Global Journal of Mathematical Analysis, 2 (4) (2014) 281-285
©Science Publishing Corporation
www.sciencepubco.com/index.php/GJMA
doi: 10.14419/gjma.v2i4.3400
Research Paper

An application of grand Furuta inequality to a type of operator equation

Jian Shi

College of Mathematics and Computer Science, Hebei University, Baoding 071002, Hebei, China
E-mail: mathematic@126.com

Abstract

The existence of positive semidefinite solutions of the operator equation $\sum_{j=1}^{n} A^{n-j}XA^{j-1} = Y$ is investigated by applying grand Furuta inequality. If there exists positive semidefinite solutions of the operator equation, one of the special types of Y is obtained, which extends the related result before. Finally, an example is given based on our result.

Keywords: *grand Furuta inequality, operator equation, matrix equation, positive semidefinite operator.*

1. Introduction

A capital letter (such as T) means a bounded linear operator on a Hilbert space. $T \geqslant 0$ and $T > 0$ mean a positive semidefinite operator and a positive definite operator, respectively.

In the middle of last century, E. Heinz et al. studied operator theory and obtained the following famous theorem:
Theorem 1.1 (Löwner-Heinz Inequality, [16] [13]). If $A \geqslant B \geqslant 0$, then $A^{\alpha} \geqslant B^{\alpha}$ holds for any $\alpha \in [0, 1]$.
It is essential to notice that Löwner-Heinz inequality does not always hold for $\alpha > 1$.

In 1987, T. Furuta proved the following result which is an important and historical extension of Löwner-Heinz inequality:
Theorem 1.2 (Furuta Inequality, [8]). If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}, \tag{1.1}$$

$$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \tag{1.2}$$

hold for $p \geqslant 0$, $q \geqslant 1$ with $(1 + r)q \geqslant p + r$.

Afterwards, the studies of the theory of operator inequalities have been developed quickly and some results related to Furuta inequality have been obtained in recent twenty-five years, such as [1, 2, 9, 17, 23, 24, 25]. It is well known that Furuta inequality has many applications. See [3, 5, 11, 14, 15, 20, 21, 22, 26].

In 1995, T. Furuta showed another operator inequality which interpolates Furuta inequality:

Theorem 1.3 (Grand Furuta Inequality, [9]). If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$,

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.3}$$

holds for $s \geqslant 1$ and $r \geqslant t$.

Consequently, some nice proofs of grand Furuta inequality were shown, such as [6] and [10]. K. Tanahashi, in [18], proved that the outer exponent value of (1.3) is the best possible. Later on, the proof was improved by T. Yamazaki and M. Fujii et al. in [19] and [7], respectively.

Recently, T. Furuta proved the following theorem by Furuta inequality:

Theorem 1.4 ([12]). Let m and n be nature numbers. If A and B are a positive definite operator and a positive semidefinite operator, respectively, then there exists positive semidefinite operator solution X satisfying the following operator equation:

$$\sum_{j=1}^{n} A^{n-j}XA^{j-1} = A^{\frac{nr}{2(m+r)}}(\sum_{i=1}^{m} A^{\frac{n(m-i)}{m+r}}BA^{\frac{n(i-1)}{m+r}})A^{\frac{nr}{2(m+r)}} \tag{1.4}$$

for r such that $\begin{cases} r \geqslant 0, & \text{if } n \geqslant m; \\ r \geqslant \frac{m-n}{n-1}, & \text{if } m \geqslant n \geqslant 2. \end{cases}$

Our purpose of the present article is to study the existence of positive semidefinite solution of operator equation $\sum_{j=1}^{n} A^{n-j}XA^{j-1} = Y$ by grand Furuta inequality, and show a more generalized special type of Y than Theorem 1.4. Although we use the same method as in [12], we think that careful argument is required, and a more generalized example, especially the expression of Y, is also required. Therefore, we have this article.

2. Positive semidefinite solutions of an operator equation

Let us recall a useful lemma first.

