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Research Paper

Inequalities for the Psi and k-Gamma functions

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Abstract

In this paper, the authors establish some inequalities involving the Psi and k-Gamma functions. The procedure utilizes some monotonicity properties of some functions associated with the Psi and k-Gamma functions.

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MSC: 33B15, 26A48.

1. Introduction

The well-known classical Gamma function, $\Gamma(t)$ is usually defined for $t > 0$ by

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx.$$

The p-analogue of the Gamma function is defined (see also [2], [3]) for $t > 0$ and $p \in N$ by

$$\Gamma_p(t) = \frac{p! p^t}{t(t+1)\dots(t+p)} = \frac{p^t}{t(1+\frac{t}{1})\dots(1+\frac{t}{p})}.$$

Also, the q-analogue of the Gamma function is defined (see [4]) for $t > 0$ and $q \in (0, 1)$ by

$$\Gamma_q(t) = (1-q)^{1-t} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^{t+n}} = (1-q)^{1-t} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+t}}.$$

Similarly, the k-analogue or the k-Gamma function is defined (see [1]) for $t > 0$ and $k > 0$ by

$$\Gamma_k(t) = \int_0^\infty e^{-\frac{x^k}{k}} x^{t-1}\, dx.$$

The psi function, $\psi(t)$ also known in literature as the digamma function is defined for $t > 0$ as the logarithmic derivative of the gamma function. That is,

$$\psi(t) = \frac{d}{dt}\ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}.$$

The p-analogue, q-analogue and k-analogue of the psi function are equivalently defined for $t > 0$ as follows.

$$\psi_p(t) = \frac{d}{dt}\ln(\Gamma_p(t)) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \quad \psi_q(t) = \frac{d}{dt}\ln(\Gamma_q(t)) = \frac{\Gamma'_q(t)}{\Gamma_q(t)} \quad \text{and} \quad \psi_k(t) = \frac{d}{dt}\ln(\Gamma_k(t)) = \frac{\Gamma'_k(t)}{\Gamma_k(t)}.$$

The following series representations for the functions $\psi(t)$ and $\psi_k(t)$ are valid and are well-known in literature.

$$\psi(t) = -\gamma - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{n(n+t)} \tag{1}$$

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)} \tag{2}$$

where γ denotes the Euler-Mascheroni's constant.

The polygamma functions, $\psi^{(m)}(t)$ are defined for $t > 0$ and $m \in N$ as the m-th derivative of the psi function. That is,

$$\psi^{(m)}(t) = \frac{d^m}{dt^m}\psi(t) = \frac{d^{m+1}}{dt^{m+1}}\ln(\Gamma(t)).$$

where $\psi^{(0)}(t) \equiv \psi(t)$. They also exhibit the series representation shown below.

$$\psi^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+t)^{m+1}} \tag{3}$$

Consequently, the following representations are trivially obtained from (3).

$$\psi'(t) = \sum_{n=0}^{\infty} \frac{1}{(n+t)^2} \tag{4}$$

$$\psi^{(m+1)}(t) = (-1)^{m+2}(m+1)! \sum_{n=0}^{\infty} \frac{1}{(n+t)^{m+2}} \tag{5}$$

By using basic analyses, the purpose of this paper is to establish some inequalities for Psi and k-Gamma functions. We present our results in the following sections.

2. Some inequalities for the Psi function

This section is devoted to some inequalities associated with the Psi function. We proceed as follows.

