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Research Paper

$\mathcal{I}$-statistically pre-Cauchy double sequences

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Abstract

In the present paper we are concerned with $\mathcal{I}$-statistically pre-Cauchy double sequences in line of of Das et al. [5]. Particularly, we prove that for double sequences, $\mathcal{I}$-statistical convergence implies $\mathcal{I}$-statistical pre-Cauchy condition and examine some main properties of these concepts.

Keywords: *Double sequences, Ideal, Filter, $\mathcal{I}$-statistical convergence, $\mathcal{I}$-statistical pre-Cauchy.*

1. Introduction and preliminaries

The notion of statistical convergence of sequences of real numbers was introduced by Fast in [6] and Steinhaus in [25] and is based on the notion of asymptotic density of a set $A \subset \mathbb{N}$. However, the first idea of statistical convergence appeared (under the name almost convergence) in the first edition (Warsaw, 1935) of the celebrated monograph [28] of Zygmund. It should be also mentioned that the notion of statistical convergence has been considered, in other contexts [8, 11, 14, 18, 22]. Statistical convergence has several applications in different fields of mathematics: summability theory [4, 23], trigonometric series [28], measure theory [17], and approximation theory [14]. Particularly, in [2], Connor et al. introduced the concept of statistically pre-Cauchy sequences and proved that statistically convergent sequences are always statistically pre-Cauchy and on the other hand under certain general conditions statistical pre-Cauchy condition implies statistical convergence of a sequence. Gürdal [10] presented statistically pre-Cauchy sequences and bounded moduli.

The idea of $\mathcal{I}$-convergence for sequences, was inspired by the concept of statistical convergence introduced in [16], see Kostyrko, Salát, and Wilczyński [16] for a comprehensive bibliography. All the results of [16] apply to sequences of functions with domains being singletons. Recently, in [4, 21], Savaş et al. studied the $\mathcal{I}$-statistically convergence of sequences and obtained some results of this concept. For more informations about $\mathcal{I}$-convergence and $\mathcal{I}$-statistically convergence, see [6, 5, 12, 15, 19, 20, 24, 27].

The notion of $\mathcal{I}$-statistically pre-Cauchy of double sequences has not been studied previously. Motivated by this fact, in this paper, we are concerned with $\mathcal{I}$-statistically pre-Cauchy double sequences, and some important results are established.

Now we recall some definitions and notations that will be used in paper.

The notion of a statistically convergent sequence can be defined using the asymptotic density of subsets of the set of positive integers $\mathbb{N} = \{1, 2, ...\}$. For any $K \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ we denote $K(n) := cardK \cap \{1, 2, ..., n\}$ and we define lower and upper asymptotic density of the set K by the formulas

$$\underline{\delta}(K) := \liminf_{n\to\infty} \frac{K(n)}{n};\ \overline{\delta}(K) := \limsup_{n\to\infty} \frac{K(n)}{n}.$$

If $\underline{\delta}(K) = \overline{\delta}(K) =: \delta(K)$, then the common value $\delta(K)$ is called the asymptotic density of the set K and

$$\delta(K) = \lim_{n\to\infty} \frac{K(n)}{n}.$$

Obviously all three densities $\underline{\delta}(K)$, $\overline{\delta}(K)$ and $\delta(K)$ (if they exist) lie in the unit interval $[0,1]$.

$$\delta(K) = \lim_{n} \frac{1}{n} |K_n| = \lim_{n} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k),$$

if it exists, where χ_K is the characteristic function of the set K [7]. We say that a number sequence $x = (x_k)_{k\in\mathbb{N}}$ statistically converges to a point L if for each $\varepsilon > 0$ we have $\delta(K(\varepsilon)) = 0$, where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ and in such situation we will write $L = st\text{-}\lim x_k$.

Statistical convergence for double sequences $x = (x_{jk})$ of real numbers was introduced and studied by Mursaleen and Edely [18].

A real double sequence $x = (x_{jk})$ is said to be statistically convergent to the number ℓ if for each $\varepsilon > 0$ the set

$$\{(j,k),\ j \leq m \text{ and } k \leq n : |x_{jk} - \ell| \geq \varepsilon\}$$

has double natural density zero. In this case we write $st_2\text{-}\lim_{j,k} x_{jk} = \ell$ and denote the set of all statistically convergent double sequences.