Lemma 2.1 ([4], [12]). Let A be a positive definite operator and B be a positive semidefinite operator. Let m be a positive integer and $x \geqslant 0$, then

$$\frac{d}{dx}[(A+xB)^m]\Big|_{x=0} = \sum_{j=1}^{m} A^{m-j}BA^{j-1}.$$

Now we give the main result as follows,

Theorem 2.1. Let m, n and k be positive integers. If A and B are a positive definite operator and a positive semidefinite operator, respectively, then for each $t \in [0,1]$, there exists positive semidefinite operator solution X which satisfies the following operator equation:

$$\begin{aligned}&\sum_{j=1}^{n} A^{n-j}XA^{j-1} \\ =&A^{\frac{nr}{2[(m-t)k+r]}}\Big(\sum_{i=1}^{k}\sum_{j=1}^{m} A^{\frac{n[2(m-t)(k-i)-t+2(m-j)]}{2[(m-t)k+r]}}BA^{\frac{n[2(j-1)-t+2(m-t)(i-1)]}{2[(m-t)k+r]}}\Big)A^{\frac{nr}{2[(m-t)k+r]}}\end{aligned} \tag{2.1}$$

for r such that $\begin{cases} r \geqslant t, & \text{if } (1-t)n \geqslant (m-t)k\ ; \\ r \geqslant max\{\frac{(m-t)k-(1-t)n}{n-1}, t\}, & \text{if } (m-t)k \geqslant (1-t)n \text{ with } n \geqslant 2\ . \end{cases}$

Proof. First, by $A + xB \geqslant A > 0$ holds for any $x \geqslant 0$, then $A^{-1} \geqslant (A+xB)^{-1} > 0$. Replacing A by A^{-1}, B by $(A+xB)^{-1}$, p by m, s by k in (1.3), and taking reverse, we have

$$(A^{\frac{r}{2}}(A^{-\frac{t}{2}}(A+xB)^mA^{-\frac{t}{2}})^kA^{\frac{r}{2}})^{\frac{1-t+r}{(m-t)k+r}} \geqslant A^{1-t+r}. \tag{2.2}$$

For any $\alpha \in [0,1]$, applying Löwner-Heinz inequality to (2.2), and taking an integer n such that $\frac{1}{n} = \frac{1-t+r}{(m-t)k+r} \cdot \alpha$, then the following inequality is obtained:

$$(A^{\frac{r}{2}}(A^{-\frac{t}{2}}(A+xB)^m A^{-\frac{t}{2}})^k A^{\frac{r}{2}})^{\frac{1}{n}} \geqslant A^{\frac{(m-t)k+r}{n}}. \tag{2.3}$$

By $\alpha \in [0,1]$ and the condition of r in grand Furuta inequality, we have to take $r \geqslant t$ if $(1-t)n \geqslant (m-t)k$, or $r \geqslant max\{\frac{(m-t)k-(1-t)n}{n-1}, t\}$ if $(m-t)k \geqslant (1-t)n$ with $n \geqslant 2$.

Put $Y(x) = (A^{\frac{r}{2}}(A^{-\frac{t}{2}}(A+xB)^m A^{-\frac{t}{2}})^k A^{\frac{r}{2}})^{\frac{1}{n}}$. According to (2.3), we have $Y(x) \geqslant Y(0) = A^{\frac{(m-t)k+r}{n}}$ for any $x \geqslant 0$. Thus, $Y'(0) \geqslant 0$. Differentiating $Y^n(x) = A^{\frac{r}{2}}(A^{-\frac{t}{2}}(A+xB)^m A^{-\frac{t}{2}})^k A^{\frac{r}{2}}$, using Lemma 2.1, and taking $x = 0$, the following equality holds.