Lemma 2.1. *Let $0 < s \leq t$, then the following statement holds true.*

$$\psi(s) \leq \psi(t). \tag{6}$$

Proof. From (1), we have the following (See also [6]).

$$\begin{aligned}
\psi(s) - \psi(t) &= (s-1)\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+s)} - (t-1)\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+t)} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)}\left(\frac{s-1}{n+s} - \frac{t-1}{n+t}\right) \\
&= \sum_{n=0}^{\infty} \frac{(s-t)}{(n+s)(n+t)} \leq 0.
\end{aligned}$$

Lemma 2.2. *Let $0 < s \leq t$, then the following statement holds true.*

$$\psi'(s) \geq \psi'(t). \tag{7}$$

Proof. From (4), we have the following.

$$\begin{aligned}\psi'(s) - \psi'(t) &= \sum_{n=0}^{\infty} \frac{1}{(n+s)^2} - \sum_{n=0}^{\infty} \frac{1}{(n+t)^2} \\ &= \sum_{n=0}^{\infty} \left[\frac{1}{(n+s)^2} - \frac{1}{(n+t)^2}\right] \\ &= \sum_{n=0}^{\infty} \frac{2n(t-s) + (t^2 - s^2)}{(n+s)^2(n+t)^2} \geq 0.\end{aligned}$$

Theorem 2.3. *Define a function U by*

$$U(t) = \frac{[\psi(a+bt)]^{\alpha}}{[\psi(c+dt)]^{\beta}}, \quad t \in [0, \infty)$$

where a, b, c, d, α, β are positive real numbers such that $a \leq c$, $b \leq d$, $\beta d \leq \alpha b$, $a + bt \leq c + dt$, $\psi(a+bt) > 0$ and $\psi(c+dt) > 0$. Then U is non-decreasing on $t \in [0, \infty)$ and the inequalities

$$\frac{[\psi(a)]^{\alpha}}{[\psi(c)]^{\beta}} \leq \frac{[\psi(a+bt)]^{\alpha}}{[\psi(c+dt)]^{\beta}} \leq \frac{[\psi(a+b)]^{\alpha}}{[\psi(c+d)]^{\beta}} \tag{8}$$

hold true for $t \in [0, 1]$.

Proof. Let $\mu(t) = \ln U(t)$ for every $t \in [0, \infty)$. Then,

$$\mu = \ln \frac{[\psi(a+bt)]^{\alpha}}{[\psi(c+dt)]^{\beta}} = \alpha \ln \psi(a+bt) - \beta \ln \psi(c+dt)$$

and

$$\begin{aligned}\mu'(t) &= \alpha b \frac{\psi'(a+bt)}{\psi(a+bt)} - \beta d \frac{\psi'(c+dt)}{\psi(c+dt)} \\ &= \frac{\alpha b \psi'(a+bt)\psi(c+dt) - \beta d \psi'(c+dt)\psi(a+bt)}{\psi(a+bt)\psi(c+dt)}.\end{aligned}$$

Since $0 < a + bt \leq c + dt$, then by Lemmas 2.1 and 2.2 we have, $\psi(a+bt) \leq \psi(c+dt)$ and $\psi'(a+bt) \geq \psi'(c+dt)$. Then that implies;
$\psi(c+dt)\psi'(a+bt) \geq \psi(c+dt)\psi'(c+dt) \geq \psi(a+bt)\psi'(c+dt)$. Further, $\alpha b \geq \beta d$ implies;
$\alpha b \psi(c+dt)\psi'(a+bt) \geq \alpha b \psi(a+bt)\psi'(c+dt) \geq \beta d \psi(a+bt)\psi'(c+dt)$. Hence,
$\alpha b \psi(c+dt)\psi'(a+bt) - \beta d \psi(a+bt)\psi'(c+dt) \geq 0$. Therefore $\mu'(t) \geq 0$.
That implies μ as well as U are non-decreasing on $t \in [0, \infty)$ and for $t \in [0, 1]$ we have,

$$U(0) \leq U(t) \leq U(1)$$

resulting to inequalities (8).