If Y is a non-empty set, then a family of subsets of Y is called an ideal in Y iff (**i**) $\emptyset \in \mathcal{I}$; (**ii**) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$; (**iii**) for each $A \in \mathcal{I}$ and $B \subset A$ we have imply $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $Y \in \mathcal{I} \neq \emptyset$ and $\mathcal{P}(Y)$ is the power set of Y.

Let Y is a non-empty set. A non-empty family of sets $F \subset \mathcal{P}(Y)$ is called a filter on Y iff (**i**) $\emptyset \notin F$; (**ii**) $A, B \in F$ implies $A \cap B \in F$; (**iii**) for each $A \in F$ and $A \subset B$ we have $B \in F$.

A nontrivial ideal $\mathcal{I}$ in Y is called an admissible ideal if it is different from $\mathcal{P}(\mathbb{N})$ and contains all singletons, i.e., $\{x\} \in \mathcal{I}$ for each $x \in Y$.

Let $\mathcal{I} \subset \mathcal{P}(Y)$ be a nontrivial ideal. Then a class

$$F(\mathcal{I}) = \{M \subset \mathbb{N} : M = Y \backslash A, \text{ for some } A \in \mathcal{I}\}$$

is a filter on Y, called the filter associated with the ideal $\mathcal{I}$.

An admissible ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is said to satisfy the condition (AP) if for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{I}$ there are sets $B_n \subset \mathbb{N}$, $n \in \mathbb{N}$ such that the symmetric difference $A_n \Delta B_n$ is a finite set for every n and $\cup_{n\in\mathbb{N}} B_n \in \mathcal{I}$.

Definition 1.1 *([15, 16]) Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be an admissible ideal in $\mathbb{N}$. The sequence $\{x_k\}_{k\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $\ell \in \mathbb{R}$ if, for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in \mathcal{I}$. In this case we write $\mathcal{I}\text{-}\lim x = \ell$.*

Throughout, $\mathcal{I}$ will stand for an admissible ideal of $\mathbb{N}$, and by a sequence we always mean a sequence of real numbers.

This concept was extended to $\mathcal{I}$-convergence of double sequences by B.C. Tripathy in [26]. In order to distinguish between the ideals of $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ we shall denote the ideals of $\mathcal{P}(\mathbb{N})$ by $\mathcal{I}$ and that of $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ by $\mathcal{I}_2$, respectively. In general, there is no connection between $\mathcal{I}$ and $\mathcal{I}_2$.

Recall that $\mathcal{I}$-statistical convergence using the notion of ideals of $\mathbb{N}$ is defined by following:

Definition 1.2 *A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is said to be $\mathcal{I}$-statistically convergent to L or $S(\mathcal{I})$-convergent to L if, for each $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|x_k - L\| \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}$$

or equivalently if for each $\varepsilon > 0$

$$\delta_{\mathcal{I}}(A(\varepsilon)) = \mathcal{I}\text{-}\lim \delta_n(A(\varepsilon)) = 0,$$

where $A(\varepsilon) = \{k \leq n : \|x_k - L\| \geq \varepsilon\}$ and $\delta_n(A(\varepsilon)) = \frac{|A(\varepsilon)|}{n}$.

In this case we write $x_k \to L(S(\mathcal{I}))$. The class of all $\mathcal{I}$-statistically convergent sequences will be denoted simply by $S(\mathcal{I})$. Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}$-statistically convergent is the statistical convergence [4, 21].

Definition 1.3 *([26]) Let $\mathcal{I}_2$ be an ideal of $\mathcal{P}(\mathbb{N}\times\mathbb{N})$. Then a double sequence (x_{jk}) is said to be $\mathcal{I}$-convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,*

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : |x_{jk} - L| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we write $\mathcal{I}_2$-$\lim x_{jk} = L$.

2. Main results

In this section, we are concerned with ideal statistical pre-Cauchy and ideal statistical convergence for double sequences.

Definition 2.1 ([5]) *A sequence $(x_k)_{k\in\mathbb{N}}$ is said to $\mathcal{I}$-statistically pre-Cauchy if, for any $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n^2}\left|\{(j,k) : |x_k - x_j| \geq \varepsilon,\ j,k \leq n\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

We now introduce the main definition of this paper.

Definition 2.2 *We say that a double sequence $x = (x_{jk})$ is $\mathcal{I}$-statistically pre-Cauchy if, for any $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2}\left|\{(j,k) : |x_{jk} - x_{pq}| \geq \varepsilon,\ j \leq m, k \leq n\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Definition 2.3 ([1]) *We say that a double sequence $x = (x_{jk})$ is said to be $\mathcal{I}$-statistically convergent to L if, for any $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\{(j,k) : |x_{jk} - L| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

The following is our main result in this section.

