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Research Paper

Absolute monotonicity of a function involving the exponential function

Feng Qi

*College of Mathematics, Inner Mongolia University for Nationalities,
Tongliao City, Inner Mongolia Autonomous Region, 028043, China
Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China
Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China
E-mail: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: http://qifeng618.wordpress.com*

Abstract

In the paper, the author verifies the absolute monotonicity of a function involving the exponential function.

Keywords: *absolute monotonicity; absolutely monotonic function; completely monotonic function; completely monotonic degree; exponential function*
MSC: *Primary 26A48; Secondary 33B10, 44A10*

1. Preliminaries and main results

Recall from [5, 15, 16] that a function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I such that $(-1)^k f^{(k)}(x) \ge 0$ for $x \in I$ and $k \ge 0$. Recall also from [5, 15, 16] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k-1)}(t) \ge 0$ for $t \in I$ and $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers.

It is easy to see that a function $f(x)$ is completely monotonic in (a,b) if and only if $f(-x)$ is absolutely monotonic in $(-b,-a)$. See [16, p. 145, Definition 2c].

Theorem 12a in [16, p. 160] reads that a necessary and sufficient condition that $f(x)$ should be completely monotonic in $0 \le x < \infty$ is that $f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\,\alpha(t)$, where $\alpha(t)$ is bounded and non-decreasing and the integral converges for $0 \le x < \infty$. Theorem 12c in [16, p. 162] states that a necessary and sufficient condition that $f(x)$ should be absolutely monotonic in $-\infty < x < 0$ is that $f(x) = \int_0^\infty e^{xt} \,\mathrm{d}\,\alpha(t)$, where $\alpha(t)$ is non-decreasing and the integral converges for $-\infty < x < 0$.

For more information on these kinds of functions, please refer to [5, Chapter XIII], [16, Chapter IV], [2, 12, 15] and closely related references therein.

The classical Euler's gamma function $\Gamma(x)$ may be defined for $\Re(z) > 0$ by $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}\,t$. The logarithmic derivative of $\Gamma(z)$, denoted by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, is called the psi or di-gamma function. As a whole, the derivatives $\psi^{(i)}(z)$ for $i \ge 0$ are called polygamma functions.

Recently, when computing the completely monotonic degree of the function

$$\Psi(x) = [\psi'(x)]^2 + \psi''(x) \tag{1.1}$$

with respect to $x \in (0, \infty)$, we encounter the function

$$\begin{aligned} R(t) = {} & e^{9t}\bigl(t^5 - 12t^4 + 70t^3 - 160t^2 + 192t - 128\bigr) - e^{8t}\bigl(16t^7 - 220t^6 + 1219t^5 - 3220t^4 + 4490t^3 - 3248t^2 \\ & + 1152t - 768\bigr) - 4e^{7t}\bigl(37t^7 - 423t^6 + 1397t^5 - 1409t^4 - 1020t^3 + 2632t^2 - 732t + 456\bigr) - 4e^{6t}\bigl(225t^7 \\ & - 1281t^6 + 1213t^5 + 3127t^4 - 4372t^3 - 2648t^2 + 1020t - 504\bigr) - 2e^{5t}\bigl(908t^7 - 1514t^6 - 6493t^5 \\ & + 8710t^4 + 12754t^3 - 1216t^2 - 1656t + 336\bigr) - 2e^{4t}\bigl(908t^7 + 1710t^6 - 5489t^5 - 12370t^4 + 594t^3 \\ & + 4880t^2 + 696t + 336\bigr) - 4e^{3t}\bigl(225t^7 + 1263t^6 + 1771t^5 - 887t^4 - 3208t^3 - 728t^2 + 12t - 168\bigr) \\ & - 4e^{2t}\bigl(37t^7 + 353t^6 + 1099t^5 + 1337t^4 + 272t^3 - 632t^2 - 108t + 24\bigr) - e^{t}\bigl(16t^7 + 180t^6 + 827t^5 \\ & + 1864t^4 + 2226t^3 + 1312t^2 + 240t + 96\bigr) + t^5 + 8t^4 + 30t^3 + 48t^2 + 48t + 32 \end{aligned} \tag{1.2}$$

on $(0, \infty)$. For more information on the notion "completely monotonic degree", please refer to [1, 3, 8, 10, 14] and closely related references therein. For more information on results and properties of the function (1.1), please refer to [6, 11, 13].

The main aim of this paper is to verify that the absolute monotonicity of the function $R(t)$ on $(0, \infty)$.

Theorem 1.1. *The function $R(t)$ defined by (1.2) is absolutely monotonic on $(0, \infty)$. In other words, the function $R(-t)$ defined by (1.2) is completely monotonic on $(-\infty, 0)$.*

2. Proof of Theorem 1.1

Now we are in a position to spend a large amount of texts to detail the proof of the absolute monotonicity of the function $R(t)$ on $(0, \infty)$.

Consecutive differentiation and straightforward simplification give

$$\begin{aligned} R'(t) = {} & 5t^4 + 32t^3 + 90t^2 + 96t + 48 - e^{t}\bigl(16t^7 + 292t^6 + 1907t^5 + 5999t^4 + 9682t^3 + 7990t^2 + 2864t + 336\bigr) \\ & - 4e^{2t}\bigl(74t^7 + 965t^6 + 4316t^5 + 8169t^4 + 5892t^3 - 448t^2 - 1480t - 60\bigr) - 4e^{3t}\bigl(675t^7 + 5364t^6 \\ & + 12891t^5 + 6194t^4 - 13172t^3 - 11808t^2 - 1420t - 492\bigr) - 2e^{4t}\bigl(3632t^7 + 13196t^6 - 11696t^5 \\ & - 76925t^4 - 47104t^3 + 21302t^2 + 12544t + 2040\bigr) - 2e^{5t}\bigl(4540t^7 - 1214t^6 - 41549t^5 + 11085t^4 \\ & + 98610t^3 + 32182t^2 - 10712t + 24\bigr) - 4e^{6t}\bigl(1350t^7 - 6111t^6 - 408t^5 + 24827t^4 - 13724t^3 - 29004t^2 \\ & + 824t - 2004\bigr) - 4e^{7t}\bigl(259t^7 - 2702t^6 + 7241t^5 - 2878t^4 - 12776t^3 + 15364t^2 + 140t + 2460\bigr) \\ & - e^{8t}\bigl(128t^7 - 1648t^6 + 8432t^5 - 19665t^4 + 23040t^3 - 12514t^2 + 2720t - 4992\bigr) + e^{9t}\bigl(9t^5 - 103t^4 \\ & + 582t^3 - 1230t^2 + 1408t - 960\bigr), \\ R''(t) = {} & 4\bigl(5t^3 + 24t^2 + 45t + 24\bigr) - e^{t}\bigl(16t^7 + 404t^6 + 3659t^5 + 15534t^4 + 33678t^3 + 37036t^2 + 18844t \\ & + 3200\bigr) - 8e^{2t}\bigl(74t^7 + 1224t^6 + 7211t^5 + 18959t^4 + 22230t^3 + 8390t^2 - 1928t - 800\bigr) - 4e^{3t}\bigl(2025t^7 \\ & + 20817t^6 + 70857t^5 + 83037t^4 - 14740t^3 - 74940t^2 - 27876t - 2896\bigr) - 8e^{4t}\bigl(3632t^7 + 19552t^6 \\ & + 8098t^5 - 91545t^4 - 124029t^3 - 14026t^2 + 23195t + 5176\bigr) - 2e^{5t}\bigl(22700t^7 + 25710t^6 - 215029t^5 \\ & - 152320t^4 + 537390t^3 + 456740t^2 + 10804t - 10592\bigr) - 8e^{6t}\bigl(4050t^7 - 13608t^6 - 19557t^5 + 73461t^4 \\ & + 8482t^3 - 107598t^2 - 26532t - 5600\bigr) - 4e^{7t}\bigl(1813t^7 - 17101t^6 + 34475t^5 + 16059t^4 - 100944t^3 \\ & + 69220t^2 + 31708t + 17360\bigr) - 4e^{8t}\bigl(256t^7 - 3072t^6 + 14392t^5 - 28790t^4 + 26415t^3 - 7748t^2 - 817t \\ & - 9304\bigr) + e^{9t}\bigl(81t^5 - 882t^4 + 4826t^3 - 9324t^2 + 10212t - 7232\bigr), \\ R^{(3)}(t) = {} & 12\bigl(5t^2 + 16t + 15\bigr) - e^{t}\bigl(16t^7 + 516t^6 + 6083t^5 + 33829t^4 + 95814t^3 + 138070t^2 + 92916t + 22044\bigr) \\ & - 8e^{2t}\bigl(148t^7 + 2966t^6 + 21766t^5 + 73973t^4 + 120296t^3 + 83470t^2 + 12924t - 3528\bigr) - 12e^{3t}\bigl(2025t^7 \\ & + 25542t^6 + 112491t^5 + 201132t^4 + 95976t^3 - 89680t^2 - 77836t - 12188\bigr) - 8e^{4t}\bigl(14528t^7 \\ & + 103632t^6 + 149704t^5 - 325690t^4 - 862296t^3 - 428191t^2 + 64728t + 43899\bigr) - 2e^{5t}\bigl(113500t^7 \end{aligned}$$

$$
\begin{aligned}
&+287450t^6-920885t^5-1836745t^4+2077670t^3+3895870t^2+967500t-42156\big)-24e^{6t}\big(8100t^7\\
&-17766t^6-66330t^5+114327t^4+114912t^3-206714t^2-124796t-20044\big)-4e^{7t}\big(12691t^7\\
&-107016t^6+138719t^5+284788t^4-642372t^3+181708t^2+360396t+153228\big)-4e^{8t}\big(2048t^7\\
&-22784t^6+96704t^5-158360t^4+96160t^3+17261t^2-22032t-75249\big)+e^{9t}\big(729t^5-7533t^4\\
&+39906t^3-69438t^2+73260t-54876\big),
\end{aligned}
$$

$$
\begin{aligned}
R^{(4)}(t)=&\,24(5t+8)-e^t\big(16t^7+628t^6+9179t^5+64244t^4+231130t^3+425512t^2+369056t+114960\big)\\
&-32e^{2t}\big(74t^7+1742t^6+15332t^5+64194t^4+134121t^3+131957t^2+48197t+1467\big)-12e^{3t}\big(6075t^7\\
&+90801t^6+490725t^5+1165851t^4+1092456t^3+18888t^2-412868t-114400\big)-16e^{4t}\big(29056t^7\\
&+258112t^6+610304t^5-277120t^4-2375972t^3-2149826t^2-298735t+120162\big)-10e^{5t}\big(113500t^7\\
&+446350t^6-575945t^5-2757630t^4+608274t^3+5142472t^2+2525848t+151344\big)-96e^{6t}\big(12150t^7\\
&-12474t^6-126144t^5+88578t^4+286695t^3-223887t^2-290551t-61265\big)-4e^{7t}\big(88837t^7\\
&-660275t^6+328937t^5+2687111t^4-3357452t^3-655160t^2+2886188t+1432992\big)-8e^{8t}\big(8192t^7\\
&-83968t^6+318464t^5-391680t^4+67920t^3+213284t^2-70867t-312012\big)+e^{9t}\big(6561t^5-64152t^4\\
&+329022t^3-505224t^2+520464t-420624\big),
\end{aligned}
$$