Remark 2.4. If in particular, $a = b = c = d = 1$ then we obtain

$$(-\gamma)^{\alpha-\beta} \leq [\psi(1+t)]^{\alpha-\beta} \leq (1-\gamma)^{\alpha-\beta}.$$

Remark 2.5. For $t \in (1, \infty)$, we have $U(t) \geq U(1)$ yielding

$$\frac{[\psi(a+bt)]^{\alpha}}{[\psi(c+dt)]^{\beta}} \geq \frac{[\psi(a+b)]^{\alpha}}{[\psi(c+d)]^{\beta}}.$$

Remark 2.6. Results similar to Theorem 2.3 can also be found in [5] for the k-analogue of the psi function.

Lemma 2.7. *Suppose that $0<\alpha\leq\beta$, $t\geq 0$ and m is a positive odd integer. Suppose further that,*

$$A=\sum_{n=0}^{\infty}\frac{1}{(n+\beta+t)^{m+2}},\qquad B=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha+t)^{m+1}},$$
$$C=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha+t)^{m+2}}\quad \textit{and}\quad D=\sum_{n=0}^{\infty}\frac{1}{(n+\beta+t)^{m+1}}$$

are such that $AB-CD\geq 0$. Then,

$$\psi^{(m+1)}(\alpha+t)\psi^{(m)}(\beta+t)-\psi^{(m+1)}(\beta+t)\psi^{(m)}(\alpha+t)\geq 0.$$

Proof. From (3) and (5) and for a positive odd integer m we have,
$\psi^{(m)}(t)=m!\sum_{n=0}^{\infty}\frac{1}{(n+t)^{m+1}}$ and $\psi^{(m+1)}(t)=-(m+1)!\sum_{n=0}^{\infty}\frac{1}{(n+t)^{m+2}}$
Then that implies,

$$\begin{aligned}
&\psi^{(m+1)}(\alpha+t)\psi^{(m)}(\beta+t)-\psi^{(m+1)}(\beta+t)\psi^{(m)}(\alpha+t)=\\
&\left(-(m+1)!\sum_{n=0}^{\infty}\frac{1}{(n+\alpha+t)^{m+2}}\right)\left(m!\sum_{n=0}^{\infty}\frac{1}{(n+\beta+t)^{m+1}}\right)\\
&-\left(-(m+1)!\sum_{n=0}^{\infty}\frac{1}{(n+\beta+t)^{m+2}}\right)\left(m!\sum_{n=0}^{\infty}\frac{1}{(n+\alpha+t)^{m+1}}\right)\\
&=m!(m+1)!\left[\left(\sum_{n=0}^{\infty}\frac{1}{(n+\beta+t)^{m+2}}\right)\left(\sum_{n=0}^{\infty}\frac{1}{(n+\alpha+t)^{m+1}}\right)\right.\\
&\left.-\left(\sum_{n=0}^{\infty}\frac{1}{(n+\alpha+t)^{m+2}}\right)\left(\sum_{n=0}^{\infty}\frac{1}{(n+\beta+t)^{m+1}}\right)\right]\\
&=m!(m+1)!\left[AB-CD\right]\geq 0.
\end{aligned}$$

Theorem 2.8. *For a positive odd integer m, define a function V by*

$$V(t)=\frac{\psi^{(m)}(\alpha+t)}{\psi^{(m)}(\beta+t)},\quad t\in[0,\infty)$$

where $0<\alpha\leq\beta$ are real numbers. If $A=\sum_{n=0}^{\infty}\frac{1}{(n+\beta+t)^{m+2}}$, $B=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha+t)^{m+1}}$, $C=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha+t)^{m+2}}$ and $D=\sum_{n=0}^{\infty}\frac{1}{(n+\beta+t)^{m+1}}$ are such that $AB-CD\geq 0$, then V is non-decreasing on $t\in[0,\infty)$ and the inequalities

$$\frac{\psi^{(m)}(\alpha)}{\psi^{(m)}(\beta)}\leq\frac{\psi^{(m)}(\alpha+t)}{\psi^{(m)}(\beta+t)}\leq\frac{\psi^{(m)}(\alpha+1)}{\psi^{(m)}(\beta+1)}\tag{9}$$

are valid for $t\in[0,1]$.

