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Research Paper

Complete monotonicity of a function involving the *p*-psi function and alternative proofs

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Abstract

In the paper, the authors prove that the function $x^\alpha\left[\ln\frac{px}{x+p+1}-\psi_p(x)\right]$ is completely monotonic on $(0,\infty)$ if and only if $\alpha\le 1$, where $p\in\mathbb{N}$ and $\psi_p(x)$ is the p-analogue of the classical psi function $\psi(x)$.

Keywords: *completely monotonic function; necessary and sufficient condition; p-gamma function; p-psi function; inequality*
MSC: *Primary 33D05; Secondary 26A48, 33B15, 33E50*

1. Introduction

Recall from [12, Chapter XIII], [16, Chapter 1] and [17, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies

$$0\le(-1)^n f^{(n)}(x)<\infty \tag{1.1}$$

for $x\in I$ and $n\ge 0$. The celebrated Bernstein-Widder's Theorem (see [16, p. 3, Theorem 1.4] or [17, p. 161, Theorem 12b]) characterizes that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0<x<\infty$ is that

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\,\alpha(t), \tag{1.2}$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0<x<\infty$. This expresses that a completely monotonic function f on $[0,\infty)$ is a Laplace transform of the measure α.

It is common knowledge that the classical Euler's gamma function $\Gamma(x)$ may be defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}\,t.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, is called psi function or digamma function.

An alternative definition of the gamma function $\Gamma(x)$ is

$$\Gamma(x)=\lim_{p\to\infty}\Gamma_p(x), \tag{1.3}$$

where

$$\Gamma_p(x)=\frac{p!p^x}{x(x+1)\cdots(x+p)}=\frac{p^x}{x(1+x/1)\cdots(1+x/p)} \tag{1.4}$$

for $x>0$ and $p\in\mathbb{N}$, the set of all positive integers. See [3, p. 250]. The p-analogue of the psi function $\psi(x)$ is defined as the logarithmic derivative of the Γ_p function, that is,

$$\psi_p(x)=\frac{\mathrm{d}}{\mathrm{d}\,x}\ln\Gamma_p(x)=\frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{1.5}$$

The function ψ_p has the following properties:

1. It has the following representations

$$\psi_p(x)=\ln p-\sum_{k=0}^{p}\frac{1}{x+k}=\ln p-\int_0^\infty\frac{1-e^{-(p+1)t}}{1-e^{-t}}e^{-xt}\,\mathrm{d}\,t. \tag{1.6}$$

2. It is increasing on $(0,\infty)$ and ψ_p' is completely monotonic on $(0,\infty)$.

The very right hand side of the formula (1.6) corrects errors appeared in [8, p. 374, Lemma 5] and [10, p. 29, Lemma 2.3].

In [2, pp. 374–375, Theorem 1], it was proved that the function

$$\theta_\alpha(x)=x^\alpha[\ln x-\psi(x)] \tag{1.7}$$

is completely monotonic on $(0,\infty)$ if and only if $\alpha\le1$. For the history, background, applications and alternative proofs of this conclusion, please refer to [4], [13, p. 8, Section 1.6.6] and closely related references therein.

The aim of this paper is to generalize [2, pp. 374–375, Theorem 1] and [4, p. 105, Theorem 1] to the case of the p-analogue $\psi_p(x)$ of the psi function $\psi(x)$ as follows.

Theorem 1.1. *The function*

$$\theta_{p,\alpha}(x)=x^\alpha\left[\ln\frac{px}{x+p+1}-\psi_p(x)\right] \tag{1.8}$$

for $p\in\mathbb{N}$ is completely monotonic on $(0,\infty)$ if and only if $\alpha\le1$.

Remark 1.1. Letting $p\to\infty$ in Theorem 1.1, we obtain [2, pp. 374–375, Theorem 1] and [4, p. 105, Theorem 1].

