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Research Paper

Fixed point theorems for hybrid contraction without continuity

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Abstract

In this paper we establish a coincidence and fixed point theorems for hybrid contraction under generalized weakly contractive condition by using the concept of (IT)-commutativity in a complete metric space without appeal to continuity of mappings. Our results extend and generalize the results of Choudhury et al. [6] and others.

Keywords: *Coincidence Point, Fixed Point, Control Function,Weak Contraction.*

1. Introduction

Study of fixed point theorem for multi-valued mappings was initiated by Nadler [22]. Subsequently a number of fixed point theorems in metric space have been proved for multi-valued mapping satisfying contractive type conditions (see, for instance [8], [9], [18], [20], [32] and references therein). Later on the study of hybrid fixed point theory for nonlinear single-valued and multi-valued mappings is a new development in the domain of contractive type multi-valued theory(see, for instance [4], [5], [12], [17], [21], [24], [28], [29], [30], [31], [33], [34] and references therein). On the other hand Alber and Guerre-Delabriere [3], defined weakly contractive mappings on a Hilbert space and established a fixed point theorem for such a mappings. Subsequently Rhoades [26] use the notion of weakly contractive mappings and obtained a fixed point theorem in complete metric space.
Afterward, weak contraction and function satisfying weak contractive type inequalities have been considered in a large number of papers, (see, for instance [1], [2], [6], [7], [11], [25] and references therein).
In this paper we will establish a fixed point theorems under generalized weak contractive condition for a pair of multi-valued and single-valued mappings by using the concept of (IT)-commutativity of mappings in complete metric space without appeal to continuity of mappings. Our results extend and generalize the results of Choudhury et al. [6] and others.

2. Preliminary notes

Let (X, d) be a metric space. Following [22], we define
$CL(X) = \{A : A \quad \text{is a non-empty closed subset of } X\}$.
$CB(X) = \{A : A \quad \text{is a non-empty closed and bounded subset of } X\}$.
$C(X) = \{A : A \quad \text{is a non-empty compact subset of } X\}$.

$BN(X) = \{A : A \quad \text{is a non-empty bounded subset of } X\}$.
For non-empty subsets A and B of X, and $x \in X$,
$D(A, B) = inf\{d(a, b) : a \in A, b \in B\}$.
$H(A, B) = max[sup\{D(a, B) : a \in A\}, sup\{D(A, b) : b \in B\}]$.
$d(x, A) = inf\{d(x, a) : a \in A\}$.
$\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}$.
Following Hadžić-Gajić [13] and Pant [23], Singh-Mishra [30] introduced the notion of R-weak commutativity of a hybrid pair of single-valued and multi-valued mappings.

Definition 2.1. [30] : Let (X, d) be a metric space. The mappings $f : X \to X$ and $T : X \to CL(X)$ are pointwise R-weakly commuting on X if given $x \in X$ and $fx \in X$, there exists $R > 0$ such that

$$d(fy, Tfx) \leq Rd(Tx, fx) \quad \text{for each } y \in X \cap Tx. \tag{A}$$

Mappings f and T will be called R-weakly commuting on X if for each $x \in X$ and (A) hold for some $R > 0$.
Following Jungck [15] and Jungck-Rhoades [16], we have the following definition.

Definition 2.2. The mappings $f : X \to X$ and $T : X \to CL(X)$ are weakly compatible if they commute at their coincidence points, that is, $Tfx = fTx$ whenever $fx \in Tx$.
Following Itoh-Takahashi [14] and Singh-Mishra [30], we have the following definition of (IT)-commutativity.

Definition 2.3. The mappings $f : X \to X$ and $T : X \to CL(X)$ are commuting at a point $x \in X$ if $fTx \subset Tfx$. f and T are commuting on X if they are commuting at each point $x \in X$.
The above commutativity is called Itoh-Takahashi commutativity or simply (IT)-commutativity (see, [30]).
The following example shows that (IT)-commutativity of f and T at a coincidence point is indeed more general than their weak compatibility at the same point.