Theorem 2.4 *An $\mathcal{I}_2$-statistically convergent double sequence is $\mathcal{I}_2$-statistically pre-Cauchy.*

Proof. For any $\varepsilon > 0$ and $\delta > 0$,

$$A = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\left\{j \leq m, k \leq n : |x_{jk} - L| \geq \frac{\varepsilon}{2}\right\}\right| \geq \delta\right\} \in \mathcal{I}_2$$

Then

$$\frac{1}{mn}\left|\left\{j \leq m, k \leq n : |x_{jk} - L| \geq \frac{\varepsilon}{2}\right\}\right| < \delta,$$

that is,

$$\frac{1}{mn}\left|\left\{j \leq m, k \leq n : |x_{jk} - L| < \frac{\varepsilon}{2}\right\}\right| > 1 - \delta$$

for all $(m,n) \in A^c$, where c stands for the complement. Writing

$$B_{mn} = \left\{j \leq m, k \leq n : |x_{jk} - L| < \frac{\varepsilon}{2}\right\},$$

we observe that for $j,k,p,q \in B_{mn}$

$$|x_{jk} - x_{pq}| \leq |x_{jk} - L| + |x_{pq} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore

$$B_{mn} \times B_{mn} \subset \{(j,k) : |x_{jk} - L| < \varepsilon,\ j \leq m, k \leq n\}$$

which implies

$$\left[\frac{|B_{mn}|}{mn}\right]^2 \leq \frac{1}{m^2n^2} \left|\{(j,k) : |x_{jk} - x_{pq}| < \varepsilon,\ j \leq m, k \leq n\}\right|.$$

Hence

$$\frac{1}{m^2n^2} \left|\{(j,k) : |x_{jk} - x_{pq}| < \varepsilon,\ j \leq m, k \leq n\}\right| \geq \left[\frac{|B_{mn}|}{mn}\right]^2 > (1-\delta)^2,$$

that is

$$\frac{1}{m^2n^2} \left|\{(j,k) : |x_{jk} - x_{pq}| \geq \varepsilon,\ j \leq m, k \leq n\}\right| < 1 - (1-\delta)^2$$

for all $(m,n) \in A^c$. Let $\delta_1 > 0$ be given. Choosing $\delta > 0$ so that $1 - (1-\delta)^2 < \delta_1$, we see that every $(m,n) \in A^c$

$$\frac{1}{m^2n^2} \left|\{(j,k) : |x_{jk} - x_{pq}| \geq \varepsilon,\ j \leq m, k \leq n\}\right| < \delta_1$$

and so

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m^2n^2} \left|\{(j,k) : |x_{jk} - x_{pq}| \geq \varepsilon,\ j \leq m, k \leq n\}\right| \geq \delta_1\right\} \subset A.$$

Since $A \in \mathcal{I}_2$, we obtain

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m^2n^2} \left|\{(j,k) : |x_{jk} - x_{pq}| \geq \varepsilon,\ j \leq m, k \leq n\}\right| \geq \delta_1\right\} \in \mathcal{I}_2$$

and this completes the proof the theorem.

The next result gives a necessary and sufficient condition for a double sequence to be $\mathcal{I}_2$-statistically pre-Cauchy.

Theorem 2.5 *Let $x = (x_{jk})$ be a bounded double sequence. A double sequence $x = (x_{jk})$ is $\mathcal{I}_2$-statistically pre-Cauchy if and only if*

$$\mathcal{I}_2\text{-}\lim_{m,n} \frac{1}{m^2n^2} \sum_{j,p \leq m} \sum_{k,q \leq n} |x_{jk} - x_{pq}| = 0.$$

Proof. First assume that

$$\mathcal{I}_2\text{-}\lim_{m,n} \frac{1}{m^2n^2} \sum_{j,p \leq m} \sum_{k,q \leq n} |x_{jk} - x_{pq}| = 0.$$

Recall that for any $\varepsilon > 0$ and $(m,n) \in \mathbb{N} \times \mathbb{N}$ we have that

$$\frac{1}{m^2n^2} \sum_{j,p \leq m} \sum_{k,q \leq n} |x_{jk} - x_{pq}| \geq \varepsilon. \left(\frac{1}{m^2n^2} \left|\{(j,k) : |x_{jk} - x_{pq}| \geq \varepsilon\},\ j \leq m, k \leq n\right|\right).$$