$$
\begin{aligned}
R^{(5)}(t)=&\,120-e^t\big(16t^7+740t^6+12947t^5+110139t^4+488106t^3+1118902t^2+1220080t+484016\big)\\
&-32e^{2t}\big(148t^7+4002t^6+41116t^5+205048t^4+525018t^3+666277t^2+360308t+51131\big)\\
&-12e^{3t}\big(18225t^7+314928t^6+2016981t^5+5951178t^4+7940772t^3+3334032t^2-1200828t\\
&-756068\big)-16e^{4t}\big(116224t^7+1235840t^6+3989888t^5+1943040t^4-10612368t^3-15727220t^2\\
&-5494592t+181913\big)-10e^{5t}\big(567500t^7+3026250t^6-201625t^5-16667875t^4-7989150t^3\\
&+27537182t^2+22914184t+3282568\big)-96e^{6t}\big(72900t^7+10206t^6-831708t^5-99252t^4\\
&+2074482t^3-483237t^2-2191080t-658141\big)-4e^{7t}\big(621859t^7-4000066t^6-1659091t^5\\
&+20454462t^4-12753720t^3-14658476t^2+18892996t+12917132\big)-8e^{8t}\big(65536t^7-614400t^6\\
&+2043904t^5-1541120t^4-1023360t^3+1910032t^2-140368t-2566963\big)+3e^{9t}\big(19683t^5\\
&-181521t^4+901530t^3-1186650t^2+1224576t-1088384\big),
\end{aligned}
$$

$$
\begin{aligned}
R^{(6)}(t)=&\,e^t\big[81e^{8t}\big(6561t^5-56862t^4+273618t^3-295380t^2+320292t-317440\big)-64e^{7t}\big(65536t^7\\
&-557056t^6+1583104t^5-263680t^4-1793920t^3+1526272t^2+337140t-2584509\big)\\
&-4e^{6t}\big(4353013t^7-23647449t^6-35614033t^5+134885779t^4-7458192t^3-140870492t^2\\
&+102934020t+109312920\big)-1728e^{5t}\big(24300t^7+31752t^6-273834t^5-264114t^4+669438t^3\\
&+184668t^2-784053t-341107\big)-10e^{4t}\big(2837500t^7+19103750t^6+17149375t^5-84347500t^4\\
&-106617250t^3+113718460t^2+169645284t+39327024\big)-64e^{3t}\big(116224t^7+1439232t^6\\
&+5843648t^5+6930400t^4-8669328t^3-23686496t^2-13358202t-1191735\big)-108e^{2t}\big(6075t^7\\
&+119151t^6+882279t^5+3104271t^4+5291892t^3+3758268t^2+340620t-385448\big)-64e^t\big(148t^7\\
&+4520t^6+53122t^5+307838t^4+935114t^3+1453804t^2+1026585t+231285\big)-16t^7-852t^6\\
&-17387t^5-174874t^4-928662t^3-2583220t^2-3457884t-1704096\big]\\
\triangleq&\,e^tR_1(t),
\end{aligned}
$$

$$
\begin{aligned}
R_1'(t)=&\,81e^{8t}\big(52488t^5-422091t^4+1961496t^3-1542186t^2+1971576t-2219228\big)-64e^{7t}\big(458752t^7\\
&-3440640t^6+7739392t^5+6069760t^4-13612160t^3+5302144t^2+5412524t-17754423\big)\\
&-4e^{6t}\big(26118078t^7-111413603t^6-355568892t^5+631244509t^4+494793964t^3-867597528t^2\\
&+335863136t+758811540\big)-1728e^{5t}\big(121500t^7+328860t^6-1178658t^5-2689740t^4+2290734t^3\\
&+2931654t^2-3550929t-2489588\big)-10e^{4t}\big(11350000t^7+96277500t^6+183220000t^5\\
&-251643125t^4-763859000t^3+135022090t^2+906018056t+326953380\big)-64e^{3t}\big(348672t^7\\
&+5131264t^6+26166336t^5+50009440t^4+1713616t^3-97067472t^2-87447598t-16933407\big)
\end{aligned}
$$

$$
\begin{aligned}
&- 108e^{2t}\big(12150t^7 + 280827t^6 + 2479464t^5 + 10619937t^4 + 23000868t^3 + 23392212t^2 + 8197776t\\
&- 430276\big) - 64e^t\big(148t^7 + 5556t^6 + 80242t^5 + 573448t^4 + 2166466t^3 + 4259146t^2 + 3934193t\\
&+ 1257870\big) - 112t^6 - 5112t^5 - 86935t^4 - 699496t^3 - 2785986t^2 - 5166440t - 3457884,\\
R_1''(t) = &-4\big[168t^5 + 6390t^4 + 86935t^3 + 524622t^2 + 1392993t + 1291610 + 16e^t\big(148t^7 + 6592t^6\\
&+ 113578t^5 + 974658t^4 + 4460258t^3 + 10758544t^2 + 12452485t + 5192063\big) + 54e^{2t}\big(12150t^7\\
&+ 323352t^6 + 3321945t^5 + 16818597t^4 + 44240742t^3 + 57893514t^2 + 31589988t + 3668612\big)\\
&+ 16e^{3t}\big(1046016t^7 + 17834496t^6 + 109286592t^5 + 280860000t^4 + 205178608t^3 - 286061568t^2\\
&- 456477738t - 138247819\big) + 10e^{4t}\big(11350000t^7 + 116140000t^6 + 327636250t^5 - 22618125t^4\\
&- 1015502125t^3 - 437872160t^2 + 973529101t + 553457894\big) + 432e^{5t}\big(607500t^7 + 2494800t^6\\
&- 3920130t^5 - 19341990t^4 + 694710t^3 + 21530472t^2 - 11891337t - 15998869\big) + 2e^{6t}\big(78354234t^7\\
&- 242827536t^6 - 1400947485t^5 + 1004811297t^4 + 2746870910t^3 - 1860601638t^2 + 139991880t\\
&+ 2444366188\big) + 16e^{7t}\big(3211264t^7 - 20873216t^6 + 33531904t^5 + 81185280t^4 - 71006080t^3\\
&- 3721472t^2 + 48491956t - 118868437\big) - 81e^{8t}\big(104976t^5 - 778572t^4 + 3500901t^3 - 1613250t^2\\
&+ 3172059t - 3945562\big)\big],\\
R_1^{(3)}(t) = &-4\big[3\big(280t^4 + 8520t^3 + 86935t^2 + 349748t + 464331\big) + 16e^t\big(148t^7 + 7628t^6 + 153130t^5\\
&+ 1542548t^4 + 8358890t^3 + 24139318t^2 + 33969573t + 17644548\big) + 54e^{2t}\big(24300t^7 + 731754t^6\\
&+ 8584002t^5 + 50246919t^4 + 155755872t^3 + 248509254t^2 + 178967004t + 38927212\big)\\
&+ 48e^{3t}\big(1046016t^7 + 20275200t^6 + 144955584t^5 + 463004320t^4 + 579658608t^3 - 80882960t^2\\
&- 647185450t - 290407065\big) + 10e^{4t}\big(45400000t^7 + 544010000t^6 + 2007385000t^5 + 1547708750t^4\\
&- 4152481000t^3 - 4797995015t^2 + 3018372084t + 3187360677\big) + 432e^{5t}\big(3037500t^7 + 16726500t^6\\
&- 4631850t^5 - 116310600t^4 - 73894410t^3 + 109736490t^2 - 16395741t - 91885682\big)\\
&+ 6e^{6t}\big(156708468t^7 - 302828526t^6 - 3287550042t^5 - 325289881t^4 + 6833490216t^3 - 974332366t^2\\
&- 960417332t + 4935396336\big) + 16e^{7t}\big(22478848t^7 - 123633664t^6 + 109484032t^5 + 735956480t^4\\
&- 172301440t^3 - 239068544t^2 + 332000748t - 783587103\big) - 81e^{8t}\big(839808t^5 - 5703696t^4\\
&+ 24892920t^3 - 2403297t^2 + 22149972t - 28392437\big)\big],\\
R_1^{(4)}(t) = &-8\big[3\big(560t^3 + 12780t^2 + 86935t + 174874\big) + 8e^t\big(148t^7 + 8664t^6 + 198898t^5\\
&+ 2308198t^4 + 14529082t^3 + 49215988t^2 + 82248209t + 51614121\big) + 108e^{2t}\big(12150t^7 + 408402t^6\\
&+ 5389632t^5 + 35853462t^4 + 128124855t^3 + 241071531t^2 + 213738129t + 64205357\big)\\
&+ 24e^{3t}\big(3138048t^7 + 68147712t^6 + 556517952t^5 + 2113790880t^4 + 3590993104t^3 + 1496326944t^2\\
&- 2103322270t - 1518406645\big) + 10e^{4t}\big(90800000t^7 + 1246920000t^6 + 5646800000t^5 + 8113880000t^4\\
&- 5209544500t^3 - 15824711530t^2 + 1238749153t + 7883907396\big) + 216e^{5t}\big(15187500t^7\\
&+ 104895000t^6 + 77199750t^5 - 604712250t^4 - 834714450t^3 + 326999220t^2 + 137494275t\\
&- 475824151\big) + 12e^{6t}\big(235062702t^7 - 180002970t^6 - 5385567852t^5 - 4597372374t^4 + 9924945443t^3\\
&+ 3663619113t^2 - 1927792181t + 7162990171\big) + 8e^{7t}\big(157351936t^7 - 708083712t^6\\
&+ 24586240t^5 + 5699115520t^4 + 1737715840t^3 - 2190384128t^2 + 1845868148t - 5153108973\big)\\
&- 81e^{8t}\big(3359232t^5 - 20715264t^4 + 88164288t^3 + 27726192t^2 + 86196591t - 102494762\big)\big],\\
R_1^{(5)}(t) = &\,8\big[81e^{8t}\big(26873856t^5 - 148925952t^4 + 622453248t^3 + 486302400t^2 + 745025112t\\
&- 733761505\big) - 8e^{7t}\big(1101463552t^7 - 3855122432t^6 - 4076398592t^5 + 40016739840t^4\\
&+ 34960472960t^3 - 10119541376t^2 + 8540308780t - 34225894663\big) - 12e^{6t}\big(1410376212t^7\\
&+ 565421094t^6 - 33393424932t^5 - 54512073504t^4 + 41160183162t^3 + 51756551007t^2 - 4239514860t\\
&+ 41050148845\big) - 1080e^{5t}\big(15187500t^7 + 126157500t^6 + 203073750t^5 - 527512500t^4\\
&- 1318484250t^3 - 173829450t^2 + 268293963t - 448325296\big) - 10e^{4t}\big(363200000t^7 + 5623280000t^6
\end{aligned}
$$