Example 2.4. [30] : Let $X = [0, \infty)$ with the usual metric d and $fx = 4x$, $Tx = [3 + x, \infty)$, $x \in X$. Then $f1 \in T1$, $fT1 \subset Tf1$, and f, T are (IT)-commuting at $x = 1$. The inequality (A) is also satisfied for $x = 1$ and f, T are R-weakly commuting at $x = 1$. Notice that f, T are not weakly compatible since $fT1 \neq Tf1$.

Notation $C(f, T)$ will stand for the set of coincidence points of the mappings f and T, that is, $C(f, T) = \{z : fz \in Tz\}$.
Following Rhoades [26], we have the following definition.

Definition 2.5. [26] : A mapping $f : X \to X$, where (X, d) is a metric space, is said to be weakly contractive if for $x, y \in X$

$$d(fx, fy) \leq d(x, y) - \phi(d(x, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$. If one takes $\phi(t) = (1 - k)t$, where $0 < k < 1$, a weak contraction reduces to a Banach contraction.

In (1976-77), Delbosco [10] and Skof [35] have established fixed point theorem for self mappings of complete metric space by altering the distances between the points and subsequently, Khan et al. [19] generalized the notion of altering distance between the point.

Definition 2.6. ([10] see also [35]): A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:
I. ψ is monotone increasing and continuous,
II. $\psi(t) = 0$ if and only if $t = 0$.
Recently Choudhury et al. [6] defined generalized weak contraction and proved the following theorem.

Definition 2.7. [6] : Let (X, d) be a metric space, f a self mapping of X. We shall call f a generalized weakly contractive mapping if for all $x, y \in X$

$$\begin{aligned} \psi(d(fx, fy)) \quad &\leq \quad \psi\left(max\left\{ \; d(x, y), d(x, fx), d(y, fy), \tfrac{1}{2}[d(x, fy) + d(y, fx)] \; \right\}\right) \\ &\quad\quad -\phi(max\{d(x, y), d(y, fy)\}), \end{aligned}$$

where ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Theorem 2.8. [6] : Let (X, d) be a complete metric space, and let $f : X \to X$ be such that

$$\begin{aligned}\psi(d(fx, fy)) &\leq \psi\left(max\left\{\ d(x, y), d(x, fx), d(y, fy), \tfrac{1}{2}[d(x, fy) + d(y, fx)]\ \right\}\right) \\ &\quad -\phi(max\{d(x, y), d(y, fy)\}),\end{aligned}$$

for some ϕ and ψ defined as in Definition 2.5 and 2.6. Then f has a unique fixed point.

3. Main results

Now we state our main result.

Theorem 3.1. Let (X, d) be a complete metric space. Let $S, T : X \to C(X)$ be two multi-valued mappings and $f : X \to X$ be a self mapping such that for all $x, y \in X$

$$S(X) \cup T(X) \subset f(X) \tag{1}$$

$$f(X) \text{ is closed} \tag{2}$$

$$\psi(H(Sx, Ty)) \leq \psi(M(x, y)) - \phi(m(x, y)), \tag{3}$$

where

$$M(x, y) = max\left\{\ d(fx, fy), D(fx, Sx), D(fy, Ty), \tfrac{1}{2}[D(fx, Ty) + D(fy, Sx)]\ \right\}$$

and

$$m(x, y) = max\{d(fx, fy), D(fx, Sx), D(fy, Ty)\},$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and $\psi : [0, \infty) \to [0, \infty)$ is an altering distance function. Then S, f and T, f have a coincidence point. Further S and f have a common fixed point fu provided $ffu = fu$ and S, f are (IT)-commuting at $u \in C(S, f)$ and if T and f have a common fixed point fu provided $ffu = fu$ and T, f are (IT)-commuting at $u \in C(T, f)$. Then S, T and f have a common fixed point.