2. Proofs of Theorem 1.1

First Proof. From the identity (1.6) and the integral expression

$$\ln\frac{b}{a}=\int_0^\infty\frac{e^{-at}-e^{-bt}}{t}\,\mathrm{d}\,t \tag{2.1}$$

in [1, p. 230, 5.1.32], we obtain

$$\theta_{p,1}(x)=x\int_0^\infty\left[1-e^{-(p+1)t}\right]\varphi(t)e^{-xt}\,\mathrm{d}\,t, \tag{2.2}$$

where

$$\varphi(t)=\frac{1}{1-e^{-t}}-\frac{1}{t}. \tag{2.3}$$

The function $\varphi(t)$ is increasing on $(0,\infty)$ with

$$\lim_{t\to0^+}\varphi(t)=\frac12 \quad\text{and}\quad \lim_{t\to\infty}\varphi(t)=1. \tag{2.4}$$

See [5, 6, 7, 11, 14, 15, 18] and related references therein. Therefore, for $x>0$ and $n\in\mathbb{N}$, we have

$$\begin{aligned}
(-1)^n\theta_{p,1}^{(n)}(x)&=x(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\int_0^\infty\bigl[1-e^{-(p+1)t}\bigr]\varphi(t)e^{-xt}\,\mathrm{d}\,t-(-1)^{n-1}n\frac{\mathrm{d}^{n-1}}{\mathrm{d}\,x^{n-1}}\int_0^\infty\bigl[1-e^{-(p+1)t}\bigr]\varphi(t)e^{-xt}\,\mathrm{d}\,t\\
&=x\int_0^\infty t^n\varphi(t)\bigl[1-e^{-(p+1)t}\bigr]e^{-xt}\,\mathrm{d}\,t-n\int_0^\infty t^{n-1}\varphi(t)\bigl[1-e^{-(p+1)t}\bigr]e^{-xt}\,\mathrm{d}\,t\\
&=\int_0^{n/x}t^{n-1}\bigl[1-e^{-(p+1)t}\bigr]\varphi(t)(tx-n)e^{-xt}\,\mathrm{d}\,t+\int_{n/x}^\infty t^{n-1}\bigl[1-e^{-(p+1)t}\bigr]\varphi(t)(tx-n)e^{-xt}\,\mathrm{d}\,t\\
&>\varphi\Bigl(\frac nx\Bigr)\int_0^{n/x}t^{n-1}\bigl[1-e^{-(p+1)t}\bigr](tx-n)e^{-xt}\,\mathrm{d}\,t+\varphi\Bigl(\frac nx\Bigr)\int_{n/x}^\infty t^{n-1}\bigl[1-e^{-(p+1)t}\bigr](tx-n)e^{-xt}\,\mathrm{d}\,t\\
&=\varphi\Bigl(\frac nx\Bigr)\int_0^\infty t^{n-1}\bigl[1-e^{-(p+1)t}\bigr](tx-n)e^{-xt}\,\mathrm{d}\,t\\
&=\varphi\Bigl(\frac nx\Bigr)\biggl[x\int_0^\infty t^n\bigl[1-e^{-(p+1)t}\bigr]e^{-xt}\,\mathrm{d}\,t-n\int_0^\infty t^{n-1}\bigl[1-e^{-(p+1)t}\bigr]e^{-xt}\,\mathrm{d}\,t\biggr]\\
&=\varphi\Bigl(\frac nx\Bigr)\biggl[x\int_0^\infty t^ne^{-xt}\,\mathrm{d}\,t-x\int_0^\infty t^ne^{-(x+p+1)t}\,\mathrm{d}\,t-n\int_0^\infty t^{n-1}e^{-xt}\,\mathrm{d}\,t+n\int_0^\infty t^{n-1}e^{-(x+p+1)t}\,\mathrm{d}\,t\biggr]\\
&=\varphi\Bigl(\frac nx\Bigr)\biggl[x\frac{n!}{x^{n+1}}-x\frac{n!}{(x+p+1)^{n+1}}-n\frac{(n-1)!}{x^n}+n\frac{(n-1)!}{(x+p+1)^n}\biggr]\\
&=\varphi\Bigl(\frac nx\Bigr)n!\biggl[\frac{1}{x^n}-\frac{x}{(x+p+1)^{n+1}}-\frac{1}{x^n}+\frac{1}{(x+p+1)^n}\biggr]\\
&=\varphi\Bigl(\frac nx\Bigr)\frac{n!}{(x+p+1)^n}\Bigl(1-\frac{x}{x+p+1}\Bigr)\\
&=\varphi\Bigl(\frac nx\Bigr)\frac{n!(p+1)}{(x+p+1)^{n+1}}\\
&>0,
\end{aligned}$$

where we used the formula

$$\frac{1}{x^\omega}=\frac{1}{\Gamma(\omega)}\int_0^\infty t^{\omega-1}e^{-xt}\,\mathrm{d}\,t \tag{2.5}$$

for real numbers $x>0$ and $\omega>0$, see [1, p. 255, 6.1.1]. So we obtain that the function $\theta_{p,1}(x)$ is completely monotonic on $(0,\infty)$.

