsilon}{2}\Big\} \in \mathcal{F}(\mathcal{I}_2)$$

and therefore $A^c(\frac{\varepsilon}{2}, z)$ is non-empty. So, we can choose positive integers k and l such that $(k,l) \notin A(\frac{\varepsilon}{2}, z)$. Then, for every $\varepsilon > 0$ and nonzero $z \in X$ we have

$$\|x_{kl} - L, z\| < \frac{\varepsilon}{2}.$$

Take

$$B(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - x_{kl}, z\| \geq \varepsilon\},$$

for each $\varepsilon > 0$ and nonzero $z \in X$. We prove that $B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z)$. Let $(m,n) \in B(\varepsilon, z)$. Then, we have

$$\varepsilon \leq \|x_{mn} - x_{kl}, z\| \leq \|x_{mn} - L, z\| + \|x_{kl} - L, z\| < \|x_{mn} - L, z\| + \frac{\varepsilon}{2}.$$

This implies that

$$\frac{\varepsilon}{2} < \|x_{mn} - L, z\|, \text{ for each nonzero } z \text{ in } X$$

and therefore $(m,n) \in A(\frac{\varepsilon}{2}, z)$. Since $B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z)$ and $A(\frac{\varepsilon}{2}, z) \in \mathcal{I}_2$, we get $B(\varepsilon, z) \in \mathcal{I}_2$. This completes the proof.

Definition 3.3 *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2^*$-Cauchy sequence, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that for each $\varepsilon > 0$ and for all $(m,n), (s,t) \in M$,*

$$\|x_{mn} - x_{st}, z\| < \varepsilon, \text{ for each nonzero } z \in X,$$

where $m, n, s, t > k_0 = k_0(\varepsilon) \in \mathbb{N}$. In this case, we write

$$\lim_{m,n,s,t\to\infty} \|x_{mn} - x_{st}, z\| = 0.$$

Theorem 3.4 *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If $x = (x_{mn})$ is an $\mathcal{I}_2^*$-Cauchy double sequence then $x = (x_{mn})$ is $\mathcal{I}_2$-Cauchy double sequence in 2-normed space $(X, \|\cdot,\cdot\|)$.*

Proof. Suppose that $x=(x_{mn})$ is an $\mathcal{I}_2^*$-Cauchy double sequence in 2-normed space. Then, there exists a set $M\in\mathcal{F}(\mathcal{I}_2)$ (i.e., $H=\mathbb{N}\times\mathbb{N}\backslash M\in\mathcal{I}_2$) such that for each $\varepsilon>0$ and for all $(m,n),(s,t)\in M$,

$$\|x_{mn}-x_{st},z\|<\varepsilon,\ \text{for each nonzero } z\in X,$$

where $m,n,s,t\geq k_0=k_0(\varepsilon)\in\mathbb{N}$. Then,

$$\begin{aligned} A(\varepsilon,z) &= \{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-x_{st},z\|\geq\varepsilon\} \\ &\subset H\cup\big[M\cap\big((\{1,2,3,\ldots,(k_0-1)\}\times\mathbb{N})\cup(\mathbb{N}\times\{1,2,3,\ldots,(k_0-1)\})\big)\big]. \end{aligned}$$

Since $\mathcal{I}_2$ be a strongly admissible ideal, then

$$H\cup\big[M\cap\big((\{1,2,3,\ldots,(k_0-1)\}\times\mathbb{N})\cup(\mathbb{N}\times\{1,2,3,\ldots,(k_0-1)\})\big)\big]\in\mathcal{I}_2.$$

Therefore, we have $A(\varepsilon,z)\in\mathcal{I}_2$. This shows that $x=(x_{mn})$ is $\mathcal{I}_2$-Cauchy double sequences in 2-normed space.

Theorem 3.5 *Let $\mathcal{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and $x=(x_{mn})$ in X. If $x=(x_{mn})$ is $\mathcal{I}_2^*$-convergent, then $x=(x_{mn})$ is an $\mathcal{I}_2$-Cauchy double sequence in 2-normed space $(X,\|\cdot,\cdot\|)$.*