Proof. Let x_0 be an arbitrary point in X. We shall construct sequences $\{x_n\}$ and $\{y_n\}$ as follows. Since $S(X) \cup T(X) \subset f(X)$, we can choose points x_1, x_2 in X such that

$$y_1 = fx_1 \in Sx_0$$

and

$$y_2 = fx_2 \in Tx_1.$$

In view of the Remark of Nadler [22, page 480], we have the following

$$d(fx_1, fx_2) \leq H(Sx_0, Tx_1).$$

So

$$\psi(d(y_1, y_2)) \leq \psi(H(Sx_0, Tx_1)) \leq \psi(M(x_0, x_1)) - \phi(m(x_0, x_1)). \tag{4}$$

Similarly choose x_3, x_4 in X such that

$$y_3 = fx_3 \in Sx_2$$

and

$$y_4 = fx_4 \in Tx_3.$$

Again in view of the Remark of Nadler [22, page 480], we have the following

$d(fx_3, fx_4) \leq H(Sx_2, Tx_3)$.

So

$$\psi(d(y_3, y_4)) \quad \leq \quad \psi(H(Sx_2, Tx_3)) \leq \psi(M(x_2, x_3)) - \phi(m(x_2, x_3)). \tag{5}$$

We continue this process to obtain a sequence $\{y_n\}$ in X such that

$y_{2n+1} = fx_{2n+1} \in Sx_{2n}$

and

$y_{2n+2} = fx_{2n+2} \in Tx_{2n+1},$ for all $n = 0, 1, 2, 3...$.

If there exists a positive integer $2n$ such that $y_{2n+1} = y_{2n+2}$, then y_{2n+1} is a coincidence point of f and T. A similar conclusion holds if $y_{2n+2} = y_{2n+3}$, for some n, then f and S have a coincidence point. Therefore we may assume that $y_n \neq y_{n+1}$, for all $n \geq 0$. Then we have the following

$$\begin{aligned}
\psi(d(y_{2n+1}, y_{2n+2})) &\leq \psi(H(Sx_{2n}, Tx_{2n+1})) \leq \psi(M(x_{2n}, x_{2n+1})) - \phi(m(x_{2n}, x_{2n+1})) \\
&\leq \psi\left(max\left\{\begin{array}{c} d(fx_{2n}, fx_{2n+1}), D(fx_{2n}, Sx_{2n}), D(fx_{2n+1}, Tx_{2n+1}), \\ \frac{1}{2}[D(fx_{2n}, Tx_{2n+1}) + D(fx_{2n+1}, Sx_{2n})] \end{array}\right\}\right) \\
&\quad -\phi\left(max\left\{\ d(fx_{2n}, fx_{2n+1}), D(fx_{2n}, Sx_{2n}), D(fx_{2n+1}, Tx_{2n+1})\ \right\}\right) \\
&\leq \psi\left(max\left\{\begin{array}{c} d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \\ \frac{1}{2}[d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})] \end{array}\right\}\right) \\
&\quad -\phi\left(max\left\{\ d(y_{2n}, y_{2n+1}), D(fx_{2n}, Sx_{2n}), D(fx_{2n+1}, Tx_{2n+1})\ \right\}\right).
\end{aligned}$$

Since $\frac{1}{2}[d(y_{2n}, y_{2n+2})] \leq max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\}$, it follows that

$$\begin{aligned}
\psi(d(y_{2n+1}, y_{2n+2})) &\leq \psi\left(max\left\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\right\}\right) \\
&\quad -\phi\left(max\left\{\ d(y_{2n}, y_{2n+1}), D(fx_{2n}, Sx_{2n}), D(fx_{2n+1}, Tx_{2n+1})\ \right\}\right).
\end{aligned} \tag{6}$$

Suppose that $d(y_{2n}, y_{2n+1}) \leq d(y_{2n+1}, y_{2n+2})$ and $d(y_{2n}, y_{2n+1}) \leq D(fx_{2n}, Sx_{2n})$, for some positive integer n. Then from (6), we have