$$
\begin{aligned}
&+30068720000t^5+60689520000t^4+11617342000t^3-78927479620t^2-26694426448t+32774378737\big)\\
&-24e^{3t}\big(9414144t^7+226409472t^6+2078440128t^5+9123962400t^4+19228142832t^3+15261960144t^2\\
&-3317312922t-6658542205\big)-108e^{2t}\big(24300t^7+901854t^6+13229676t^5+98655084t^4\\
&+399663558t^3+866517627t^2+909619320t+342148843\big)-8e^t\big(148t^7+9700t^6+250882t^5\\
&+3302688t^4+23761874t^3+92803234t^2+180680185t+133862330\big)-15\big(336t^2+5112t+17387\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_1^{(6)}(t)=64\big[&162e^{8t}\big(13436928t^5-66064896t^4+273995136t^3+359861184t^2+433300356t\\
&-320316683\big)-e^{7t}\big(7710244864t^7-19275612160t^6-51665524736t^5+259735185920t^4\\
&+404790270080t^3+34044629248t^2+39543078708t-231040953861\big)-9e^{6t}\big(1410376212t^7\\
&+2210860008t^6-32828003838t^5-82339927614t^4+4818800826t^3+72336642588t^2+13012668809t\\
&+40343563035\big)-135e^{5t}\big(75937500t^7+737100000t^6+1772313750t^5-1622193750t^4\\
&-8702471250t^3-4824600000t^2+993810915t-1973332517\big)-5e^{4t}\big(363200000t^7+6258880000t^6\\
&+38503640000t^5+98275420000t^4+72306862000t^3-70214473120t^2-66158166258t+26100772125\big)\\
&-9e^{3t}\big(9414144t^7+248375808t^6+2531259072t^5+12588029280t^4+31393426032t^3+34490102976t^2\\
&+6857327174t-7764313179\big)-27e^{2t}\big(24300t^7+986904t^6+15935238t^5+131729274t^4\\
&+596973726t^3+1466012964t^2+1776136947t+796958503\big)-e^t\big(148t^7+10736t^6+309082t^5\\
&+4557098t^4+36972626t^3+164088856t^2+366286653t+314542515\big)-45(28t+213)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_1^{(7)}(t)=64\big[&648e^{8t}\big(26873856t^5-115333632t^4+481925376t^3+925218720t^2+1046531304t-532308277\big)\\
&-7e^{7t}\big(7710244864t^7-11565367296t^6-68187478016t^5+222831239680t^4+553210376320t^3\\
&+207526173568t^2+49270115636t-225391942617\big)-9e^{6t}\big(8462257272t^7+23137793532t^6\\
&-183702862980t^5-658179584874t^4-300446905500t^3+448476258006t^2+222749298030t\\
&+255074047019\big)-675e^{5t}\big(75937500t^7+843412500t^6+2656833750t^5+150120000t^4\\
&-10000226250t^3-10046082750t^2-936029085t-1774570334\big)-10e^{4t}\big(726400000t^7\\
&+13788960000t^6+95783920000t^5+292809940000t^4+341164564000t^3-31968653240t^2\\
&-202530805636t+19122461121\big)-9e^{3t}\big(28242432t^7+811026432t^6+9084032064t^5\\
&+50420383200t^4+144532395216t^3+197650587024t^2+89552187474t-16435612363\big)\\
&-27e^{2t}\big(48600t^7+2143908t^6+37791900t^5+343134738t^4+1720864548t^3+4722947106t^2\\
&+6484299822t+3370053953\big)-e^t\big(148t^7+11772t^6+373498t^5+6102508t^4+55201018t^3\\
&+275006734t^2+694464365t+680829168\big)-1260\big],
\end{aligned}
$$

$$
\begin{aligned}
R_1^{(8)}(t)=64e^t\big[&41472e^{7t}\big(3359232t^5-12317184t^4+53032320t^3+138242592t^2+159729498t-50186483\big)\\
&-49e^{6t}\big(7710244864t^7-3855122432t^6-78100649984t^5+174125898240t^4+680542513280t^3\\
&+444616334848t^2+108563308084t-218353354669\big)-432e^{5t}\big(1057782159t^7+4126303377t^6\\
&-20070633681t^5-101408163003t^4-92404161927t^3+37281600657t^2+46530173004t\\
&+36524866253\big)-3375e^{4t}\big(75937500t^7+949725000t^6+3668928750t^5+2806953750t^4\\
&-9880130250t^3-16046218500t^2-4954462185t-1961776151\big)-1280e^{3t}\big(22700000t^7\\
&+470630000t^6+3639605000t^5+12891870000t^4+19811703250t^3+6997024055t^2-6828597883t\\
&-984695009\big)-27e^{2t}\big(28242432t^7+876925440t^6+10706084928t^5+65560436640t^4\\
&+211759572816t^3+342182982240t^2+221319245490t+13415116795\big)-432e^t\big(6075t^7+289251t^6\\
&+5527953t^5+54701811t^4+300891753t^3+913030491t^2+1400905866t+826525483\big)-148t^7\\
&-12808t^6-444130t^5-7969998t^4-79611050t^3-440609788t^2-1244477833t-1375293533\big]\\
\triangleq\ &64e^tR_2(t),
\end{aligned}
$$

$$
R_2'(t)=41472e^{7t}\big(23514624t^5-69424128t^4+321957504t^3+1126795104t^2+1394591670t-191575883\big)
$$

$$
\begin{aligned}
&- 98e^{6t}\big(23130734592t^7 + 15420489728t^6 - 245867317248t^5 + 327126069760t^4 + 2389879336320t^3\\
&+ 2354662774464t^2 + 770306259100t - 600778409965\big) - 432e^{5t}\big(5288910795t^7 + 28035991998t^6\\
&- 75595348143t^5 - 607393983420t^4 - 867653461647t^3 - 90804482496t^2 + 307214066334t\\
&+ 229154504269\big) - 3375e^{4t}\big(303750000t^7 + 4330462500t^6 + 20374065000t^5 + 29572458750t^4\\
&- 28292706000t^3 - 93825264750t^2 - 51910285740t - 12801566789\big) - 1280e^{3t}\big(68100000t^7\\
&+ 1570790000t^6 + 13742595000t^5 + 56873635000t^4 + 111002589750t^3 + 80426181915t^2\\
&- 6491745539t - 9782682910\big) - 108e^{2t}\big(14121216t^7 + 487886976t^6 + 6668430624t^5 + 46162824480t^4\\
&+ 171440223048t^3 + 329911170732t^2 + 281751113865t + 62037369770\big) - 432e^{t}\big(6075t^7 + 331776t^6\\
&+ 7263459t^5 + 82341576t^4 + 519698997t^3 + 1815705750t^2 + 3226966848t + 2227431349\big) - 1036t^6\\
&- 76848t^5 - 2220650t^4 - 31879992t^3 - 238833150t^2 - 881219576t - 1244477833,
\end{aligned}
$$

$$
\begin{aligned}
R_2''(t) = &-4\big[1554t^5 + 96060t^4 + 2220650t^3 + 23909994t^2 + 119416575t + 220304894 + 108e^{t}\big(6075t^7\\
&+ 374301t^6 + 9254115t^5 + 118658871t^4 + 849065301t^3 + 3374802741t^2 + 6858378348t + 5454398197\big)\\
&+ 27e^{2t}\big(28242432t^7 + 1074622464t^6 + 16264183104t^5 + 125667802080t^4 + 527531744016t^3\\
&+ 1174143010608t^2 + 1223324569194t + 405825853405\big) + 320e^{3t}\big(204300000t^7 + 5189070000t^6\\
&+ 50652525000t^5 + 239333880000t^4 + 560502309250t^3 + 574286314995t^2 + 141377127213t\\
&- 35839794269\big) + 3375e^{4t}\big(303750000t^7 + 4862025000t^6 + 26869758750t^5 + 55040040000t^4\\
&+ 1279752750t^3 - 115044794250t^2 - 98822918115t - 25779138224\big) + 108e^{5t}\big(26444553975t^7\\
&+ 177202335555t^6 - 209760788727t^5 - 3414946657815t^4 - 6767843241915t^3 - 3056982797421t^2\\
&+ 1354461366678t + 1452986587679\big) + 49e^{6t}\big(69392203776t^7 + 127219040256t^6 - 691340482560t^5\\
&+ 366709916160t^4 + 7823890148480t^3 + 10648807327872t^2 + 4665581551764t - 1417182100345\big)\\
&- 10368e^{7t}\big(164602368t^5 - 368395776t^4 + 1976006016t^3 + 8853438240t^2 + 12015731898t\\
&+ 53560489\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_2^{(3)}(t) = &-12\big[2590t^4 + 128080t^3 + 2220650t^2 + 15939996t + 39805525 + 36e^{t}\big(6075t^7 + 416826t^6\\
&+ 11499921t^5 + 164929446t^4 + 1323700785t^3 + 5921998644t^2 + 13607983830t + 12312776545\big)\\
&+ 36e^{2t}\big(14121216t^7 + 586735488t^6 + 9744025248t^5 + 83164129920t^4 + 389433674088t^3\\
&+ 982720313316t^2 + 1198733789901t + 508744069001\big) + 320e^{3t}\big(204300000t^7 + 5665770000t^6\\
&+ 61030665000t^5 + 323754755000t^4 + 879614149250t^3 + 1134788624245t^2 + 524234670543t\\
&+ 11285914802\big) + 1125e^{4t}\big(1215000000t^7 + 21574350000t^6 + 136651185000t^5\\
&+ 354508953750t^4 + 225279171000t^3 - 456339918750t^2 - 625381260960t - 201939471011\big)\\
&+ 36e^{5t}\big(132222769875t^7 + 1071123555600t^6 + 14410069695t^5 - 18123537232710t^4\\
&- 47499002840835t^3 - 35588443712850t^2 + 658341238548t + 8619394305073\big)\\
&+ 98e^{6t}\big(69392203776t^7 + 208176611328t^6 - 564121442304t^5 - 209407152640t^4 + 8068363425920t^3\\
&+ 14560752402112t^2 + 8215183994388t - 639585175051\big) - 3456e^{7t}\big(1152216576t^5\\
&- 1755758592t^4 + 12358459008t^3 + 67902085728t^2 + 101816999766t + 12390655321\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_2^{(4)}(t) = &-48\big[2590t^3 + 96060t^2 + 1110325t + 3984999 + 9e^{t}\big(6075t^7 + 459351t^6 + 14000877t^5\\
&+ 222429051t^4 + 1983418569t^3 + 9893100999t^2 + 25451981118t + 25920760375\big) + 9e^{2t}\big(28242432t^7\\
&+ 1272319488t^6 + 23008463424t^5 + 215048386080t^4 + 1111523867856t^3 + 3133741648896t^2\\
&+ 4362908206434t + 2216221927903\big) + 80e^{3t}\big(612900000t^7 + 18427410000t^6 + 217086615000t^5\\
&+ 1276417590000t^4 + 3933861467750t^3 + 6043208320485t^2 + 3842281260119t + 558092414949\big)\\
&+ 1125e^{4t}\big(1215000000t^7 + 23700600000t^6 + 169012710000t^5 + 525322935000t^4 + 579788124750t^3\\
&- 287380540500t^2 - 853551220335t - 358284786251\big) + 9e^{5t}\big(661113849375t^7 + 6281177167125t^6\\
&+ 6498791682075t^5 - 90545635815075t^4 - 309989163135015t^3 - 320439227086755t^2
\end{aligned}
$$

