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Research Paper

The (q, k)-analogues of some inequalities involving the Psi function

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Abstract

In this paper, the (q, k)-analogues of some inequalities involving the Psi function are presented. These results generalize some earlier results presented by W. T. Sulaiman. The approach is based on some mononicity properties of some functions involving the $\psi_{q,k}$ function.

Keywords: *Psi function, (q, k)-analogue, Inequality.*
MSC: 33B15, 26A48.

1. Introduction and preliminaries

The Psi function $\psi(t)$ otherwise known as the digamma function is defined as

$$\psi(t) = \frac{d}{dt}\ln\Gamma(t) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0$$

where $\Gamma(t)$ is the well-known classical Euler's Gamma function defined by

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}\,dx, \qquad t > 0.$$

Equivalently, the (q, k)-analogue of the Psi function, $\psi_{q,k}(t)$ is defined as

$$\psi_{q,k}(t) = \frac{d}{dt}\ln\Gamma_{q,k}(t) = \frac{\Gamma'_{q,k}(t)}{\Gamma_{q,k}(t)}, \quad t > 0, \quad k > 0, \quad q \in (0, 1).$$

where $\Gamma_{q,k}(t)$ is the (q, k)-analogue of the gamma function given by (see [1],[2])

$$\Gamma_{q,k}(t) = \frac{(1-q^k)_{q,k}^{\frac{t}{k}-1}}{(1-q)^{\frac{t}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^t)_{q,k}^{\infty}(1-q)^{\frac{t}{k}-1}}, \quad t > 0.$$

The functions $\psi(t)$ and $\psi_{q,k}(t)$ as defined above exhibit the following series representations.

$$\psi(t) = -\gamma + (t-1)\sum_{n=0}^{\infty}\frac{1}{(1+n)(n+t)}, \quad t>0$$
$$\psi_{q,k}(t) = \frac{-\ln(1-q)}{k} + (\ln q)\sum_{n=1}^{\infty}\frac{q^{nkt}}{1-q^{nk}}, \qquad t>0.$$

where γ is the Euler-Mascheroni's constant.

By taking the m-th derivative of these functions, it is easy to demonstrate that the following statements are true for each $m \in N$.

$$\psi^{(m)}(t) = (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+t)^{m+1}}, \quad t>0$$
$$\psi^{(m)}_{q,k}(t) = (\ln q)^{m+1}\sum_{n=1}^{\infty}\frac{n^m k^m q^{nkt}}{1-q^{nk}}, \quad k>0, \quad q\in(0,1), \quad t>0.$$

In 2011, Sulaiman [7], established the following results for the Psi function.

$$\psi(s+t) \geq \psi(s) + \psi(t) \tag{1}$$

for $t>0$ and $0<s<1$.

$$\psi^{(m)}(s+t) \leq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{2}$$

for $s,t>0$ and for a positive odd integer m.

$$\psi^{(m)}(s+t) \geq \psi^{(m)}(s) + \psi^{(m)}(t) \tag{3}$$

for $s,t>0$ and for a positive even integer m.

$$\psi^{(m)}(s)\psi^{(m)}(t) \geq \left[\psi^{(m)}(s+t)\right]^2 \tag{4}$$

for $s,t>0$ and for a positive odd integer m.

Recently, the p-analogues, q-analogues, k-analogues and (q,k)-analogues of these inequalities have been respectively established in the papers [3], [4],[5] and [6].

The purpose of this paper is to establish that the inequalities (1), (2), (3) and (4) still hold true for the (q,k)-analogue of the Psi function by using the same techniques as in [3], [4],[5] and [6].