Proof. Let $f(t)=\ln V(t)$ for every $t\in[0,\infty)$. Then,

$$f(t)=\ln\frac{\psi^{(m)}(\alpha+t)}{\psi^{(m)}(\beta+t)}=\ln\psi^{(m)}(\alpha+t)-\ln\psi^{(m)}(\beta+t).\quad\text{Then,}$$

$$\begin{aligned}
f'(t)&=\frac{\psi^{(m+1)}(\alpha+t)}{\psi^{(m)}(\alpha+t)}-\frac{\psi^{(m+1)}(\beta+t)}{\psi^{(m)}(\beta+t)}\\
&=\frac{\psi^{(m+1)}(\alpha+t)\psi^{(m)}(\beta+t)-\psi^{(m+1)}(\beta+t)\psi^{(m)}(\alpha+t)}{\psi^{(m)}(\alpha+t)\psi^{(m)}(\beta+t)}.
\end{aligned}$$

Since m is odd, then $\psi^{(m)}(t) = m!\sum_{n=0}^{\infty} \frac{1}{(n+t)^{m+1}} > 0$ for each $t > 0$. That implies $\psi^{(m)}(\alpha+t) > 0$ and $\psi^{(m)}(\beta+t) > 0$ for each $t \geq 0$. Also by Lemma 2.7 we have $\psi^{(m+1)}(\alpha+t)\psi^{(m)}(\beta+t) - \psi^{(m+1)}(\beta+t)\psi^{(m)}(\alpha+t) \geq 0$. Therefore $f'(t) \geq 0$. That implies f as well as V are nondecreasing on $t \in [0,\infty)$ and for $t \in [0,1]$ we have,

$$V(0) \leq V(t) \leq V(1)$$

resulting to inequalities (9).

Remark 2.9. For $t \in (1,\infty)$, we have $V(t) \geq V(1)$ yielding

$$\frac{\psi^{(m)}(\alpha+t)}{\psi^{(m)}(\beta+t)} \geq \frac{\psi^{(m)}(\alpha+1)}{\psi^{(m)}(\beta+1)}.$$

3. Some inequalities for the k-Gamma Function

This section is dedicated to some inequalities associated with the k-Gamma function.
In 2010, Krasniqi and Shabani [3] proved that,

$$\frac{p^{-t}e^{-\gamma t}\Gamma(\alpha)}{\Gamma_p(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{p^{1-t}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_p(\alpha+1)}$$

for $p \in N$, $t \in (0,1)$, where α is a positive real number such that $\alpha + t > 1$.

Also in that same year, Krasniqi, Mansour and Shabani [2] proved the following:

$$\frac{(1-q)^{t}e^{-\gamma t}\Gamma(\alpha)}{\Gamma_q(\alpha)} < \frac{\Gamma(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{(1-q)^{t-1}e^{\gamma(1-t)}\Gamma(\alpha+1)}{\Gamma_q(\alpha+1)}$$

for $q \in (0,1)$, $t \in (0,1)$, where α is a positive real number such that $\alpha + t > 1$.

In this section, our interest is to establish similar inequalities for the k-Gamma function. We also present some new results involving products of certain ratios of the k-Gamma function. We proceed as follows.

Lemma 3.1. *Let $k \geq 1$ and $\alpha > 0$ such that $\alpha + t > 0$. Then,*

$$\gamma + \frac{\ln k - \gamma}{k} + \psi(\alpha+t) - \psi_k(\alpha+t) \geq 0.$$

Proof. Using equations (1) and (2) we obtain,

$$\gamma + \frac{\ln k - \gamma}{k} + \psi(t) - \psi_k(t) = t\left[\sum_{n=1}^{\infty} \frac{1}{n(n+t)} - \sum_{n=1}^{\infty} \frac{1}{nk(nk+t)}\right] \geq 0.$$

Substituting t by $\alpha + t$ completes the proof.