$$
\begin{aligned}
&- 67885181232960t + 43755312763913\big) + 49e^{6t}\big(208176611328t^7 + 867402547200t^6 \\
&- 1067834492928t^5 - 2038525063680t^4 + 23786275972480t^3 + 55784802345216t^2 \\
&+ 39206304385276t + 2188836472041\big) - 864e^{7t}\big(8065516032t^5 - 6529227264t^4 \\
&+ 79486178688t^3 + 512389977120t^2 + 848523169818t + 188551587013\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_2^{(5)}(t) = 48\big[&864e^{7t}\big(56458612224t^5 - 5377010688t^4 + 530286341760t^3 + 3825188375904t^2 \\
&+ 6964442142966t + 2168384278909\big) - 9e^{t}\big(6075t^7 + 501876t^6 + 16756983t^5 + 292433436t^4 \\
&+ 2873134773t^3 + 15843356706t^2 + 45238183116t + 51372741493\big) - 36e^{2t}\big(14121216t^7 \\
&+ 685584000t^6 + 13412710944t^5 + 136284772320t^4 + 770810320008t^3 + 2400513725340t^2 \\
&+ 3748324927665t + 2198838015560\big) - 80e^{3t}\big(1838700000t^7 + 59572530000t^6 + 761824305000t^5 \\
&+ 4914685845000t^4 + 16907254763250t^3 + 29931209364705t^2 + 23613260421327t + 5516558504966\big) \\
&- 1125e^{4t}\big(4860000000t^7 + 103307400000t^6 + 818254440000t^5 + 2946355290000t^4 + 4420444239000t^3 \\
&+ 589842212250t^2 - 3988965962340t - 2286690365339\big) - 45e^{5t}\big(661113849375t^7 + 7206736556250t^6 \\
&+ 14036204282625t^5 - 84046844133000t^4 - 382425671787075t^3 - 506432724967764t^2 \\
&- 196060872067662t + 30178276517321\big) - 98e^{6t}\big(624529833984t^7 + 3330825781248t^6 \\
&- 601295837184t^5 - 8785161423360t^4 + 67281777790080t^3 + 203033820994368t^2 \\
&+ 173403715501044t + 26169661608761\big) - 5\big(1554t^2 + 38424t + 222065\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_2^{(6)}(t) = 144\big[&2016e^{7t}\big(56458612224t^5 + 34950569472t^4 + 527213764224t^3 + 4052453950944t^2 \\
&+ 8057353107510t + 3163304585047\big) - 196e^{6t}\big(624529833984t^7 + 4059443920896t^6 \\
&+ 2729529944064t^5 - 9286241287680t^4 + 61425003507840t^3 + 236674709889408t^2 \\
&+ 241081655832500t + 55070280858935\big) - 15e^{5t}\big(3305569246875t^7 + 40661479726875t^6 \\
&+ 113421440750625t^5 - 350053199251875t^4 - 2248315735467375t^3 - 3679440640200045t^2 \\
&- 1993169810273838t - 45169489481057\big) - 1500e^{4t}\big(4860000000t^7 + 111812400000t^6 \\
&+ 973215540000t^5 + 3969173340000t^4 + 7366799529000t^3 + 3905175391500t^2 - 3694044856215t \\
&- 3283931855924\big) - 80e^{3t}\big(1838700000t^7 + 63862830000t^6 + 880969365000t^5 \\
&+ 6184393020000t^4 + 23460169223250t^3 + 46838464127955t^2 + 43567399997797t + 13387645312075\big) \\
&- 12e^{2t}\big(28242432t^7 + 1470016512t^6 + 30938925888t^5 + 339633099360t^4 + 2086759729296t^3 \\
&+ 7113458410704t^2 + 12297677306010t + 8146000958785\big) - 3e^{t}\big(6075t^7 + 544401t^6 + 19768239t^5 \\
&+ 376218351t^4 + 4042868517t^3 + 24462761025t^2 + 76924896528t + 96610924609\big) - 20(259t + 3202)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_2^{(7)}(t) = 144\big[&98784e^{7t}\big(8065516032t^5 + 10754021376t^4 + 78169359744t^3 + 611200386720t^2 + 1316456727642t \\
&+ 616336432711\big) - 392e^{6t}\big(1873589501952t^7 + 14364186181632t^6 + 20366921594880t^5 \\
&- 21034899002880t^4 + 165702527948160t^3 + 802161634929984t^2 + 959919677386908t \\
&+ 285751670493055\big) - 15e^{5t}\big(16527846234375t^7 + 226446383362500t^6 + 811076082114375t^5 \\
&- 1183158792506250t^4 - 12641791474344375t^3 - 25142150407402350t^2 - 17324730331769280t \\
&- 2219017257679123\big) - 1500e^{4t}\big(19440000000t^7 + 481269600000t^6 + 4563736560000t^5 \\
&+ 20742771060000t^4 + 45343891476000t^3 + 37721100153000t^2 - 6965828641860t \\
&- 16829772279911\big) - 80e^{3t}\big(5516100000t^7 + 204459390000t^6 + 3026085075000t^5 \\
&+ 22958025885000t^4 + 95118079749750t^3 + 210895900053615t^2 + 224379128249301t \\
&+ 83730335934022\big) - 48e^{2t}\big(14121216t^7 + 784432512t^6 + 17674487712t^5 + 208490207040t^4 \\
&+ 1383012964008t^3 + 5121799002324t^2 + 9705567858357t + 7147419805895\big) - 3e^{t}\big(6075t^7 \\
&+ 586926t^6 + 23034645t^5 + 475059546t^4 + 5547741921t^3 + 36591366576t^2 + 125850418578t \\
&+ 173535821137\big) - 5180\big],
\end{aligned}
$$

$$\begin{aligned}
R_2^{(8)}(t) &= 432e^t\big[230496e^{6t}\big(8065516032t^5 + 16515104256t^4 + 84314514816t^3 + 644701540896t^2 \\
&\quad + 1491085409562t + 804401679517\big) - 784e^{5t}\big(1873589501952t^7 + 16550040600576t^6 \\
&\quad + 34731107776512t^5 - 4062464340480t^4 + 151679261946240t^3 + 885012898904064t^2 \\
&\quad + 1227306889030236t + 445738283390873\big) - 25e^{4t}\big(16527846234375t^7 + 249585368090625t^6 \\
&\quad + 1082811742149375t^5 - 372082710391875t^4 - 13588318508349375t^3 - 32727225292008975t^2 \\
&\quad - 27381590494730220t - 5683963324032979\big) - 16000e^{3t}\big(2430000000t^7 + 64411200000t^6 \\
&\quad + 660705120000t^5 + 3305930220000t^4 + 8260832817000t^3 + 8966127345000t^2 \\
&\quad + 1486840179330t - 2321403680047\big) - 80e^{2t}\big(5516100000t^7 + 217330290000t^6 + 3435003855000t^5 \\
&\quad + 28001501010000t^4 + 125728780929750t^3 + 306013979803365t^2 + 364976394951711t \\
&\quad + 158523378683789\big) - 16e^t\big(28242432t^7 + 1667713536t^6 + 40055570496t^5 + 505352852640t^4 \\
&\quad + 3599986756176t^3 + 14392636896672t^2 + 29654733721362t + 24000407470147\big) - 6075t^7 \\
&\quad - 629451t^6 - 26556201t^5 - 590232771t^4 - 7447980105t^3 - 53234592339t^2 - 199033151730t \\
&\quad - 299386239715\big] \\
&\triangleq 432e^tR_3(t), \\
R_3'(t) &= 2765952e^{6t}\big(4032758016t^5 + 11618183808t^4 + 47662292160t^3 + 343429399152t^2 + 852992961597t \\
&\quad + 526457957222\big) - 784e^{5t}\big(9367947509760t^7 + 95865329516544t^6 + 272955782486016t^5 \\
&\quad + 153343217180160t^4 + 742146452369280t^3 + 4880102280359040t^2 + 7906560242959308t \\
&\quad + 3455998305984601\big) - 25e^{4t}\big(66111384937500t^7 + 1114036396003125t^6 + 5828759177141250t^5 \\
&\quad + 3925727869179375t^4 - 55841604874965000t^3 - 171673856693084025t^2 - 174980812562938830t \\
&\quad - 50117443790862136\big) - 48000e^{3t}\big(2430000000t^7 + 70081200000t^6 + 789527520000t^5 \\
&\quad + 4407105420000t^4 + 12668739777000t^3 + 17226960162000t^2 + 7464258409330t - 1825790286937\big) \\
&\quad - 80e^{2t}\big(11032200000t^7 + 473273280000t^6 + 8173989450000t^5 + 73178021295000t^4 \\
&\quad + 363463565899500t^3 + 989214302395980t^2 + 1341980749510152t + 682023152319289\big) \\
&\quad - 16e^t\big(28242432t^7 + 1865410560t^6 + 50061851712t^5 + 705630705120t^4 + 5621398166736t^3 \\
&\quad + 25192597165200t^2 + 58440007514706t + 53655141191509\big) - 3\big(14175t^6 + 1258902t^5 + 44260335t^4 \\
&\quad + 786977028t^3 + 7447980105t^2 + 35489728226t + 66344383910\big), \\
R_3''(t) &= -2\big[3\big(42525t^5 + 3147255t^4 + 88520670t^3 + 1180465542t^2 + 7447980105t + 17744864113\big) \\
&\quad + 8e^t\big(28242432t^7 + 2063107584t^6 + 61254315072t^5 + 955939963680t^4 + 8443920987216t^3 \\
&\quad + 42056791665408t^2 + 108825201845106t + 112095148706215\big) + 80e^{2t}\big(11032200000t^7 \\
&\quad + 511885980000t^6 + 9593809290000t^5 + 93612994920000t^4 + 509819608489500t^3 \\
&\quad + 1534409651245230t^2 + 2331195051906132t + 1353013527074365\big) + 24000e^{3t}\big(7290000000t^7 \\
&\quad + 227253600000t^6 + 2789069760000t^5 + 17168953860000t^4 + 55634641011000t^3 + 89687099817000t^2 \\
&\quad + 56846695551990t + 1986887548519\big) + 25e^{4t}\big(132222769875000t^7 + 2459462639287500t^6 \\
&\quad + 14999627542291875t^5 + 22423353681211875t^4 - 103831754011571250t^3 - 427110120698615550t^2 \\
&\quad - 521635481818961685t - 187725293863193687\big) + 392e^{5t}\big(46839737548800t^7 + 544902280151040t^6 \\
&\quad + 1939970889529344t^5 + 2131494998330880t^4 + 4324105130567040t^3 + 26626950758903040t^2 \\
&\quad + 49293005775514620t + 25186551772882313\big) - 4148928e^{6t}\big(8065516032t^5 + 29957630976t^4 \\
&\quad + 110815496064t^3 + 734521090464t^2 + 1934938855962t + 1337246901643\big)\big], \\
R_3^{(3)}(t) &= -2\big[81\big(7875t^4 + 466260t^3 + 9835630t^2 + 87441892t + 275851115\big) + 8e^t\big(28242432t^7 + 2260804608t^6 \\
&\quad + 73632960576t^5 + 1262211539040t^4 + 12267680841936t^3 + 67388554627056t^2 + 192938785175922t \\
&\quad + 220920350551321\big) + 160e^{2t}\big(11032200000t^7 + 550498680000t^6 + 11129467230000t^5 \\
&\quad + 117597518145000t^4 + 697045598329500t^3 + 2299139063979480t^2 + 3865604703151362t
\end{aligned}$$

$$+ 2518611053027431\big) + 648000e^{3t}\big(810000000t^7 + 27140400000t^6 + 360397440000t^5 + 2424155940000t^4 + 8725175499000t^3 + 16146860092000t^2 + 12959788381110t + 2326198451761\big) + 25e^{4t}\big(528891079500000t^7 + 10763409946275000t^6 + 74755286004892500t^5 + 164691552436306875t^4 - 325633601321437500t^3 - 2019935744829175950t^2 - 2940762168673077840t - 1272536657271736433\big) + 1960e^{5t}\big(46839737548800t^7 + 610477912719360t^6 + 2593853625710592t^5 + 4071465887860224t^4 + 6029301129231744t^3 + 29221413837243264t^2 + 59943786079075836t + 35045152927985237\big) - 49787136e^{6t}\big(4032758016t^5 + 18339447168t^4 + 65393625024t^3 + 394964419248t^2 + 1089889609725t + 829868355485\big)\big],$$