$\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(d(y_{2n+1}, y_{2n+2})) - \phi(D(fx_{2n}, Sx_{2n})),$

that is , $\phi(D(fx_{2n}, Sx_{2n})) \leq 0$, which implies that $D(fx_{2n}, Sx_{2n}) = 0$, that is $fx_{2n} \in Sx_{2n}$ or $y_{2n} \in Sx_{2n}$ contradicting the formation of the sequence. Therefore $D(fx_{2n}, Sx_{2n}) < d(y_{2n}, y_{2n+1})$, for all $n \geq 0$. Again suppose that $d(y_{2n}, y_{2n+1}) \leq d(y_{2n+1}, y_{2n+2})$ and $d(y_{2n}, y_{2n+1}) \leq D(fx_{2n+1}, Tx_{2n+1})$, for some positive integer n. Then from (6), we have

$\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(d(y_{2n+1}, y_{2n+2})) - \phi(D(fx_{2n+1}, Tx_{2n+1})),$

that is, $\phi(D(fx_{2n+1}, Tx_{2n+1})) \leq 0$, which implies that $D(fx_{2n+1}, Tx_{2n+1}) = 0$, that is $fx_{2n+1} \in Tx_{2n+1}$ or $y_{2n+1} \in Tx_{2n+1}$ contradicting the formation of the sequence.
Therefore $D(fx_{2n+1}, Tx_{2n+1}) < d(y_{2n}, y_{2n+1})$, for all $n \geq 0$.
Now

$$\begin{aligned}
\psi(d(y_{2n+3}, y_{2n+2})) &\leq \psi(H(Sx_{2n+2}, Tx_{2n+1})) \leq \psi(M(x_{2n+2}, x_{2n+1})) - \phi(m(x_{2n+2}, x_{2n+1})) \\
&\leq \psi\left(max\left\{\begin{array}{c} d(fx_{2n+2}, fx_{2n+1}), D(fx_{2n+2}, Sx_{2n+2}), D(fx_{2n+1}, Tx_{2n+1}), \\ \frac{1}{2}[D(fx_{2n+2}, Tx_{2n+1}) + D(fx_{2n+1}, Sx_{2n+2})] \end{array}\right\}\right) \\
&\quad -\phi\left(max\left\{\ d(fx_{2n+2}, fx_{2n+1}), D(fx_{2n+2}, Sx_{2n+2}), D(fx_{2n+1}, Tx_{2n+1})\ \right\}\right) \\
&\leq \psi\left(max\left\{\begin{array}{c} d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+3}), d(y_{2n+1}, y_{2n+2}), \\ \frac{1}{2}[d(y_{2n+2}, y_{2n+2}) + d(y_{2n+1}, y_{2n+3})] \end{array}\right\}\right) \\
&\quad -\phi\left(max\left\{\ d(y_{2n+2}, y_{2n+1}), D(fx_{2n+2}, Sx_{2n+2}), D(fx_{2n+1}, Tx_{2n+1})\ \right\}\right).
\end{aligned}$$

Since $\frac{1}{2}[d(y_{2n+1}, y_{2n+3})] \leq max\{d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3})\}$, it follows that

$$\begin{aligned}
\psi(d(y_{2n+2}, y_{2n+3})) &\leq \psi\left(max\left\{d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3})\right\}\right) \\
&\quad -\phi\left(max\left\{\ d(y_{2n+1}, y_{2n+2}), D(fx_{2n+2}, Sx_{2n+2}), D(fx_{2n+1}, Tx_{2n+1})\ \right\}\right).
\end{aligned} \tag{7}$$