Theorem 3.2. *Define a function W for $k \geq 1$ by*

$$W(t) = \frac{k^{\frac{t}{k}} e^{t\left(\frac{k\gamma-\gamma}{k}\right)}\Gamma(\alpha+t)}{\Gamma_k(\alpha+t)}, \quad t \in (0,\infty)$$

where α is a positive real number. Then W is increasing on $t \in (0,\infty)$ and for $t \in (0,1)$, the following inequalities are valid.

$$\frac{k^{-\frac{t}{k}} e^{-t\left(\frac{k\gamma-\gamma}{k}\right)}\Gamma(\alpha)}{\Gamma_k(\alpha)} \leq \frac{\Gamma(\alpha+t)}{\Gamma_k(\alpha+t)} \leq \frac{k^{\frac{1-t}{k}} e^{(1-t)\left(\frac{k\gamma-\gamma}{k}\right)}\Gamma(\alpha+1)}{\Gamma_k(\alpha+1)}. \tag{10}$$

Proof. Let $v(t) = \ln W(t)$ for every $t \in (0,\infty)$. Then,

$$\begin{aligned} v(t) &= \ln \frac{k^{\frac{t}{k}} e^{t\left(\frac{k\gamma-\gamma}{k}\right)}\Gamma(\alpha+t)}{\Gamma_k(\alpha+t)} \\ &= \frac{t}{k}\ln k + t\left(\frac{k\gamma-\gamma}{k}\right) + \ln\Gamma(\alpha+t) - \ln\Gamma_k(\alpha+t) \end{aligned}$$

Then,

$$\begin{aligned} v'(t) &= \frac{\ln k}{k} + \frac{k\gamma-\gamma}{k} + \psi(\alpha+t) - \psi_k(\alpha+t) \\ &= \gamma + \frac{\ln k-\gamma}{k} + \psi(\alpha+t) - \psi_k(\alpha+t) \geq 0. \quad \text{(by Lemma 3.1)}. \end{aligned}$$

That implies v is increasing on $t \in (0,\infty)$. Hence $W = e^{v(t)}$ is increasing on $t \in (0,\infty)$ and for $t \in (0,1)$ we have,

$W(0) \leq W(t) \leq W(1)$

resulting to inequalities (10).

Remark 3.3. For $t \in [1,\infty)$, we have $W(1) \leq W(t)$ yielding

$$\frac{k^{\frac{1-t}{k}} e^{(1-t)\left(\frac{k\gamma-\gamma}{k}\right)}\Gamma(\alpha+1)}{\Gamma_k(\alpha+1)} \leq \frac{\Gamma(\alpha+t)}{\Gamma_k(\alpha+t)}.$$

Lemma 3.4. *Let $k > 0$, $s > 0$, $t > 0$ with $s \leq t$, then*

$$\psi_k(s) \leq \psi_k(t). \tag{11}$$

Proof. From (2), we have the following.

$$\begin{aligned} \psi_k(s) - \psi_k(t) &= \frac{1}{t} - \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{nk} - \frac{1}{s+nk}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{nk} - \frac{1}{t+nk}\right) \\ &= \frac{s-t}{st} + \sum_{n=1}^{\infty}\left(\frac{1}{t+nk} - \frac{1}{s+nk}\right) \\ &= \frac{s-t}{st} + \sum_{n=1}^{\infty}\frac{(s-t)}{(s+nk)(t+nk)} \leq 0. \end{aligned}$$

Lemma 3.5. *Let a, b, α_i, β_i, $i = 1, ..., n$ be real numbers such that $at + \alpha_i > 0$, $bt + \beta_i > 0$. Then, $at + \alpha_i \leq bt + \beta_i$ implies $\psi_k(at + \alpha_i) \leq \psi_k(bt + \beta_i)$.*

Proof. A direct consequence of Lemma 3.4.