$$R_3^{(4)}(t) = -8\big[81\big(7875t^3 + 349695t^2 + 4917815t + 21860473\big) + 2e^t\big(28242432t^7 + 2458501632t^6 + 87197788224t^5 + 1630376341920t^4 + 17316526998096t^3 + 104191597152864t^2 + 327715894430034t + 413859135727243\big) + 160e^{2t}\big(5516100000t^7 + 294555690000t^6 + 6390481635000t^5 + 72710593110000t^4 + 466120317309750t^3 + 1672353730736865t^2 + 3082371883565421t + 2225706702301556\big) + 162000e^{3t}\big(2430000000t^7 + 87091200000t^6 + 1244034720000t^5 + 9074455020000t^4 + 35872150257000t^3 + 74616106773000t^2 + 71173085327330t + 19938383736393\big) + 25e^{4t}\big(528891079500000t^7 + 11688969335400000t^6 + 90900400924305000t^5 + 258135659942422500t^4 - 160942048885130625t^3 - 2264160945820254075t^2 - 3950730041087665815t - 2007727199440005893\big) + 490e^{5t}\big(234198687744000t^7 + 3380267726438400t^6 + 16632135604869120t^5 + 33326597567854080t^4 + 46432369197599616t^3 + 164194972573911552t^2 + 358161758069865708t + 235169550719002021\big) - 37340352e^{6t}\big(8065516032t^5 + 43400157696t^4 + 155239846272t^3 + 855322463520t^2 + 2443088832282t + 2023033247545\big)\big],$$

$$R_3^{(5)}(t) = 8\big[448084224e^{6t}\big(4032758016t^5 + 25060710528t^4 + 92086642368t^3 + 466471193328t^2 + 1364098160061t + 1215107359796\big) - 490e^{5t}\big(1170993438720000t^7 + 18540729446400000t^6 + 103442284382976000t^5 + 249793665863616000t^4 + 365468236259414400t^3 + 960271970462356608t^2 + 2119198735497151644t + 1534009511664875813\big) - 25e^{4t}\big(2115564318000000t^7 + 50458114898100000t^6 + 433735419709620000t^5 + 1487044644391215000t^4 + 388774444229167500t^3 - 9539469929936408175t^2 - 20331242055991171410t - 11981638838847689387\big) - 162000e^{3t}\big(7290000000t^7 + 278283600000t^6 + 4254651360000t^5 + 33443538660000t^4 + 143914270851000t^3 + 331464771090000t^2 + 362751469527990t + 130988236536509\big) - 160e^{2t}\big(11032200000t^7 + 627724080000t^6 + 14548297410000t^5 + 177373594395000t^4 + 1223083007059500t^3 + 4743068413402980t^2 + 9509451228604572t + 7533785288168533\big) - 2e^t\big(28242432t^7 + 2656198656t^6 + 101948798016t^5 + 2066365283040t^4 + 23838032365776t^3 + 156141178147152t^2 + 536099088735762t + 741575030157277\big) - 405\big(4725t^2 + 139878t + 983563\big)\big],$$

$$R_3^{(6)}(t) = 16\big[672126336e^{6t}\big(8065516032t^5 + 56842684416t^4 + 217587565440t^3 + 1025029029024t^2 + 3039177115674t + 2884914106279\big) - 245e^{5t}\big(5854967193600000t^7 + 100900601303040000t^6 + 628455798593280000t^5 + 1766179751232960000t^4 + 2826515844751536000t^3 + 5897764561090026240t^2 + 12516537618410471436t + 9789246293821530709\big) - 25e^{4t}\big(4231128636000000t^7 + 108320704909200000t^6 + 1018845184113540000t^5 + 4058427838056480000t^4 + 3751638177240765000t^3 - 18495778193529065100t^2$$

$$
\begin{aligned}
&- 5020195404191875099 5t - 34128898705690964479\big) - 729000e^{3t}\big(2430000000t^7\\
&+ 98431200000t^6 + 1603739520000t^5 + 13511541420000t^4 + 62835218577000t^3\\
&+ 158459680647000t^2 + 194575994529330t + 83968464348613\big) - 160e^{2t}\big(11032200000t^7\\
&+ 666336780000t^6 + 16431469650000t^5 + 213744337920000t^4 + 1577830195849500t^3\\
&+ 6577692923992230t^2 + 14252519642007552t + 12288510902470819\big) - e^t\big(28242432t^7\\
&+ 2853895680t^6 + 117885989952t^5 + 2576109273120t^4 + 32103493497936t^3\\
&+ 227655275244480t^2 + 848381445030066t + 1277674118893039\big) - 3645(525t + 7771)\big],\\
R_3^{(7)}(t) =\ &16\big[8065516032e^{6t}\big(4032758016t^5 + 31781973888t^4 + 127741344192t^3 + 566911405872t^2\\
&+ 1690426729341t + 1695721812779\big) - 245e^{5t}\big(29274835968000000t^7 + 545487776870400000t^6\\
&+ 3747682600784640000t^5 + 11973177749131200000t^4 + 21197298228689520000t^3\\
&+ 37968370339704739200t^2 + 74378217214232409660t + 61462769087518124981\big)\\
&- 25e^{4t}\big(16924514544000000t^7 + 462900720088800000t^6 + 4725304965909360000t^5\\
&+ 21327937272793620000t^4 + 31240264061188980000t^3 - 62728198242393965400t^2\\
&- 237799372554733134180t - 186717548864682608911\big) - 2187000e^{3t}\big(2430000000t^7\\
&+ 104101200000t^6 + 1800601920000t^5 + 16184440620000t^4 + 80850607137000t^3\\
&+ 221294899224000t^2 + 300215781627330t + 148827129191723\big) - 320e^{2t}\big(11032200000t^7\\
&+ 704949480000t^6 + 18430479990000t^5 + 254823012045000t^4 + 2005318871689500t^3\\
&+ 8944438217766480t^2 + 20830212565999782t + 19414770723474595\big) - e^t\big(28242432t^7\\
&+ 3051592704t^6 + 135009364032t^5 + 3165539222880t^4 + 42407930590416t^3\\
&+ 323965755738288t^2 + 1303691995519026t + 2126055563923105\big) - 1913625\big],\\
R_3^{(8)}(t) =\ &16e^t\big[24196548096e^{5t}\big(8065516032t^5 + 70285211136t^4 + 297858653568t^3 + 1261564155936t^2\\
&+ 3758794395930t + 3954919202005\big) - 1225e^{4t}\big(29274835968000000t^7 + 586472547225600000t^6\\
&+ 4402267933029120000t^5 + 15720860349915840000t^4 + 30775840427994480000t^3\\
&+ 50686749276918451200t^2 + 89565565350114305340t + 76338412530364606913\big)\\
&- 800e^{3t}\big(2115564318000000t^7 + 61564827567600000t^6 + 677457005755320000t^5\\
&+ 3404321060022540000t^4 + 6571025166747825000t^3 - 4912250024562778800t^2\\
&- 33645433959491264610t - 30770924000420736557\big) - 19683000e^{2t}\big(810000000t^7\\
&+ 36590400000t^6 + 669601440000t^5 + 6395147940000t^4 + 34143287099000t^3\\
&+ 100715168787000t^2 + 149248571481110t + 82966352133611\big) - 1280e^t\big(5516100000t^7\\
&+ 371781090000t^6 + 10272664215000t^5 + 150449606010000t^4 + 1257482447889750t^3\\
&+ 5976208262650365t^2 + 14887325391883131t + 14914938503237243\big) - 28242432t^7\\
&- 3249289728t^6 - 153318920256t^5 - 3840586043040t^4 - 55070087481936t^3\\
&- 451189547509536t^2 - 1951623506995602t - 3429747559442131\big]\\
\triangleq\ &16e^tR_4(t),\\
R_4'(t) =\ &-2\big[98848512t^6 + 9747869184t^5 + 383297300640t^4 + 7681172086080t^3 + 82605131222904t^2\\
&+ 451189547509536t + 975811753497801 + 640e^t\big(5516100000t^7 + 410393790000t^6\\
&+ 12503350755000t^5 + 201812927085000t^4 + 1859280871929750t^3 + 9748655606319615t^2\\
&+ 26839741917183861t + 29802263895120374\big) + 39366000e^{2t}\big(405000000t^7 + 19712700000t^6\\
&+ 389686320000t^5 + 4034575770000t^4 + 23466791489500t^3 + 75965049717750t^2\\
&+ 124981870134055t + 78795318937083\big) + 1200e^{3t}\big(2115564318000000t^7 + 66501144309600000t^6\\
&+ 800586660890520000t^5 + 4533416069614740000t^4 + 11110119913444545000t^3\\
&+ 1658775142185046200t^2 - 36920267309199783810t - 41986068653584491427\big)
\end{aligned}
$$

$$\begin{aligned}
&+ 9800e^{4t}\big(7318708992000000t^7 + 159425877542400000t^6 + 1320494188466880000t^5\\
&+ 5305923816550560000t^4 + 11624175194477580000t^3 + 18442157399478577800t^2\\
&+ 28727234997143382735t + 24682450966973295812\big) - 12098274048e^{5t}\big(40327580160t^5\\
&+ 391753635840t^4 + 1770434112384t^3 + 7201396740384t^2 + 21317100291522t + 23533390405955\big)\big],
\end{aligned}$$

$$\begin{aligned}
R_4''(t) = -16\big[&18\big(4118688t^5 + 338467680t^4 + 10647147240t^3 + 160024418460t^2 + 1147293489207t\\
&+ 3133260746594\big) + 80e^{t}\big(5516100000t^7 + 449006490000t^6 + 14965713495000t^5\\
&+ 264329680860000t^4 + 2666532580269750t^3 + 15326498222108865t^2 + 46337053129823091t\\
&+ 56642005812304235\big) + 4920750e^{2t}\big(810000000t^7 + 42260400000t^6 + 897648840000t^5\\
&+ 10017583140000t^4 + 63071886059000t^3 + 222330473904000t^2 + 401893839703610t\\
&+ 282572508008221\big) + 450e^{3t}\big(2115564318000000t^7 + 71437461051600000t^6\\
&+ 933588949509720000t^5 + 5867727171098940000t^4 + 17154674672930865000t^3\\
&+ 12768895055629591200t^2 - 35814417214409753010t - 54292824423317752697\big)\\
&+ 1225e^{4t}\big(29274835968000000t^7 + 688934473113600000t^6 + 6238532019121920000t^5\\
&+ 27826166208536640000t^4 + 67720396044112560000t^3 + 108641155181347051200t^2\\
&+ 151793254787530686540t + 127457038865036565983\big) - 1512284256e^{5t}\big(201637900800t^5\\
&+ 2160406080000t^4 + 10419185105280t^3 + 41318286039072t^2 + 120988294938378t\\
&+ 138984052321297\big)\big],
\end{aligned}$$