$$\begin{aligned}
&\frac{d}{dx}[Y^n(x)]\Big|_{x=0} = \sum_{j=1}^{n} Y(0)^{n-j} Y'(0) Y(0)^{j-1} \\
=\ & \frac{d}{dx}[A^{\frac{r}{2}}(A^{-\frac{t}{2}}(A+xB)^m A^{-\frac{t}{2}})^k A^{\frac{r}{2}}]\Big|_{x=0} \\
=\ & A^{\frac{r}{2}}\{\sum_{i=1}^{k}[(A^{-\frac{t}{2}}(A+xB)^m A^{-\frac{t}{2}})^{k-i}\Big|_{x=0}] \cdot [(A^{-\frac{t}{2}}(A+xB)^m A^{-\frac{t}{2}})'\Big|_{x=0}] \\
& \cdot[(A^{-\frac{t}{2}}(A+xB)^m A^{-\frac{t}{2}})^{i-1}\Big|_{x=0}]\}A^{\frac{r}{2}} \\
=\ & A^{\frac{r}{2}}\{\sum_{i=1}^{k}[A^{(m-t)(k-i)}(A^{-\frac{t}{2}}(\sum_{j=1}^{m} A^{m-j}BA^{j-1})A^{-\frac{t}{2}})A^{(m-t)(i-1)}]\}A^{\frac{r}{2}} \\
=\ & A^{\frac{r}{2}}\Big(\sum_{i=1}^{k}\sum_{j=1}^{m} A^{(m-t)(k-i)-\frac{t}{2}+(m-j)}BA^{(j-1)-\frac{t}{2}+(m-t)(i-1)}\Big)A^{\frac{r}{2}}.
\end{aligned}$$

Replacing $Y(0)$ by $A^{\frac{(m-t)k+r}{n}}$, $Y'(0)$ by X, we have

$$\begin{aligned}
&\sum_{j=1}^{n} A^{\frac{(m-t)k+r}{n}(n-j)} X A^{\frac{(m-t)k+r}{n}(j-1)} \\
&= A^{\frac{r}{2}}\Big(\sum_{i=1}^{k}\sum_{j=1}^{m} A^{(m-t)(k-i)-\frac{t}{2}+(m-j)}BA^{(j-1)-\frac{t}{2}+(m-t)(i-1)}\Big)A^{\frac{r}{2}}.
\end{aligned} \tag{2.4}$$

Replacing A by $A^{\frac{n}{(m-t)k+r}}$ in (2.4), (2.1) is obtained. $\square$

Remark 2.1. If we take $t = 0$ and $k = 1$ in Theorem 2.1, the theorem is just Theorem 1.4, which is the main result of [12].

Remark 2.2. According to the related result before, if A and Y are positive semidefinite matrices in matrix equation $\sum_{j=1}^{n} A^{n-j}XA^{j-1} = Y$, then X is also a positive semidefinite matrix, see [4]. However, by Theorem 2.1, in some special cases, if Y can be expressed as the right hand of (2.1) without being a positive semidefinite matrix, there still exists positive semidefinite solution satisfying the matrix equation $\sum_{j=1}^{n} A^{n-j}XA^{j-1} = Y$.

For example, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2\times 2^{\frac{1}{3}} \end{pmatrix},\ Y = \begin{pmatrix} 4 & 3\times 2^{\frac{1}{4}} + 6\times 2^{\frac{3}{4}} \\ 3\times 2^{\frac{1}{4}} + 6\times 2^{\frac{3}{4}} & 32 \end{pmatrix}.$$

Although Y is not a positive semidefinite matrix (because its eigenvalues are $\{37.5589\ldots, -1.5589\ldots\}$), by simple calculation, the solution of the following matrix equation

$$A^2X + AXA + XA^2 = Y$$

is

$$X = \begin{pmatrix} \frac{4}{3} & \frac{3\times 2^{\frac{1}{4}}+6\times 2^{\frac{3}{4}}}{1+2\times 2^{\frac{1}{3}}+4\times 2^{\frac{2}{3}}} \\ \frac{3\times 2^{\frac{1}{4}}+6\times 2^{\frac{3}{4}}}{1+2\times 2^{\frac{1}{3}}+4\times 2^{\frac{2}{3}}} & \frac{4\times 2^{\frac{1}{3}}}{3} \end{pmatrix},$$

which is still a definite matrix whose eigenvalues are $\{2.9013\ldots, 0.1119\ldots\}$. The critical reason is that Y can be expressed as follows,

$$Y = A^{\frac{3}{8}}\Big(\sum_{i=1}^{2}\sum_{j=1}^{2} A^{\frac{3[3(2-i)-\frac{1}{2}+2(2-j)]}{8}} B A^{\frac{3[2(j-1)-\frac{1}{2}+3(i-1)]}{8}}\Big)A^{\frac{3}{8}},$$

which is the right hand of (2.1) under the condition of $m=2$, $n=3$, $k=2$, $t=\frac{1}{2}$, $r=1$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

References

[1] T. Ando, *On some operator inequalities*, Math. Ann. **279** (1987), 157-159.