2. Main results

Theorem 2.1. *Let $t>0$, $0<s\leq 1$, $q\in(0,1)$ and $k>0$. Then the following inequality is valid.*

$$\psi_{q,k}(s+t) \geq \psi_{q,k}(s) + \psi_{q,k}(t). \tag{5}$$

Proof. Let $\mu(t) = \psi_{q,k}(s+t) - \psi_{q,k}(s) - \psi_{q,k}(t)$. Then fixing s we have,

$$\mu'(t) = \psi'_{q,k}(s+t) - \psi'_{q,k}(t) = (\ln q)^2\sum_{n=1}^{\infty}\left[\frac{nkq^{nk(s+t)}}{1-q^{nk}} - \frac{nkq^{nkt}}{1-q^{nk}}\right]$$
$$= (\ln q)^2\sum_{n=1}^{\infty}\frac{nkq^{nkt}(q^{nks}-1)}{1-q^{nk}} \leq 0.$$

That implies μ is non-increasing. In addition,

$$\begin{aligned}
\lim_{t\to\infty}\mu(t) &= \lim_{t\to\infty}\left[\psi_{q,k}(s+t)-\psi_{q,k}(s)-\psi_{q,k}(t)\right]\\
&= \frac{\ln(1-q)}{k} + (\ln q)\lim_{t\to\infty}\sum_{n=1}^{\infty}\left[\frac{q^{nk(s+t)}}{1-q^{nk}} - \frac{q^{nks}}{1-q^{nk}} - \frac{q^{nkt}}{1-q^{nk}}\right]\\
&= \frac{\ln(1-q)}{k} + (\ln q)\lim_{t\to\infty}\sum_{n=1}^{\infty}\left[\frac{q^{nks}.q^{nkt}-q^{nks}-q^{nkt}}{1-q^{nk}}\right]\\
&= \frac{\ln(1-q)}{k} - (\ln q)\sum_{n=1}^{\infty}\frac{q^{nks}}{1-q^{nk}} \geq 0.
\end{aligned}$$

Therefore $\mu(t)\geq 0$ concluding the proof.

Theorem 2.2. *Let $s,t>0$, $q\in(0,1)$ and $k>0$. Suppose that m is a positive odd integer, then the following inequality is valid.*

$$\psi_{q,k}^{(m)}(s+t)\leq \psi_{q,k}^{(m)}(s)+\psi_{q,k}^{(m)}(t). \tag{6}$$

Proof. Let $\eta(t)=\psi_{q,k}^{(m)}(s+t)-\psi_{q,k}^{(m)}(s)-\psi_{q,k}^{(m)}(t)$. Then fixing s we have,

$$\begin{aligned}
\eta'(t) &= \psi_{q,k}^{(m+1)}(s+t)-\psi_{q,k}^{(m+1)}(t)\\
&= (\ln q)^{m+2}\sum_{n=1}^{\infty}\left[\frac{n^{m+1}k^{m+1}q^{nk(s+t)}}{1-q^{nk}} - \frac{n^{m+1}k^{m+1}q^{nkt}}{1-q^{nk}}\right]\\
&= (\ln q)^{m+2}\sum_{n=1}^{\infty}\left[\frac{n^{m+1}k^{m+1}q^{nkt}(q^{nks}-1)}{1-q^{nk}}\right] \geq 0. \text{ (since } m \text{ is odd)}
\end{aligned}$$

That implies η is non-decreasing. In addition,

$$\begin{aligned}
\lim_{t\to\infty}\eta(t) &= (\ln q)^{m+1}\lim_{t\to\infty}\sum_{n=1}^{\infty}\left[\frac{n^mk^mq^{nk(s+t)}}{1-q^{nk}} - \frac{n^mk^mq^{nks}}{1-q^{nk}} - \frac{n^mk^mq^{nkt}}{1-q^{nk}}\right]\\
&= (\ln q)^{m+1}\lim_{t\to\infty}\sum_{n=1}^{\infty}\left[\frac{n^mk^mq^{nks}.q^{nkt}}{1-q^{nk}} - \frac{n^mk^mq^{nks}}{1-q^{nk}} - \frac{n^mk^mq^{nkt}}{1-q^{nk}}\right]\\
&= -(\ln q)^{m+1}\sum_{n=1}^{\infty}\frac{n^mk^mq^{nks}}{1-q^{nk}} \leq 0. \text{ (since } m \text{ is odd)}
\end{aligned}$$

Therefore $\eta(t)\leq 0$ concluding the proof.