$$\begin{aligned}
R_4^{(3)}(t) = -32\big[&27\big(6864480t^4 + 451290240t^3 + 10647147240t^2 + 106682945640t + 382431163069\big)\\
&+ 40e^{t}\big(5516100000t^7 + 487619190000t^6 + 17659752435000t^5 + 339158248335000t^4\\
&+ 3723851303709750t^3 + 23326095962918115t^2 + 76990049574040821t + 102979058942127326\big)\\
&+ 9841500e^{2t}\big(405000000t^7 + 22547700000t^6 + 512215020000t^5 + 6130852620000t^4\\
&+ 41553526169500t^3 + 158469151496250t^2 + 312112156803805t + 241759713930013\big)\\
&+ 675e^{3t}\big(2115564318000000t^7 + 76373777793600000t^6 + 1076463871612920000t^5\\
&+ 7423708753615140000t^4 + 24978310901062785000t^3 + 29923569728560456200t^2\\
&- 27301820510656692210t - 66230963494787670367\big) + 4900e^{4t}\big(14637417984000000t^7\\
&+ 370082718028800000t^6 + 3635966864396160000t^5 + 17812165616219520000t^4\\
&+ 47773281126324600000t^3 + 79715726107215735600t^2 + 103056916189102106070t\\
&+ 82702676280959618809\big) - 756142128e^{5t}\big(1008189504000t^5 + 11810219904000t^4\\
&+ 60737549846400t^3 + 237848985511200t^2 + 687578046770034t + 815908556544863\big)\big],
\end{aligned}$$

$$\begin{aligned}
R_4^{(4)}(t) = -32\big[&3240\big(228816t^3 + 11282256t^2 + 177452454t + 889024547\big) + 40e^{t}\big(5516100000t^7\\
&+ 526231890000t^6 + 20585467575000t^5 + 427457010510000t^4 + 5080484297049750t^3\\
&+ 34497649874047365t^2 + 123642241499877051t + 179969108516168147\big)\\
&+ 9841500e^{2t}\big(810000000t^7 + 47930400000t^6 + 1159716240000t^5 + 14822780340000t^4\\
&+ 107630462819000t^3 + 441598881501000t^2 + 941162616600110t + 795631584663831\big)\\
&+ 2025e^{3t}\big(2115564318000000t^7 + 81310094535600000t^6 + 1229211427200120000t^5\\
&+ 9217815206303340000t^4 + 34876589239216305000t^3 + 54901880629623241200t^2\\
&- 7352774024949721410t - 75331570331673234437\big) + 9800e^{4t}\big(29274835968000000t^7\\
&+ 791396399001600000t^6 + 8382181882878720000t^5 + 44714248393429440000t^4\\
&+ 131170893485088240000t^3 + 231091373903918371200t^2 + 285829558485419947740t\\
&+ 216933810656470290653\big) - 756142128e^{5t}\big(5040947520000t^5 + 64092047040000t^4\\
&+ 350928628848000t^3 + 1371457577095200t^2 + 3913588204872570t + 4767120829494349\big)\big],
\end{aligned}$$

$$R_4^{(5)}(t) = 160\big[756142128e^{5t}\big(5040947520000t^5 + 69132994560000t^4 + 402202266480000t^3$$

$$+ 1582014754404000t^2 + 4462171235710650t + 5549838470468863\big)$$
$$- 31360e^{4t}\big(7318708992000000t^7 + 210656840486400000t^6 + 2392319120345280000t^5$$
$$+ 13797993936756960000t^4 + 43971285469629420000t^3 + 82367386004433637800t^2$$
$$+ 100343811359344783335t + 72097800069456319397\big) - 1215e^{3t}\big(2115564318000000t^7$$
$$+ 86246411277600000t^6 + 1391831616271320000t^5 + 11266500918303540000t^4$$
$$+ 47167009514287425000t^3 + 89778469868839546200t^2 + 29248479728132439390t$$
$$- 77782495006656474907\big) - 7873200e^{2t}\big(405000000t^7 + 25382700000t^6 + 651753720000t^5$$
$$+ 8861035470000t^4 + 68638011749500t^3 + 301522287864750t^2 + 691380749050555t$$
$$+ 633106446481943\big) - 8e^{t}\big(5516100000t^7 + 564844590000t^6 + 23742858915000t^5$$
$$+ 530384348385000t^4 + 6790312339089750t^3 + 49739102765196615t^2 + 192637541247971781t$$
$$+ 303611350016045198\big) - 3888\big(114408t^2 + 3760752t + 29575409\big)\big],$$

$$R_4^{(6)}(t) = 160\big[3780710640e^{5t}\big(5040947520000t^5 + 74173942080000t^4 + 457508662128000t^3$$
$$+ 1823336114292000t^2 + 5094977137472250t + 6442272717610993\big)$$
$$- 31360e^{4t}\big(29274835968000000t^7 + 893858324889600000t^6 + 10833217524299520000t^5$$
$$+ 67153571348754240000t^4 + 231077117625545520000t^3 + 461383400426622811200t^2$$
$$+ 566110017446246408940t + 388735011637170060923\big) - 3645e^{3t}\big(2115564318000000t^7$$
$$+ 91182728019600000t^6 + 1564324438826520000t^5 + 13586220278755740000t^4$$
$$+ 62189010738692145000t^3 + 136945479383126971200t^2 + 89100792974025470190t$$
$$- 68033001763945661777\big) - 7873200e^{2t}\big(810000000t^7 + 53600400000t^6 + 1455803640000t^5$$
$$+ 20980839540000t^4 + 172720165379000t^3 + 808958610978000t^2 + 1985806073830610t$$
$$+ 1957593642014441\big) - 8e^{t}\big(5516100000t^7 + 603457290000t^6 + 27131926455000t^5$$
$$+ 649098642960000t^4 + 8911849732629750t^3 + 70110039782465865t^2 + 292115746778365011t$$
$$+ 496248891264016979\big) - 186624(4767t + 78349)\big],$$

$$R_4^{(7)}(t) = 160\big[18903553200e^{5t}\big(5040947520000t^5 + 79214889600000t^4 + 516847815792000t^3$$
$$+ 2097841311568800t^2 + 5824311583189050t + 7461268145105443\big)$$
$$- 250880e^{4t}\big(14637417984000000t^7 + 472544643916800000t^6 + 6087002505816960000t^5$$
$$+ 40347546627064320000t^4 + 149115344487149880000t^3 + 317345619322890975600t^2$$
$$+ 398400858829778907270t + 265131257999365831579\big) - 10935e^{3t}\big(2115564318000000t^7$$
$$+ 96119044761600000t^6 + 1746689894865720000t^5 + 16193427676799940000t^4$$
$$+ 80303971110366465000t^3 + 199134490121819116200t^2 + 180397779229443450990t$$
$$- 38332737439270505047\big) - 31492800e^{2t}\big(405000000t^7 + 28217700000t^6 + 808302420000t^5$$
$$+ 12310174320000t^4 + 107340922229500t^3 + 534019429523250t^2 + 1397382342404305t$$
$$+ 1475248339464873\big) - 8e^{t}\big(5516100000t^7 + 642069990000t^6 + 30752670195000t^5$$
$$+ 784758275235000t^4 + 11508244304469750t^3 + 96845588980355115t^2 + 432335826343296741t$$
$$+ 788364638042381990\big) - 889636608\big],$$

$$R_4^{(8)}(t) = 160e^{t}\big[94517766000e^{4t}\big(5040947520000t^5 + 84255837120000t^4 + 580219727472000t^3$$
$$+ 2407950001044000t^2 + 6663448107816570t + 8626130461743253\big)$$
$$- 501760e^{3t}\big(29274835968000000t^7 + 996320250777600000t^6 + 13591638943384320000t^5$$
$$+ 95912599518671040000t^4 + 378925782228428400000t^3 + 858364255376506771200t^2$$
$$+ 1114147336982448790140t + 729462945413621116793\big) - 98415e^{2t}\big(705188106000000t^7$$
$$+ 33685120501200000t^6 + 646309328129640000t^5 + 6368192500525380000t^4$$
$$+ 33965069337588795000t^3 + 93146153744061860400t^2 + 104384701992440953930t$$

$$
\begin{aligned}
&+7266618545736881761\big)-31492800e^{t}\big(810000000t^{7}+59270400000t^{6}+1785911040000t^{5}\\
&+28661860740000t^{4}+263922541739000t^{3}+1390061625735000t^{2}+3862803543855110t\\
&+4347879021334051\big)-8\big(5516100000t^{7}+680682690000t^{6}+34605090135000t^{5}\\
&+938521626210000t^{4}+14647277405409750t^{3}+131370321893764365t^{2}+626027004304006971t\\
&+1220700464385678731\big)\big]\\
&\triangleq 160e^{t}R_5(t),
\end{aligned}
$$