[2] T. Ando and F. Hiai, *Log majorization and complementary Golden-Thompson type inequalities*, Linera Algebra Appl. **197** (1994), 113-131.

[3] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integr. Equat. Oper. Th. **13** (1990), 307-315.

[4] R. Bhatia and M. Uchiyama, *The operator equation* $\sum_{i=0}^{n} A^{n-i}XB^{i} = Y$, Expo. Math. **27** (2009), 251-255.

[5] M. Fujii, T. Furuta and E. Kamei, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl. **179** (1993), 161-169.

[6] M. Fujii and E. Kamei, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc. **124** (1996), 2751-2756.

[7] M. Fujii, A. Matsumoto and R. Nakamoto, *A short proof of the best possibility for the grand Furuta inequality*, J. Inequal. Appl. **4** (1999), 339-344.

[8] T. Furuta, $A \geqslant B \geqslant 0$ *assures* $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ *for* $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ *with* $(1+2r)q \geqslant p+2r$, Proc. Amer. Math. Soc. **101** (1987), 85-88.

[9] T. Furuta, *An extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl. **219** (1995), 139-155.

[10] T. Furuta, *Simplified proof of an order preserving operator inequality*, Proc. Japan Acad. **74** (1998), 114.

[11] T. Furuta and M. Yanagida, *On powers of p-hyponormal and log-hyponormal operators*, J. Inequal. Appl. **5** (2000), 367-380.

[12] T. Furuta, *Positive semidefinite solutions of the operator equation* $\sum_{j=1}^{n} A^{n-j}XA^{j-1} = B$, Linear Algebra Appl. **432** (2010), 949-955.

[13] E. Heinz, *Beiträge zur Störungsteorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415-438.

[14] M. Ito and T. Yamazaki, *Relations between two inequalities* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *and* $(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{p}{p+r}} \geq A^r$ *and their applications*, Integr. Equat. Oper. Th. **44** (2002), 442-450.

[15] C.-S. Lin, *On operator order and chaotic operator order for two operators*, Linear Algebra Appl. **425** (2007), 1-6.

[16] K. Löwner, *Über monotone MatrixFunktionen*, Math. Z. **38** (1934), 177-216.

[17] K. Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc. **124** (1996), 141-146.

[18] K. Tanahashi, *The best possibility of the grand Furuta inequality*, Proc. Amer. Math. Soc. **128** (2000), 511-519.

[19] T. Yamazaki, *Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality*, Math. Inequal. Appl. **2** (1999), 473-477.

[20] C. Yang and H. Dai, *An application of Furuta inequality and its best possibility*, Applied Mathematics, A Journal of Chinese Universities Series B. **23** (3)(2008), 326-330.

[21] C. Yang and J. Yuan, *Extensions of the results on powers of* p*-hyponormal and log-hyponormal operators*, J. Inequal. Appl. **1** (2006), 1-14.

[22] J. Yuan and Z. Gao, *Structure on powers of* p*-hyponormal and log-hyponormal operators*, Integr. Equat. Oper. Th. **59** (2007), 437-448.

[23] J. Yuan and Z. Gao, *Classified construction of generalized Furuta type operator functions*, Math. Inequal. Appl. **11** (2008), 189-202.

[24] J. Yuan and Z. Gao, *Complete form of Furuta inequality*, Proc. Amer. Math. Soc. **136** (2008), 2859-2867.

[25] J. Yuan, *Classified construction of generalized Furuta type operator functions*, II, Math. Inequal. Appl. **13** (2010), 775-784.

[26] J. Yuan, *Furuta inequality and* q*-hyponormal operators*, Oper. Matrices **4** (2010), 405-415.