Suppose that $d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n+2}, y_{2n+3})$ and $d(y_{2n+1}, y_{2n+2}) \leq D(fx_{2n+2}, Sx_{2n+2})$, for some positive integer n.
Then from (7), we have

$$\psi(d(y_{2n+2}, y_{2n+3})) \leq \psi(d(y_{2n+2}, y_{2n+3})) - \phi(D(fx_{2n+2}, Sx_{2n+2})),$$

that is , $\phi(D(fx_{2n+2}, Sx_{2n+2})) \leq 0$ which implies that $D(fx_{2n+2}, Sx_{2n+2}) = 0$, that is $fx_{2n+2} \in Sx_{2n+2}$ or $y_{2n+2} \in Sx_{2n+2}$ contradicting the formation of the sequence. Therefore $D(fx_{2n+2}, Sx_{2n+2}) < d(y_{2n+1}, y_{2n+2})$, for all $n \geq 0$. Again suppose that $d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n+2}, y_{2n+3})$ and $d(y_{2n+1}, y_{2n+2}) \leq D(fx_{2n+1}, Tx_{2n+1})$, for some positive integer n. Then from (7), we have

$$\psi(d(y_{2n+2}, y_{2n+3})) \leq \psi(d(y_{2n+2}, y_{2n+3})) - \phi(D(fx_{2n+1}, Tx_{2n+1})),$$

that is, $\phi(D(fx_{2n+1}, Tx_{2n+1})) \leq 0$, which implies that $D(fx_{2n+1}, Tx_{2n+1}) = 0$, that is $fx_{2n+1} \in Tx_{2n+1}$ or $y_{2n+1} \in Tx_{2n+1}$ contradicting the formation of the sequence.
Therefore $D(fx_{2n+1}, Tx_{2n+1}) < d(y_{2n+1}, y_{2n+2})$, for all $n \geq 0$.
Thus $\{d(y_n, y_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers.
Hence there exists an $r \geq 0$ such that

$$lim_{n\to\infty} d(y_n, y_{n+1}) = r. \tag{8}$$

In view of (6), for all $n \geq 0$,

$$\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(d(y_{2n}, y_{2n+1})) - \phi(d(y_{2n}, y_{2n+1})).$$

Taking the limit as $n \to \infty$ in the above inequality and using the continuity of ϕ and ψ, we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$.
Hence we have

$$lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{9}$$

Now we shall show that $\{y_n\}$ is a Cauchy sequence. It is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ such that for each integer $2(k)$ there exists an even integer,

$$2m(k) > 2n(k) > 2(k)$$

such that

$$d(y_{2n(k)}, y_{2m(k)}) \geq \epsilon, \tag{10}$$

for every integer $2(k)$. Let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (10), such that

$$d(y_{2n(k)}, y_{2m(k)-2}) < \epsilon.$$

Using the triangle inequality, we have

$$\epsilon \leq d(y_{2n(k)}, y_{2m(k)}) \quad \leq \quad d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}),$$

that is,

$$\epsilon \leq d(y_{2n(k)}, y_{2m(k)}) < \epsilon + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).$$

Letting $k \to \infty$ in the above inequality and using (9), we have

$$lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{11}$$

Again

$$d(y_{2n(k)}, y_{2m(k)}) \quad \leq \quad d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)+1}) + d(y_{2m(k)+1}, y_{2m(k)})$$

and

$$d(y_{2n(k)+1}, y_{2m(k)+1}) \quad \leq \quad d(y_{2n(k)+1}, y_{2n(k)}) + d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)}, y_{2m(k)+1}).$$

Letting $k \to \infty$ in the above inequality and using (9) and (11), we have

$$lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)+1}) = \epsilon. \tag{12}$$

Again

$$d(y_{2n(k)}, y_{2m(k)+2}) \leq d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)+1}) + d(y_{2m(k)+1}, y_{2m(k)+2}).$$

Letting $k \to \infty$ in the above inequality and using (9) and (12), we have

$$lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)+2}) = \epsilon. \tag{13}$$

Further

$$d(y_{2n(k)}, y_{2m(k)+1}) \leq d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)+1}).$$

Letting $k \to \infty$ in the above inequality and using (9) and (12), we have

$$lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)+1}) = \epsilon. \tag{14}$$