$$
\begin{aligned}
R_5'(t)=12\big[&15752961000e^{4t}\big(10081895040000t^{5}+181114043040000t^{4}+1328951129184000t^{3}\\
&+5686229593296000t^{2}+15734846216677140t+20583984977394791\big)\\
&-125440e^{3t}\big(29274835968000000t^{7}+1064628201369600000t^{6}+15584279444939520000t^{5}\\
&+118565331090978240000t^{4}+506809248253323120000t^{3}+1237290037604935171200t^{2}\\
&+1686390173900119970940t+1100845391074437380173\big)-32805e^{2t}\big(352594053000000t^{7}\\
&+18076639436100000t^{6}+373682344816620000t^{5}+3991982910424740000t^{4}\\
&+23350727169319777500t^{3}+72046878875222526450t^{2}+98765427868251407165t\\
&+29729484770978679363\big)-2624400e^{t}\big(810000000t^{7}+64940400000t^{6}+2141533440000t^{5}\\
&+37591415940000t^{4}+378569984699000t^{3}+2181829250952000t^{2}+6642926795325110t\\
&+8210682565189161\big)-2\big(12870900000t^{6}+1361365380000t^{5}+57675150225000t^{4}\\
&+1251362168280000t^{3}+14647277405409750t^{2}+87580214595842910t+208675668101335657\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_5''(t)=180\big[&16803158400e^{4t}\big(2520473760000t^{5}+48429102960000t^{4}+377516293056000t^{3}\\
&+1670735735046000t^{2}+4644490253331285t+6129424132891019\big)\\
&-25088e^{3t}\big(29274835968000000t^{7}+1132936151961600000t^{6}+17713535847678720000t^{5}\\
&+144539130165877440000t^{4}+664896356374627440000t^{3}+1744099285858258291200t^{2}\\
&+2511250198970076751740t+1662975449041144037153\big)-2187e^{2t}\big(705188106000000t^{7}\\
&+38621437243200000t^{6}+855824526249840000t^{5}+9852377544932580000t^{4}\\
&+62669385980338515000t^{3}+214145939258404385400t^{2}+341624613486947867230t\\
&+158224397410208765891\big)-174960e^{t}\big(810000000t^{7}+70610400000t^{6}+2531175840000t^{5}\\
&+48299083140000t^{4}+528935648459000t^{3}+3317539205049000t^{2}+11006585297229110t\\
&+14853609360514271\big)-4\big(2574180000t^{5}+226894230000t^{4}+7690020030000t^{3}\\
&+125136216828000t^{2}+976485160360650t+2919340486528097\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_5^{(3)}(t)=2160\big[&1400263200e^{4t}\big(10081895040000t^{5}+206318780640000t^{4}+1703781584064000t^{3}\\
&+7815491819352000t^{2}+21919432483417140t+29162186784895361\big)\\
&-6272e^{3t}\big(29274835968000000t^{7}+1201244102553600000t^{6}+19979408151601920000t^{5}\\
&+174061689912008640000t^{4}+857615196595797360000t^{3}+2408995642232885731200t^{2}\\
&+3673983056208915612540t+2500058848697836287733\big)-729e^{2t}\big(352594053000000t^{7}\\
&+20544797807100000t^{6}+485844418989720000t^{5}+5995969430278590000t^{4}\\
&+41187070535101837500t^{3}+154075009114456078950t^{2}+277885276372676126315t\\
&+164518352076841349753\big)-14580e^{t}\big(810000000t^{7}+76280400000t^{6}+2954838240000t^{5}\\
&+60954962340000t^{4}+722131981019000t^{3}+4904346150426000t^{2}+17641663707327110t\\
&+25860194657743381\big)-50\big(85806000t^{4}+6050512800t^{3}+153800400600t^{2}\\
&+1668482891040t+6509901069071\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_5^{(4)}(t)=6480\big[&3734035200e^{4t}\big(5040947520000t^{5}+109460574720000t^{4}+955050182352000t^{3}\\
&+4546664003700000t^{2}+12913589196546570t+17321022452874823\big)\\
&-6272e^{3t}\big(29274835968000000t^{7}+1269552053145600000t^{6}+22381896356709120000t^{5}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2073607034980118400000t^4 + 10896974498118088880000t^3 + 32666108388286830912000t^2 \\
&+ 52799801510308394333400t + 37247198674341414919130\big) - 243e^{2t}\big(705188106000000t^7 \\
&+ 43557753985200000t^6 + 1094957624822040000t^5 + 14421160955505780000t^4 \\
&+ 106358018791318035000t^3 + 431711229834217670400t^2 + 863920570974264410530t \\
&+ 606921980526358825821\big) - 4860e^{t}\big(810000000t^7 + 81950400000t^6 + 3412520640000t^5 \\
&+ 75729153540000t^4 + 965951830379000t^3 + 7070742093483000t^2 + 27450356008179110t \\
&+ 43501858365070491\big) - 4000\big(1430100t^3 + 75631410t^2 + 1281670005t + 6952012046\big)\big], \\
R_5^{(5)}(t) &= 77760\big[622339200e^{4t}\big(10081895040000t^5 + 231523518240000t^4 + 2129021514144000t^3 \\
&+ 10525903280928000t^2 + 30373842396793140t + 41098839504022931\big) \\
&- 1568e^{3t}\big(29274835968000000t^7 + 1337860003737600000t^6 + 24921000463000320000t^5 \\
&+ 244663864092527040000t^4 + 1366178387809158000000t^3 + 4356308288640491971200t^2 \\
&+ 7457720710249961494140t + 5484713251111087969693\big) - 81e^{2t}\big(352594053000000t^7 \\
&+ 23012956178100000t^6 + 612815443388820000t^5 + 8579277508780440000t^4 \\
&+ 67600170351164797500t^3 + 295624129010597361450t^2 + 647815900404241040465t \\
&+ 519441133006745515543\big) - 405e^{t}\big(810000000t^7 + 87620400000t^6 + 3904223040000t^5 \\
&+ 92791756740000t^4 + 1268868444539000t^3 + 9968597584620000t^2 + 41591840195145110t \\
&+ 70952214373249601\big) - 5000\big(286020t^2 + 10084188t + 85444667\big)\big], \\
R_5^{(6)}(t) &= 233280\big[3319142400e^{4t}\big(2520473760000t^5 + 61031471760000t^4 + 590136258096000t^3 \\
&+ 3030667354134000t^2 + 8909198509314285t + 12173075025805304\big) \\
&- 1568e^{3t}\big(29274835968000000t^7 + 1406167954329600000t^6 + 27596720470475520000t^5 \\
&+ 286198864864194240000t^4 + 1692396873265860720000t^3 + 5722486676449649971200t^2 \\
&+ 10361926236010289474940t + 7970620154527741801073\big) - 27e^{2t}\big(705188106000000t^7 \\
&+ 48494070727200000t^6 + 1363708623846240000t^5 + 20222632234504980000t^4 \\
&+ 169517450737451355000t^3 + 794048769074689115400t^2 + 1886880058829676803830t \\
&+ 1686698166417732071551\big) - 135e^{t}\big(810000000t^7 + 93290400000t^6 + 4429945440000t^5 \\
&+ 112312871940000t^4 + 1640035471499000t^3 + 13775202918237000t^2 + 61529035364385110t \\
&+ 112544054568394711\big) - 20000(47670t + 840349)\big], \\
R_5^{(7)}(t) &= 699840\big[1106380800e^{4t}\big(10081895040000t^5 + 256728255840000t^4 + 2604670919424000t^3 \\
&+ 13893078190824000t^2 + 41698128745525140t + 57601498612535501\big) \\
&- 1568e^{3t}\big(29274835968000000t^7 + 1474475904921600000t^6 + 30409056379134720000t^5 \\
&+ 332193398981653440000t^4 + 2073995359751453040000t^3 + 7414883549715510691200t^2 \\
&+ 14176917353643389455740t + 11424595566531171626053\big) - 36e^{2t}\big(352594053000000t^7 \\
&+ 25481114549100000t^6 + 754595418013920000t^5 + 11815951897060290000t^4 \\
&+ 104981357603230657500t^3 + 524162472590433073950t^2 + 1340464413952182959615t \\
&+ 1315069097916285236733\big) - 45e^{t}\big(810000000t^7 + 98960400000t^6 + 4989687840000t^5 \\
&+ 134462599140000t^4 + 2089286959259000t^3 + 18695309332734000t^2 + 89079441200859110t \\
&+ 174073089932779821\big) - 317800000\big], \\
R_5^{(8)}(t) &= 6298560e^{t}\big[983449600e^{3t}\big(5040947520000t^5 + 134665312320000t^4 + 1430699587632000t^3 \\
&+ 7923290690196000t^2 + 24322333920468570t + 34013015399458393\big) \\
&- 1568e^{2t}\big(9758278656000000t^7 + 514261285171200000t^6 + 11119336062992640000t^5 \\
&+ 127625053204514880000t^4 + 838973297242330320000t^3 + 3162959636488987910400t^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 6373391017817909972180t + 538341156147076714821 1\big) - 4e^t\big(705188106000000t^7 \\
&+ 53430387469200000t^6 + 1662077523322440000t^5 + 27404880884190180000t^4 \\
&+ 257226522794702475000t^3 + 1363269017990558120400t^2 + 3729253773085232067130t \\
&+ 3970602609784753433081\big) - 5\big(810000000t^7 + 104630400000t^6 + 5583450240000t^5 \\
&+ 159411038340000t^4 + 2627137355819000t^3 + 24963170210511000t^2 + 126470059866327110t \\
&+ 263152531133638931\big)\big] \\
&\triangleq 6298560e^t R_6(t), \\
R_6'(t) &= 2\big[1475174400e^{3t}\big(5040947520000t^5 + 143066891520000t^4 + 1610253337392000t^3 \\
&+ 9353990277828000t^2 + 29604527713932570t + 42120460039614583\big) \\
&- 1568e^{2t}\big(9758278656000000t^7 + 548415260467200000t^6 + 12662119918506240000t^5 \\
&+ 155423393361996480000t^4 + 1094223403651360080000t^3 + 4421419582352483390400t^2 \\
&+ 9536350654306897882580t + 8570107070379722134301\big) - 2e^t\big(705188106000000t^7 \\
&+ 58366704211200000t^6 + 1982659848137640000t^5 + 35715268500802380000t^4 \\
&+ 366846046331463195000t^3 + 2134948586374665545400t^2 + 6455791809066348307930t \\
&+ 7699856382869985500211\big) - 25\big(567000000t^6 + 62778240000t^5 + 2791725120000t^4 \\
&+ 63764415336000t^3 + 788141206745700t^2 + 4992634042102200t + 12647005986632711\big)\big], \\
R_6''(t) &= 4\big[2212761600e^{3t}\big(5040947520000t^5 + 151468470720000t^4 + 1801009192752000t^3 \\
&+ 10964243615220000t^2 + 35840521232484570t + 51988635944258773\big) \\
&- 1568e^{2t}\big(9758278656000000t^7 + 582569235763200000t^6 + 14307365699907840000t^5 \\
&+ 187078693158262080000t^4 + 1405070190375353040000t^3 + 6062754687829523510400t^2 \\
&+ 13957770236659381272980t + 13338282397533171075591\big) - e^t\big(705188106000000t^7 \\
&+ 63303020953200000t^6 + 2332860073404840000t^5 + 45628567741490580000t^4 \\
&+ 509707120334672715000t^3 + 3235486725369055130400t^2 + 10725688981815679398730t \\
&+ 14155648191936333808141\big) - 7500\big(5670000t^5 + 523152000t^4 + 18611500800t^3 \\
&+ 318822076680t^2 + 2627137355819t + 8321056736837\big)\big], \\
R_6^{(3)}(t) &= 4\big[6638284800e^{3t}\big(5040947520000t^5 + 159870049920000t^4 + 2002967153712000t^3 \\
&+ 12765252807972000t^2 + 43150016975964570t + 63935476355086963\big) \\
&- 3136e^{2t}\big(9758278656000000t^7 + 616723211059200000t^6 + 16055073407197440000t^5 \\
&+ 222847107408031680000t^4 + 1779227576691877200000t^3 + 8170359973392553070400t^2 \\
&+ 20020524924488904783380t + 20317167515862861712081\big) - e^t\big(705188106000000t^7 \\
&+ 68239337695200000t^6 + 2712678199124040000t^5 + 57292868108514780000t^4 \\
&+ 692221391300635035000t^3 + 4764608086373073275400t^2 + 17196662432553789659530t \\
&+ 24881337173752013206871\big) - 7500\big(28350000t^4 + 2092608000t^3 + 55834502400t^2 \\
&+ 637644153360t + 2627137355819\big)\big], \\
R_6^{(4)}(t) &= 4\big[19914854400e^{3t}\big(5040947520000t^5 + 168271629120000t^4 + 2216127220272000t^3 \\
&+ 14768219961684000t^2 + 51660185514612570t + 78318815347075153\big) \\
&- 6272e^{2t}\big(9758278656000000t^7 + 650877186355200000t^6 + 17905243040375040000t^5 \\
&+ 262984790926025280000t^4 + 2224921791507940560000t^3 + 10839201338430368870400t^2 \\
&+ 28190884897881457853780t + 30327429978107314103771\big) - e^t\big(705188106000000t^7 \\
&+ 73175654437200000t^6 + 3122114225295240000t^5 + 70856259104134980000t^4 \\
&+ 921392863734694155000t^3 + 6841272260274978380400t^2 + 26725878605299936210330t \\
&+ 42077999606305802866401\big) - 1800000\big(472500t^3 + 26157600t^2 + 465287520t + 2656850639\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
R_6^{(5)}(t) = 4\big[&59744563200e^{3t}\big(5040947520000t^5 + 176673208320000t^4 + 2440489392432000t^3\\
&+ 16984347181956000t^2 + 61505665489068570t + 95538877185279343\big)\\
&- 12544e^{2t}\big(9758278656000000t^7 + 685031161651200000t^6 + 19857874599440640000t^5\\
&+ 307747898526962880000t^4 + 2750891373359991120000t^3 + 14176584025692279710400t^2\\
&+ 39030086236311826724180t + 44422872427048043030661\big) - e^t\big(705188106000000t^7\\
&+ 78111971179200000t^6 + 3561168151918440000t^5 + 86466830230611180000t^4\\
&+ 1204817900151234075000t^3 + 9605450851479060845400t^2 + 40408423125849892971130t\\
&+ 68803878211605739076731\big) - 108000000\big(23625t^2 + 871920t + 7754792\big)\big],\\
R_6^{(6)}(t) = 4\big[&179233689600e^{3t}\big(5040947520000t^5 + 185074787520000t^4 + 2676053670192000t^3\\
&+ 19424836574388000t^2 + 72828563610372570t + 116040765681635533\big)\\
&- 25088e^{2t}\big(9758278656000000t^7 + 719185136947200000t^6 + 21912968084394240000t^5\\
&+ 357392585025564480000t^4 + 3366387170413916880000t^3 + 18302921085732266390400t^2\\
&+ 53206670262004106434580t + 63937915545203956392751\big) - e^t\big(705188106000000t^7\\
&+ 83048287921200000t^6 + 4029839978993640000t^5 + 104272670990203380000t^4\\
&+ 1550685221073678795000t^3 + 13219904551932763070400t^2 + 59619324828808014661930t\\
&+ 109212301337455632047861\big) - 68040000000(75t + 1384)\big],\\
R_6^{(7)}(t) = 4\big[&537701068800e^{3t}\big(5040947520000t^5 + 193476366720000t^4 + 2922820053552000t^3\\
&+ 22100890244580000t^2 + 85778454659964570t + 140316953551759723\big)\\
&- 50176e^{2t}\big(9758278656000000t^7 + 753339112243200000t^6 + 24070523495235840000t^5\\
&+ 412175005236550080000t^4 + 4081172340465045840000t^3 + 23352501841353141710400t^2\\
&+ 71509591347736372824980t + 90541250676206009610041\big) - e^t\big(705188106000000t^7\\
&+ 87984604663200000t^6 + 4528129706520840000t^5 + 124421870885171580000t^4\\
&+ 1967775905034492315000t^3 + 17871960215153799455400t^2 + 86059133932673540802730t\\
&+ 168831626166263646709791\big) - 5103000000000\big],\\
R_6^{(8)}(t) = 4e^t\big[&1613103206400e^{2t}\big(5040947520000t^5 + 201877945920000t^4 + 3180788542512000t^3\\
&+ 25023710298132000t^2 + 100512381489684570t + 168909771771747913\big)\\
&- 100352e^t\big(9758278656000000t^7 + 787493087539200000t^6 + 26330540831965440000t^5\\
&+ 472351313974639680000t^4 + 4905522350938146000000t^3 + 29474260352050710470400t^2\\
&+ 94862093189089514535380t + 126296046350074196022531\big) - 705188106000000t^7\\
&- 92920921405200000t^6 - 5056037334500040000t^5 - 147062519417775780000t^4\\
&- 2465463388575178635000t^3 - 23775287930257276400400t^2 - 121803054362981139713530t\\
&- 254890760098937187512521\big]\\
\triangleq{} & 4e^t R_7(t),\\
R_7'(t) = 14\big[&460886630400e^{2t}\big(2520473760000t^5 + 107240157360000t^4 + 1792272217176000t^3\\
&+ 14897446555950000t^2 + 62768045893908285t + 109582981258295099\big)\\
&- 7168e^t\big(9758278656000000t^7 + 855801038131200000t^6 + 31055499357200640000t^5\\
&+ 604004018134466880000t^4 + 6794927606836704720000t^3 + 44190827404865148470400t^2\\
&+ 153810613893190935476180t + 221158139539163710557911\big) - 5\big(70518810600000t^6\\
&+ 7964650406160000t^5 + 361145523892860000t^4 + 8403572538158616000t^3\\
&+ 105662716653221941500t^2 + 679293940864493611440t + 1740043633756873424479\big)\big],\\
R_7''(t) = 784\big[&8230118400e^{2t}\big(5040947520000t^5 + 227082683520000t^4 + 4013505063792000t^3
\end{aligned}
$$