Proof. Suppose that $x=(x_{mn})$ is $\mathcal{I}_2^*$-convergent to L in X. Then, there exists a set $M\in\mathcal{F}(\mathcal{I}_2)$ (i.e., $H=\mathbb{N}\times\mathbb{N}\backslash M\in\mathcal{I}_2$) such that for each $\varepsilon>0$ and for all $(m,n)\in M$,

$$\|x_{mn}-L,z\|<\frac{\varepsilon}{2},\ \text{for each nonzero } z \text{ in } X,$$

where $m,n\geq k_0=k_0(\varepsilon)\in\mathbb{N}$. Since, for each $\varepsilon>0$ and for all $(m,n),(s,t)\in M$,

$$\|x_{mn}-x_{st},z\|\leq\|x_{mn}-L,z\|+\|x_{st}-L,z\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\ \text{for each nonzero } z \text{ in } X,$$

where $m,n,s,t\geq k_0=k_0(\varepsilon)\in\mathbb{N}$, we have

$$\|x_{mn}-x_{st},z\|<\varepsilon.$$

This shows that $x=(x_{mn})$ is an $\mathcal{I}_2^*$-Cauchy double sequence in X. Hence, by Theorem 3.4 $x=(x_{mn})$ is an $\mathcal{I}_2$-Cauchy double sequence in X.

Theorem 3.6 *Let $(X,\|\cdot,\cdot\|)$ be a linear 2-normed space. If $\mathcal{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal with the property $(AP2)$ then the concepts $\mathcal{I}_2$-Cauchy double sequence and $\mathcal{I}_2^*$-Cauchy double sequence coincide in X.*

Proof. It is known by Theorem 3.4 that an $\mathcal{I}_2^*$-Cauchy double sequence is also an $\mathcal{I}_2$-Cauchy, where $\mathcal{I}_2$ need not have the property $(AP2)$.

Now, it is sufficient to prove that a double sequence $x=(x_{mn})$ in X is a $\mathcal{I}_2^*$-Cauchy double sequence under assumption that it is an $\mathcal{I}_2$-Cauchy double sequence. Let $x=(x_{mn})$ in X be an $\mathcal{I}_2$-Cauchy double sequence. Then, there exists $s=s(\varepsilon,z)$, $t=t(\varepsilon,z)\in\mathbb{N}$ such that

$$A(\varepsilon,z)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-x_{st},z\|\geq\varepsilon\}\in\mathcal{I}_2,$$

for each $\varepsilon>0$ and nonzero z in X. Let

$$P_i=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|x_{mn}-x_{s_it_i},z\|<\frac{1}{i}\right\};\quad (i=1,2,\ldots),$$

where $s_i=s(\frac{1}{i}),\ t_i=t(\frac{1}{i})$. It is clear that

$$P_i\in\mathcal{F}(\mathcal{I}_2),\quad (i=1,2,\ldots).$$

Since $\mathcal{I}_2$ has the property $(AP2)$, then by Lemma 2.2 there exists a set $P\subset\mathbb{N}\times\mathbb{N}$ such that $P\in\mathcal{F}(\mathcal{I}_2)$, and $P\backslash P_i$ is finite for all i. Now we show that

$$\lim_{\substack{m,n,s,t\to\infty\\(m,n),(s,t)\in P}}\|x_{mn}-x_{st},z\|=0,\ \text{for each nonzero } z \text{ in } X.$$

To prove this, let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > \frac{2}{\varepsilon}$. If $(m,n),(s,t) \in P$ then $P \backslash P_j$ is a finite set, so there exists $k = k(j)$ such that $(m,n),(s,t) \in P_j$, for all $m,n,s,t > k(j)$. Therefore,

$$\|x_{mn} - x_{s_j t_j}, z\| < \frac{1}{j} \text{ and } \|x_{st} - x_{s_j t_j}, z\| < \frac{1}{j}, \text{ for each nonzero } z \text{ in } X,$$

for all $m,n,s,t > k(j)$. Hence, it follows that

$$\|x_{mn} - x_{st}, z\| \leq \|x_{mn} - x_{s_j t_j}, z\| + \|x_{st} - x_{s_j t_j}, z\| < \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon,$$

for all $n,m,s,t > k(j)$ and each nonzero z in X. Thus, for any $\varepsilon > 0$ there exists $k = k(\varepsilon)$ such that for $m,n,s,t > k(\varepsilon)$ and $(m,n),(s,t) \in P \in \mathcal{F}(\mathcal{I}_2)$

$$\|x_{mn} - x_{st}, z\| < \varepsilon, \text{ for each nonzero } z \text{ in } X.$$

This shows that the double sequence $x = (x_{mn}) \in X$ is an $\mathcal{I}_2^*$-Cauchy double sequence.

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