Since

$$(-1)^n[u(x)v(x)]^{(n)}=\sum_{i=0}^n\binom{n}{i}\bigl[(-1)^iu^{(i)}(x)\bigr]\bigl[(-1)^{n-i}v^{(n-i)}(x)\bigr],$$

the product of any two completely monotonic function is also completely monotonic on their common domain. On the other hand, the function $x^{\alpha-1}$ for $\alpha<1$ is clearly completely monotonic on $(0,\infty)$. Consequently the function

$$\theta_{p,\alpha}(x)=x^{\alpha-1}\theta_{p,1}(x)$$

for $\alpha\le1$ is completely monotonic on $(0,\infty)$.

Conversely, if $\theta_{p,\alpha}(x)$ is completely monotonic on $(0,\infty)$, then

$$\frac{\mathrm{d}\,\theta_{p,\alpha}(x)}{\mathrm{d}\,x}=x^{\alpha-1}\biggl\{\alpha\biggl[\ln\frac{px}{x+p+1}-\psi_p(x)\biggr]+\frac{p+1}{x+p+1}-x\psi_p'(x)\biggr\}\le0$$

for $x>0$, equivalently,

$$\alpha\le\frac{x\psi_p'(x)-\frac{p+1}{x+p+1}}{\ln\frac{px}{x+p+1}-\psi_p(x)}.$$

Employing L'Hôspital's rule and (1.6) results in

$$\lim_{x\to\infty}\frac{x\psi_p'(x)-\frac{p+1}{x+p+1}}{\ln\frac{px}{x+p+1}-\psi_p(x)}=\lim_{x\to\infty}\frac{x\psi_p''(x)+\psi_p'(x)+\frac{p+1}{(x+p+1)^2}}{\frac1x-\frac{1}{x+p+1}-\psi_p'(x)}=\lim_{x\to\infty}\frac{\frac{p+1}{(x+p+1)^2}-x\sum_{k=0}^p\frac{2}{(x+k)^3}+\sum_{k=0}^p\frac{1}{(x+k)^2}}{\frac1x-\frac{1}{x+p+1}-\sum_{k=0}^p\frac{1}{(x+k)^2}}=1,$$

so it is necessary that $\alpha\le1$. The proof is complete. □

Second Proof. From (2.2) and by integration by part lead to

$$\begin{aligned}\theta_{p,1}(x)&=-\int_0^\infty\big[1-e^{-(p+1)t}\big]\varphi(t)\frac{\mathrm{d}\,e^{-xt}}{\mathrm{d}\,t}\,\mathrm{d}\,t\\&=\int_0^\infty\big\{\big[1-e^{-(p+1)t}\big]\varphi(t)\big\}'e^{-xt}\,\mathrm{d}\,t-\big\{\big[1-e^{-(p+1)t}\big]\varphi(t)e^{-xt}\big\}\Big|_{t=0}^{t=\infty}\\&=\int_0^\infty\big\{\big[1-e^{-(p+1)t}\big]\varphi'(t)+(p+1)e^{-(p+1)t}\varphi(t)\big\}e^{-xt}\,\mathrm{d}\,t.\end{aligned}$$

Therefore, for showing that the function $\theta_{p,1}(x)$ is completely monotonic on $(0,\infty)$ for all $p\in\mathbb{N}$, it suffices to prove that the function

$$\big[1-e^{-(p+1)t}\big]\varphi'(t)+(p+1)e^{-(p+1)t}\varphi(t) \tag{2.6}$$

is positive. Since the function $\varphi(t)$ is increasing on $(0,\infty)$, the derivative $\varphi'(t)$ is positive on $(0,\infty)$. Further considering the limits in (2.4), the positivity of $\varphi(t)$ follows. As a result, the function (2.6) is positive.

The rest of the proof is the same as the first proof. □

Remark 2.1. This paper is a slightly modified version of the preprint [9].

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