Putting $x = y_{2n(k)}$ and $y = y_{2m(k)+1}$ in (3), we have

$$\begin{aligned}
\psi(d(y_{2n(k)+1}, y_{2m(k)+2})) &\leq \psi(H(Sx_{2n(k)}, Tx_{2m(k)+1})) \leq \psi(M(x_{2n(k)}, x_{2m(k)+1})) - \phi(m(x_{2n(k)}, x_{2m(k)+1})) \\
&\leq \psi\left(max\left\{\begin{array}{c} d(fx_{2n(k)}, fx_{2m(k)+1}), D(fx_{2n(k)}, Sx_{2n(k)}), D(fx_{2m(k)+1}, Tx_{2m(k)+1}), \\ \frac{1}{2}[D(fx_{2n(k)}, Tx_{2m(k)+1}) + D(fx_{2m(k)+1}, Sx_{2n(k)})] \end{array}\right\}\right) \\
&\quad -\phi\left(max\left\{ d(fx_{2n(k)}, fx_{2m(k)+1}), D(fx_{2n(k)}, Sx_{2n(k)}), D(fx_{2m(k)+1}, Tx_{2m(k)+1}) \right\}\right) \\
&\leq \psi\left(max\left\{\begin{array}{c} d(y_{2n(k)}, y_{2m(k)+1}), d(y_{2n(k)}, y_{2n(k)+1}), d(y_{2m(k)+1}, y_{2m(k)+2}), \\ \frac{1}{2}[d(y_{2n(k)}, y_{2m(k)+2}) + d(y_{2m(k)+1}, y_{2n(k)+1})] \end{array}\right\}\right) \\
&\quad -\phi\left(max\left\{ d(y_{2n(k)}, y_{2m(k)+1}), d(y_{2n(k)}, y_{2n(k)+1}), d(y_{2m(k)+1}, y_{2m(k)+2}) \right\}\right).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using $(9), (11-14)$ and using the continuity of ϕ and ψ, we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction by virtue of a property of ϕ.
Therefore $\{y_{2n}\}$ is a Cauchy sequence. In view of (9), $\{y_n\}$ is a Cauchy sequence in X.
Since X is complete, then there exists a point z in X such that

$$lim_{n\to\infty} y_{2n+1} = z = lim_{n\to\infty} fx_{2n+1} \in Sx_{2n}$$

and

$$lim_{n\to\infty} y_{2n+2} = z = lim_{n\to\infty} fx_{2n+2} \in Tx_{2n+1}.$$

Since $f(X)$ is closed, then there exist a point u in X such that $fu = z$.
Now firstly we have

$$\begin{aligned}
\psi(D(Su, fx_{2n+2})) &\leq \psi(H(Su, Tx_{2n+1})) \\
&\leq \psi(M(u, x_{2n+1})) - \phi(m(u, x_{2n+1})) \\
&\leq \psi\left(max\left\{\begin{array}{c} d(fu, fx_{2n+1}), D(fu, Su), D(fx_{2n+1}, Tx_{2n+1}), \\ \frac{1}{2}[D(fu, Tx_{2n+1}) + D(fx_{2n+1}, Su)] \end{array}\right\}\right) \\
&\quad -\phi(max\{d(fu, fx_{2n+1}), D(fu, Su), D(fx_{2n+1}, Tx_{2n+1})\}).
\end{aligned}$$

Taking limit $n \to \infty$, we have

$$\begin{aligned}
\psi(D(Su, z)) &\leq \psi(max\{d(z,z), D(z, Su), d(z,z), \frac{1}{2}[d(z,z) + D(z, Su)]\}) \\
&\quad -\phi(max\{d(z,z), D(z, Su), d(z,z)\}) \\
&\leq \psi(max\{0, D(Su, z), 0, \frac{1}{2}[0 + D(z, Su)]\}) \\
&\quad -\phi(max\{0, D(z, Su), 0\}) \\
&\leq \psi(D(Su, z)) - \phi(D(Su, z)),
\end{aligned}$$