Therefore for any $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m^2n^2} \left|\{(j,k) : |x_{jk} - x_{pq}| \geq \varepsilon,\ j \leq m, k \leq n\}\right| \geq \delta\right\}$$

$$\subset \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{m^2n^2} \sum_{j,p \leq m} \sum_{k,q \leq n} |x_{jk} - x_{pq}| \geq \delta\varepsilon\right\}.$$

Since $\mathcal{I}_2$-$\lim_{m,n} \frac{1}{m^2n^2} \sum\limits_{j,p\leq m} \sum\limits_{k,q\leq n} |x_{jk} - x_{pq}| = 0$, so the set on the right hand side belongs to $\mathcal{I}_2$ which implies that

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2} |\{(j,k) : |x_{jk} - x_{pq}| \geq \varepsilon,\ j \leq m, k \leq n\}| \geq \delta\right\} \in \mathcal{I}_2.$$

This shows that x is $\mathcal{I}_2$-statistically pre-Cauchy.

Conversely assume that x is $\mathcal{I}_2$-statistically pre-Cauchy and that $\varepsilon > 0$ has been given. Since x is bounded there exists an integer B such that $|x_{jk}| \leq B\ \forall j,k \in \mathbb{N}$. For each $m,n \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{m^2n^2} \sum_{j,p\leq m} \sum_{k,q\leq n} |x_{jk} - x_{pq}| &= \frac{1}{m^2n^2} \sum_{|x_{jk}-x_{pq}|<\frac{\varepsilon}{2}} |x_{jk} - x_{pq}| \\ &+ \frac{1}{m^2n^2} \sum_{|x_{jk}-x_{pq}|\geq\frac{\varepsilon}{2}} |x_{jk} - x_{pq}| \\ &\leq \frac{\varepsilon}{2} + 2B\left(\frac{1}{m^2n^2} |\{(j,k) : |x_{jk} - x_{pq}| \geq \varepsilon,\ j \leq m, k \leq n\}|\right).\end{aligned}$$

Since x is $\mathcal{I}_2$-statistically pre-Cauchy, for $\delta > 0$

$$A = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2} \left|\left\{(j,k) : |x_{jk} - x_{pq}| \geq \frac{\varepsilon}{2},\ j \leq m, k \leq n\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Then for $(m,n) \in A^c$

$$\frac{1}{m^2n^2} \left|\left\{(j,k) : |x_{jk} - x_{pq}| \geq \frac{\varepsilon}{2},\ j \leq m, k \leq n\right\}\right| < \delta$$

and so

$$\frac{1}{m^2n^2} \sum_{j,p\leq m} \sum_{k,q\leq n} |x_{jk} - x_{pq}| \leq \frac{\varepsilon}{2} + 2B\delta.$$

Let $\delta_1 > 0$ be given. Then choosing $\varepsilon, \delta > 0$ so that $\frac{\varepsilon}{2} + 2B\delta < \delta_1$ we see that every $(m,n) \in A^c$

$$\frac{1}{m^2n^2} \sum_{j,p\leq m} \sum_{k,q\leq n} |x_{jk} - x_{pq}| < \delta_1,$$

that is

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2} \sum_{j,p\leq m} \sum_{k,q\leq n} |x_{jk} - x_{pq}| \geq \delta_1\right\} \subset A \in \mathcal{I}_2.$$

This proves the necessity of the condition.

Now we give a sufficient condition under which an $\mathcal{I}_2$-statistically pre-Cauchy double sequence can be $\mathcal{I}_2$-statistically convergent.

Prior to proving the next result, we recall the following definition of $\mathcal{I}_2$-limit inferior [13].

Definition 2.6 *Let $\mathcal{I}_2$ be an admissible ideal of $\mathbb{N}\times\mathbb{N}$ and $x = (x_{jk})$ be a real double sequence. Let*

$$A_x = \{\alpha \in \mathbb{R} : \{(j,k) : x_{jk} < \alpha\} \notin \mathcal{I}_2\}.$$

Then the $\mathcal{I}_2$-limit inferior of x is given by

$$\mathcal{I}_2 - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset \\ \infty, & \text{if } A_x = \emptyset \end{cases}$$