Theorem 2.3. *Let $s,t>0$, $q\in(0,1)$ and $k>0$. Suppose that m is a positive even integer, then the following inequality is valid.*

$$\psi_{q,k}^{(m)}(s+t)\geq \psi_{q,k}^{(m)}(s)+\psi_{q,k}^{(m)}(t). \tag{7}$$

Proof. Let $\lambda(t)=\psi_{q,k}^{(m)}(s+t)-\psi_{q,k}^{(m)}(s)-\psi_{q,k}^{(m)}(t)$. Then fixing s we have,

$$\begin{aligned}
\lambda'(t) &= \psi_{q,k}^{(m+1)}(s+t)-\psi_{q,k}^{(m+1)}(t)\\
&= (\ln q)^{m+2}\sum_{n=1}^{\infty}\left[\frac{n^{m+1}k^{m+1}q^{nk(s+t)}}{1-q^{nk}} - \frac{n^{m+1}k^{m+1}q^{nkt}}{1-q^{nk}}\right]\\
&= (\ln q)^{m+2}\sum_{n=1}^{\infty}\left[\frac{n^{m+1}k^{m+1}q^{nkt}(q^{nks}-1)}{1-q^{nk}}\right] \leq 0. \text{ (since } m \text{ is even)}
\end{aligned}$$

That implies λ is non-increasing. Further,

$$\lim_{t\to\infty}\lambda(t) = (\ln q)^{m+1}\lim_{t\to\infty}\sum_{n=1}^{\infty}\left[\frac{n^m k^m q^{nk(s+t)}}{1-q^{nk}} - \frac{n^m k^m q^{nks}}{1-q^{nk}} - \frac{n^m k^m q^{nkt}}{1-q^{nk}}\right]$$
$$= -(\ln q)^{m+1}\sum_{n=1}^{\infty}\frac{n^m k^m q^{nks}}{1-q^{nk}} \leq 0. \text{ (since } m \text{ is even)}$$

Therefore $\lambda(t) \geq 0$ concluding the proof.

Theorem 2.4. *Let $s, t > 0$, $q \in (0,1)$ and $k > 0$. Suppose m is a positive odd integer, then the following inequality holds true.*

$$\psi_{q,k}^{(m)}(s)\psi_{q,k}^{(m)}(t) \geq \left[\psi_{q,k}^{(m)}(s+t)\right]^2. \tag{8}$$

Proof. We proceed as follows.

$$\psi_{q,k}^{(m)}(s) - \psi_{q,k}^{(m)}(s+t) = (\ln q)^{m+1}\sum_{n=1}^{\infty}\left[\frac{n^m k^m q^{nks}}{1-q^{nk}} - \frac{n^m k^m q^{nk(s+t)}}{1-q^{nk}}\right]$$
$$= (\ln q)^{m+1}\sum_{n=1}^{\infty}\left[\frac{n^m k^m q^{nks} - n^m k^m q^{nks}.q^{nkt}}{1-q^{nk}}\right]$$
$$= (\ln q)^{m+1}\sum_{n=1}^{\infty}\left[\frac{n^m k^m q^{nks}(1-q^{nkt})}{1-q^{nk}}\right] \geq 0. \text{ (since } m \text{ is odd)}$$

Hence,

$$\psi_{q,k}^{(m)}(s) \geq \psi_{q,k}^{(m)}(s+t) \geq 0.$$

Similarly we have,

$$\psi_{q,k}^{(m)}(t) \geq \psi_{q,k}^{(m)}(s+t) \geq 0.$$

Multiplying these inequalities yields,

$$\psi_{q,k}^{(m)}(s)\psi_{q,k}^{(m)}(t) \geq \left[\psi_{q,k}^{(m)}(s+t)\right]^2.$$

3. Concluding remarks

Remark 3.1. If in inequalities (5), (6), (7) and (8) we allow $q \to 1$ and $k \to 1$, then we repectively obtain the inequalities (1), (2), (3) and (4) as was presented by W. T. Sulaiman.

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