which implies that $\phi(D(Su,z)) = 0$. Hence $D(Su,z) = 0$, that is $z \in Su$.
Therefore $z = fu \in Su$.
Now if $ffu = fu$, then $fz = z$ and from the (IT)- commutativity of S and f, we have

$$z = fz = ffu \in fSu \subset Sfu = Sz,$$

implies $z = fz \in Sz$.
Further

$$\begin{aligned}\psi(D(fx_{2n+1},Tu)) &\leq \psi(H(Sx_{2n},Tu))\\ &\leq \psi(M(x_{2n},u)) - \phi(m(x_{2n},u))\\ &\leq \psi\left(max\left\{\begin{array}{c} d(fx_{2n},fu), D(fx_{2n},Sx_{2n}), D(fu,Tu),\\ \frac{1}{2}[D(fx_{2n},Tu) + D(fu,Sx_{2n})]\end{array}\right\}\right)\\ &\quad -\phi(max\{d(fx_{2n},fu), D(fx_{2n},Sx_{2n}), D(fu,Tu)\}).\end{aligned}$$

Taking limit $n \to \infty$, we have

$$\begin{aligned}\psi(D(z,Tu)) &\leq \psi(max\{d(z,z), d(z,z), D(z,Tu), \frac{1}{2}[D(z,Tu) + d(z,z)]\})\\ &\quad -\phi(max\{d(z,z), d(z,z), D(z,Tu)\})\\ &\leq \psi(max\{0,0,D(z,Tu), \frac{1}{2}[D(z,Tu) + 0]\})\\ &\quad -\phi(max\{0,0,D(z,Tu)\})\\ &\leq \psi(D(z,Tu)) - \phi(D(z,Tu)),\end{aligned}$$

which implies that $\phi(D(z,Tu)) = 0$. Hence $D(z,Tu) = 0$, that is $z \in Tu$.
Therefore $z = fu \in Tu$.
Now if $ffu = fu$, then $fz = z$ and from the (IT)-commutativity of T and f, we have

$$z = fz = ffu \in fTu \subset Tfu = Tz,$$

implies $z = fz \in Tz$.
Thus z is a common fixed point of S, T and f.

Corollary 3.2. Let (X,d) be a complete metric space. Let $S,T : X \to C(X)$ be two multi-valued mappings and $f : X \to X$ be a self-mapping such that for all $x, y \in X$

$$S(X) \cup T(X) \subset f(X) \tag{15}$$

$$f(X) \text{ is closed} \tag{16}$$

$$\begin{aligned}H(Sx,Ty) &\leq max\{\ d(fx,fy), D(fx,Sx), D(fy,Ty), \tfrac{1}{2}[D(fx,Ty) + D(fy,Sx)]\ \}\\ &\quad -\phi(max\{d(fx,fy), D(fx,Sx), D(fy,Ty)\}),\end{aligned} \tag{17}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then S, f and T, f have a coincidence point. Further, if S and f have a common fixed point fu provided $ffu = fu$ and S, f are (IT)-commuting at $u \in C(S,f)$ and if T and f have a common fixed point fu provided $ffu = fu$ and T, f are (IT)-commuting at $u \in C(T,f)$. Then S, T and f have a common fixed point.
Proof. By taking ψ as an identity function in the proof of Theorem 3.1, we can get the proof.

Corollary 3.3. Let (X,d) be a complete metric space. Let $S : X \to C(X)$ be a multi-valued mapping and $f : X \to X$ be a single-valued mapping such that for all $x, y \in X$

$$S(X) \subset f(X) \tag{18}$$

$$f(X) \text{ is closed} \tag{19}$$

$$\begin{aligned}\psi(H(Sx,Sy)) \quad \leq \quad & \psi\left(max\left\{\ d(fx,fy), D(fx,Sx), D(fy,Sy), \tfrac{1}{2}[D(fx,Sy)+D(fy,Sx)]\ \right\}\right) \\ & -\phi(max\{d(fx,fy), D(fx,Sx), D(fy,Sy)\}),\end{aligned} \tag{20}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function. Then S and f have a coincidence point. Further , if S and f have a common fixed point fu provided $ffu = fu$ and S, f are (IT)-commuting at $u \in C(S,f)$.Then S and f have a common fixed point.