It is known (Theorem 3, [13]) that $\mathcal{I}_2$-$\liminf x = \alpha$ (finite) if and only if for arbitrary $\varepsilon > 0$,

$$\{(j,k) : x_{jk} < \alpha + \varepsilon\} \notin \mathcal{I}_2 \text{ and } \{(j,k) : x_{jk} < \alpha - \varepsilon\} \in \mathcal{I}_2.$$

Theorem 2.7 *Let $x=(x_{jk})$ be $\mathcal{I}_2$-statistically pre-Cauchy. If $x=(x_{jk})$ has a subsequence $\left(x_{t_jt_k}\right)$ which converges to L and*

$$0<\mathcal{I}_2-\liminf_{m,n}\frac{1}{mn}\left|\{(t_j,t_k):j,k\in\mathbb{N}\}\right|<\infty,$$

then x is $\mathcal{I}_2$-statistically convergent to L.

Proof. Let $\varepsilon>0$ be given and select $n_0\in\mathbb{N}$ such that if $t_j>n_0$ or $t_k>n_0$ for some j,k then $\left|x_{t_jt_k}-L\right|<\frac{\varepsilon}{2}$. Let $A=\{(t_j,t_k):t_j>n_0 \text{ or } t_k>n_0,\ j,k\in\mathbb{N}\}$ and $B=\{(j,k):|x_{jk}-L|\geq\varepsilon\}$. Since

$$\begin{aligned}&\frac{1}{m^2n^2}\left|\left\{(j,k):|x_{jk}-x_{pq}|\geq\frac{\varepsilon}{2},\ j\leq m,k\leq n\right\}\right|\\ &\geq\frac{1}{m^2n^2}\sum_{j,p\leq m}\sum_{k,q\leq n}\chi_{A\times B}(j,k)\\ &=\frac{1}{mn}\left|\{t_j\leq m,t_k\leq n:(t_j,t_k)\in A\}\right|\frac{1}{mn}\left|\{j\leq m,k\leq n:|x_{jk}-L|\geq\varepsilon\}\right|.\end{aligned}$$

Since x is $\mathcal{I}_2$-statistically pre-Cauchy, for $\delta>0$

$$C=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{m^2n^2}\left|\left\{(j,k):|x_{jk}-x_{pq}|\geq\frac{\varepsilon}{2},\ j\leq m,k\leq n\right\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$

Therefore for every $(m,n)\in C^c$

$$\frac{1}{m^2n^2}\left|\left\{(j,k):|x_{jk}-x_{pq}|\geq\frac{\varepsilon}{2},\ j\leq m,k\leq n\right\}\right|<\delta. \tag{1}$$

Again since

$$\mathcal{I}_2-\liminf_{m,n}\frac{1}{mn}\left|\{t_j\leq m,t_k\leq n:(j,k)\in\mathbb{N}\times\mathbb{N}\}\right|=b>0,$$

so

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\left|\{t_j\leq m,t_k\leq n:(j,k)\in\mathbb{N}\times\mathbb{N}\}\right|<\frac{b}{2}\right\}=D\in\mathcal{I}_2$$

and so every $(m,n)\in D^c$

$$\frac{1}{mn}\left|\{t_j\leq m,t_k\leq n:(j,k)\in\mathbb{N}\times\mathbb{N}\}\right|\geq\frac{b}{2}. \tag{2}$$

From (1) and (2) it follows that every $(m,n)\in C^c\cap D^c=(C\cup D)^c$,

$$\frac{1}{mn}\left|\{j\leq m,k\leq n:|x_{jk}-L|\geq\varepsilon\}\right|<\frac{2\delta}{b}.$$

Let $\delta_1>0$ be given. Then choosing $\delta>0$ such that $\frac{2\delta}{b}<\delta_1$ we see that every $(m,n)\in(C\cup D)^c$

$$\frac{1}{mn}\left|\{j\leq m,k\leq n:|x_{jk}-L|\geq\varepsilon\}\right|<\delta_1,$$

that is,

$$\begin{aligned}&\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\left|\{j\leq m,k\leq n:|x_{jk}-L|\geq\varepsilon\}\right|\geq\delta\right\}\\ &\subset C\cup D.\end{aligned}$$

As $C,D\in\mathcal{I}_2$ so $C\cup D\in\mathcal{I}_2$ and consequently the set on the left hand side also belongs to $\mathcal{I}_2$. This shows that x is $\mathcal{I}_2$-statistically convergent to L.

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References

[1] C. Belen, M. Yildirim, On generalized statistical convergence of double sequences via ideals, *Ann. Univ. Ferrara Sez. VII Sci. Mat.*, 58(1)(2012) 11–20.