Lemma 3.6. *Let a, b, α_i, β_i, $i = 1, ..., n$ be real numbers such that $0 < a \leq b$, $at+\alpha_i > 0$, $bt+\beta_i > 0$, $at+\alpha_i \leq bt+\beta_i$ and $\psi_k(at + \alpha_i) > 0$. Then, $a\psi_k(at + \alpha_i) \leq b\psi_k(bt + \beta_i)$.*

Proof. From Lemma 3.5, we have $\psi_k(at + \alpha_i) \leq \psi_k(bt + \beta_i)$. This together with the fact that $0 < a \leq b$ yields, $a\psi_k(at + \alpha_i) \leq a\psi_k(bt + \beta_i) \leq b\psi_k(bt + \beta_i)$ concluding the proof.

Theorem 3.7. *Define a function X for $k > 0$ by*

$$X(t) = \prod_{i=1}^{n}\frac{\Gamma_k(at+\alpha_i)}{\Gamma_k(bt+\beta_i)}, \quad t \in [0,\infty)$$

where a, b, α_i, β_i, $i = 1, ..., n$ are real numbers such that $0 < a \leq b$, $\alpha_i > 0$, $\beta_i > 0$, $at + \alpha_i > 0$, $bt + \beta_i > 0$, $at + \alpha_i \leq bt + \beta_i$ and $\psi_k(at + \alpha_i) > 0$. Then X is decreasing and for $t \in [0,1]$, the following inequalities hold true.

$$\prod_{i=1}^{n}\frac{\Gamma_k(a+\alpha_i)}{\Gamma_k(b+\beta_i)} \le \prod_{i=1}^{n}\frac{\Gamma_k(at+\alpha_i)}{\Gamma_k(bt+\beta_i)} \le \prod_{i=1}^{n}\frac{\Gamma_k(\alpha_i)}{\Gamma_k(\beta_i)}. \tag{12}$$

Proof. Let $u(t)=\ln X(t)$ for every $t\in[0,\infty)$. Then,

$$\begin{aligned} u(t) &= \ln\prod_{i=1}^{n}\frac{\Gamma_k(at+\alpha_i)}{\Gamma_k(bt+\beta_i)} \\ &= \sum_{i=1}^{n}\left[\ln\Gamma_k(at+\alpha_i)-\ln\Gamma_k(bt+\beta_i)\right] \end{aligned}$$

Then,

$$\begin{aligned} u'(t) &= \sum_{i=1}^{n}\left[a\frac{\Gamma_k'(at+\alpha_i)}{\Gamma_k(at+\alpha_i)} - b\frac{\Gamma_k'(bt+\beta_i)}{\Gamma_k(bt+\beta_i)}\right] \\ &= \sum_{i=1}^{n}\left[a\psi_k(at+\alpha_i)-b\psi_k(bt+\beta_i)\right] \le 0. \quad \text{(by Lemma 3.6)}. \end{aligned}$$

That implies u is decreasing on $t\in[0,\infty)$. Hence, $X=e^{u(t)}$ is decreasing for each $t\in[0,\infty)$. Then for $t\in[0,1]$ we have,

$X(1)\le X(t)\le X(0)$

resulting to inequalities (12).

Remark 3.8. For $t\in(1,\infty)$, we have $X(t)\le X(1)$ yielding

$$\prod_{i=1}^{n}\frac{\Gamma_k(at+\alpha_i)}{\Gamma_k(bt+\beta_i)} \le \prod_{i=1}^{n}\frac{\Gamma_k(a+\alpha_i)}{\Gamma_k(b+\beta_i)}.$$

Remark 3.9. If $0<b\le a$, $at+\alpha_i\ge bt+\beta_i$ and $\psi_k(bt+\beta_i)>0$, then for $t\in[0,1]$ the inequalities (12) are reversed.

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