Proof. It may be completed following the proof of Theorem 3.1 by taking $S = T$.

Corollary 3.4. Let (X,d) be a complete metric space. Let $S : X \to C(X)$ be a multi-valued mapping such that for all $x, y \in X$

$$\begin{aligned}\psi(H(Sx,Sy)) \quad \leq \quad & \psi\left(max\left\{\ d(x,y), D(x,Sx), D(y,Sy), \tfrac{1}{2}[D(x,Sy)+D(y,Sx)]\ \right\}\right) \\ & -\phi(max\{d(x,y), D(x,Sx), D(y,Sy)\}),\end{aligned} \tag{21}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function.Then S has a unique fixed point.

Proof. If we take $S = T$ and f as an identity mapping in Theorem 3.1, then we can get the proof.
Now taking clue from Example 3.1 of [6] we have an example.

Example 3.5. Let $X = \{0, 1, 2, 3, ...\}$. Let $d : X \times X \to R$ be given as

$$d(x,y) = \begin{cases} x+y, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

Then (X,d) is a complete metric space.
Let $\psi : [0,\infty) \to [0,\infty)$ be defined as follows:

$$\psi(t) = t^2, \quad \text{for t} \in [0,\infty).$$

Let $\phi : [0,\infty) \to [0,\infty)$ be defined as follows:

$$\phi(s) = \begin{cases} \frac{s^2}{2}, & \text{if } s \leq 1 \\ \frac{1}{2}, & \text{if } s > 1 \end{cases} \qquad \text{for} \quad s \in [0,\infty)\,.$$

Then ϕ and ψ have the properties mentioned in Theorem 3.1.
Let $S : X \to C(X)$ be defined as follows:

$$Sx = \begin{cases} \{x-1\}, & \text{if } x \neq 0 \\ \{0\}, & \text{if } x = 0. \end{cases}$$

Sol. We can see that mapping S is satisfying the contractive condition (21) but it is not satisfying the condition (6) of [8, Page 266].

Note. In the above example, we set $x = n+1$ and $y = n$, where n is a positive integer.
Then according to the case $x \neq y$, if $y \neq 0$ and $x > y$,

$$H(Sx,Sy) = 2n-1,$$

and

$$max\left\{\ d(x,y), d(x,fx), d(y,fy), \tfrac{1}{2}[d(x,fy)+d(y,fx)]\ \right\} = 2n+1.$$

Clearly

$$H(Sx,Sy) = k_n max\left\{\ d(x,y), d(x,fx), d(y,fy), \tfrac{1}{2}[d(x,fy)+d(y,fx)]\ \right\},$$

where

$$k_n = \frac{2n-1}{2n+1}.$$

Since $k_n \to 1$ as $n \to \infty$, there dose not exist any k with $0 \leq k < 1$ such that

$$H(Sx, Sy) \leq kmax\left\{ d(x,y), D(x,Sx), D(y,Sy), \tfrac{1}{2}[D(x,Sy)+D(y,Sx)] \right\},$$

for each $x, y \in X$.

Hence Example 3.5 does not satisfy condition (6) of [8]. This show that condition (21) is more general than (6) of [8].

Remark 3.6. In Corollary 3.4, we obtain slightly generalized version of Theorem 3.1 of [6] and Theorem 2.1 of [11].

Remark 3.7. If we take S and T are single-valued mappings and f as an identity mapping in Theorem 3.1, then we can get Theorem 3.2 of [6].

Remark 3.8. As it is shown in [6] that a generalized weakly contractive condition 2.1 of [6] is more general than that (21) of Rhoades [27], so we can say that the contractive condition (3) and (17) are more general than the contractive condition used in [33].

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