[2] J. Connor, J. Fridy, and J. Kline, Statistically Pre-Cauchy Sequences, *Analysis*, 14(1994) 311-317.

[3] P. Das, S. Ghosal, Some further results on $\mathcal{I}$-Cauchy sequences and condition (AP), *Comput. Math. Appl.*, 59(2010) 2597–2600.

[4] P. Das, E. Savaş, S.Kr. Ghosal, On generalizations of certain summability methods using ideals, *Appl. Math. Lett.*, 24(2011) 1509–1514.

[5] P. Das, E. Savaş, On $\mathcal{I}$-statistically pre-Cauchy sequences, *Taiwanese J. Math.*, 18(1)(2014) 115-126.

[6] H. Fast, Sur la convergence statistique, *Colloq. Math.*, 2(1951), 241-244.

[7] A.R. Freedman, J.J. Sember, Densities and summability, *Pacitific J. Math.*, 95(1981) 10–11.

[8] J.A. Fridy, On statistical convergence, *Analysis*, 5(1985) 301–313.

[9] J.A. Fridy, M.K. Khan, Tauberian theorems via statistical convergence, *J. Math. Anal. Appl.*, 228 (1998) 73–95.

[10] M. Gürdal, Statistically pre-Cauchy sequences and bounded moduli, *Acta Comm. Tartu. Math.*, 7(2003) 3–7.

[11] M. Gürdal, S. Pehlivan, The statistical convergence in 2-Banach spaces, *Thai. J. Math.*, 2(1)(2004) 107–113.

[12] M. Gürdal, I. Açık, On $\mathcal{I}$-cauchy sequences in 2-normed spaces, *Math. Inequal. Appl.*, 11(2)(2008) 349-354.

[13] M. Gürdal, A. Şahiner, Extremal $\mathcal{I}$-limit points of double sequences, *Appl. Math. E-Notes*, 8 (2008) 131–137.

[14] M. Gürdal, A. Şahiner, I. Açık, Approximation theory in 2-Banach spaces, *Nonlinear Analysis*, 71(5-6)(2009) 1654–1661.

[15] P. Kostyrko, M. Macaj, T. Salat, Statistical convergence and $\mathcal{I}$-convergence, 1999. Unpublished; http://thales.doa.fmph.uniba.sk/macaj/ICON.pdf.

[16] P. Kostyrko, T. Salat, W. Wilczynki, $\mathcal{I}$-convergence, *Real Anal. Exchange*, 26(2)(2000-2001) 669–685.

[17] H.I. Miller, A measure theoretical subsequence characterization of statistical convergence, *Trans. Amer. Math. Soc.*, 347 (1995) 1811–1819.

[18] M. Mursaleen, O.H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, 288(2003) 223-231.

[19] M. Mursaleen, S.A. Mohiuddine, On Ideal Convergence of Double Sequences in Probabilistic Normed Spaces, *Math. Reports*, 12(62)(2010) 359-371.

[20] A. Nabiev, S. Pehlivan, M. Gürdal, On $\mathcal{I}$-Cauchy sequences, *Taiwanese J. Math.*, 11(2)(2007) 569–576.

[21] E. Savaş, P. Das, A generalized statistical convergence via ideals, *Appl. Math. Lett.*, 24(2011) 826-830.

[22] E. Savaş, S.A. Mohiuddine, λ-statistically convergent double sequences in probabilistic normed spaces, *Math. Slovaca*, 62(1) (2012) 99–108.

[23] E. Savaş, M. Gürdal, Certain summability methods in intuitionistic fuzzy normed spaces, *Journal of Intelligent and Fuzzy Systems*, 27(4)(2014) 1621–1629.

[24] A. Şahiner, M. Gürdal, S. Saltan, H. Gunawan, Ideal convergence in 2-normed spaces, *Taiwanese J. Math.*, 11(5)(2007) 1477–1484.

[25] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.*, 2(1951) 73–74.

[26] B.C. Tripathy, On $\mathcal{I}$-convergent double sequences, *Soochow J. Math.*, 31(4)(2005) 549–560.

[27] U. Yamancı, M. Gürdal, $\mathcal{I}$-statistical convergence in 2-normed space, *Arab Journal of Mathematical Sciences*, 20(1)(2014) 41–47.

[28] A. Zygmund, Trigonometric Series, second ed., *Cambridge Univ. Press*, 1979.