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\begin{aligned}
&+ 35171709763428000t^2 + 155330984899716570t + 281934008410498483\big)\\
&- 128e^t\big(9758278656000000t^7 + 924108988723200000t^6 + 36190305585987840000t^5\\
&+ 759281514920470080000t^4 + 9210943679374572240000t^3 + 64575610225375262630400t^2\\
&+ 242192268702921232416980t + 374968753432354646034091\big) - 225\big(167901930000t^5\\
&+ 15802877790000t^4 + 573246863322000t^3 + 10004253021617400t^2 + 83859298931128525t\\
&+ 269561087644640322\big)\big],\\
R_7^{(3)}(t) &= 784\big[32920473600e^{2t}\big(2520473760000t^5 + 119842526160000t^4 + 2233835215416000t^3\\
&+ 20595983679558000t^2 + 95251347331572285t + 179799750430178384\big)\\
&- 128e^t\big(9758278656000000t^7 + 992416939315200000t^6 + 41734959518327040000t^5\\
&+ 940233042850409280000t^4 + 12248069739056452560000t^3 + 92208441263498979350400t^2\\
&+ 371343489153671757677780t + 617161022135275878451071\big) - 39375\big(4797198000t^4\\
&+ 361208635200t^3 + 9827089085520t^2 + 114334320247056t + 479195993892163\big)\big],\\
R_7^{(4)}(t) &= 12544\big[2057529600e^{2t}\big(5040947520000t^5 + 252287421120000t^4 + 4947040535472000t^3\\
&+ 47893473005364000t^2 + 231694662022260570t + 454850848191929053\big)\\
&- 8e^t\big(9758278656000000t^7 + 1060724889907200000t^6 + 47689461154218240000t^5\\
&+ 1148907840442044480000t^4 + 16009001910458089680000t^3 + 128952650480668337030400t^2\\
&+ 555760371680669716378580t + 988504511288947636128851\big) - 2480625\big(19036500t^3\\
&+ 1075025700t^2 + 19498192630t + 113426905007\big)\big],\\
R_7^{(5)}(t) &= 25088\big[4115059200e^{2t}\big(2520473760000t^5 + 132444894960000t^4 + 2725807688856000t^3\\
&+ 27657016904286000t^2 + 139794067513812285t + 285349089601529669\big)\\
&- 4e^t\big(9758278656000000t^7 + 1129032840499200000t^6 + 54053810493661440000t^5\\
&+ 1387355146213135680000t^4 + 20604633272226267600000t^3 + 176979656212042606070400t^2\\
&+ 813665672642006390439380t + 1544264882969617352507431\big) - 86821875\big(815850t^2\\
&+ 30715020t + 278545609\big)\big],\\
R_7^{(6)}(t) &= 100352\big[1028764800e^{2t}\big(5040947520000t^5 + 277492158720000t^4 + 5981394957552000t^3\\
&+ 63491456875140000t^2 + 334902168836196570t + 710492246716871623\big)\\
&- e^t\big(9758278656000000t^7 + 1197340791091200000t^6 + 60828007536656640000t^5\\
&+ 1657624198681442880000t^4 + 26154053857078810320000t^3 + 238793556028721408870400t^2\\
&+ 1167624985066091602580180t + 2357930555611623742946811\big) - 27348890625(1295t + 24377)\big],\\
R_7^{(7)}(t) &= 100352\big[4115059200e^{2t}\big(2520473760000t^5 + 145047263760000t^4 + 3268189637496000t^3\\
&+ 36231774655734000t^2 + 199196812855668285t + 438971665567484954\big)\\
&- e^t\big(9758278656000000t^7 + 1265648741683200000t^6 + 68012052283203840000t^5\\
&+ 1961764236364726080000t^4 + 32784550651804581840000t^3 + 317255717599957839830400t^2\\
&+ 1645212097123534420320980t + 3525555540677715345526991\big) - 35416813359375\big],\\
R_7^{(8)}(t) &= 100352e^t\big[4115059200e^t\big(5040947520000t^5 + 302696896320000t^4 + 7116568330032000t^3\\
&+ 82268118223956000t^2 + 470857175022804570t + 1077140143990638193\big) - 9758278656000000t^7\\
&- 1333956692275200000t^6 - 75605944733303040000t^5 - 2301824497780745280000t^4\\
&- 40631607597263486160000t^3 - 415609369555371585350400t^2 - 2279723532323450099981780t\\
&- 5170767637801249765847971\big]\\
&\triangleq 100352e^tR_8(t),\\
R_8'(t) &= 980\big[4199040e^t\big(5040947520000t^5 + 327901633920000t^4 + 8327355915312000t^3
\end{aligned}
$$

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\begin{aligned}
&+ 103617823214052000t^2 + 635393411470716570t + 1547997319013442763\big)\\
&- 69701990400000t^6 - 8167081789440000t^5 - 385744615986240000t^4 - 9395202031758144000t^3\\
&- 124382472236520876000t^2 - 848182386847697112960t - 2326248502370867448961\big],\\
R_8''(t) &= 2822400\big[1458e^t\big(5040947520000t^5 + 353106371520000t^4 + 9638962450992000t^3\\
&+ 128599890959988000t^2 + 842629057898820570t + 2183390730484159333\big) - 49\big(2963520000t^5\\
&+ 289366560000t^4 + 10933804308000t^3 + 199727934348600t^2 + 1762790139406475t\\
&+ 6010362718591958\big)\big],\\
R_8^{(3)}(t) &= 2822400\big[1458e^t\big(5040947520000t^5 + 378311109120000t^4 + 11051387937072000t^3\\
&+ 157516778312964000t^2 + 1099828839818796570t + 3026019788382979903\big) - 1225\big(592704000t^4\\
&+ 46298649600t^3 + 1312056516960t^2 + 15978234747888t + 70511605576259\big)\big],\\
R_8^{(4)}(t) &= 50803200\big[81e^t\big(5040947520000t^5 + 403515846720000t^4 + 12564632373552000t^3\\
&+ 190670942124180000t^2 + 1414862396444724570t + 4125848628201776473\big)\\
&- 68600\big(2352000t^3 + 137793600t^2 + 2603286740t + 15851423361\big)\big],\\
R_8^{(5)}(t) &= 50803200\big[81e^t\big(5040947520000t^5 + 428720584320000t^4 + 14178695760432000t^3\\
&+ 228364839244836000t^2 + 1796204280693084570t + 5540711024646501043\big)\\
&- 1372000\big(352800t^2 + 13779360t + 130164337\big)\big],\\
R_8^{(6)}(t) &= 457228800\big[9e^t\big(5040947520000t^5 + 453925321920000t^4 + 15893578097712000t^3\\
&+ 270900926526132000t^2 + 2252933959182756570t + 7336915305339585613\big)\\
&- 1536640000(70t + 1367)\big],\\
R_8^{(7)}(t) &= 457228800\big[9e^t\big(5040947520000t^5 + 479130059520000t^4 + 17709279385392000t^3\\
&+ 318581660819268000t^2 + 2794735812235020570t + 9589849264522342183\big) - 107564800000\big],\\
R_8^{(8)}(t) &= 4115059200e^t\big(5040947520000t^5 + 504334797120000t^4 + 19625799623472000t^3\\
&+ 371709498975444000t^2 + 3431899133873556570t + 12384585076757362753\big)\\
&> 0,
\end{aligned}
$$

and

$R^{(k)}(0) = 0, \quad 0 \le k \le 6; \quad R_1^{(k)}(0) = 0, \quad 0 \le k \le 8; \quad R_2(0) = 0, \quad R_2'(0) = 94594500,$

$R_2''(0) = 5675670000, \quad R_2^{(3)}(0) = 186994407600, \quad R_2^{(4)}(0) = 4870671886080,$

$R_2^{(5)}(0) = 115977230617248, \quad R_2^{(6)}(0) = 2568045873032640, \quad R_2^{(7)}(0) = 51468046246929312,$

$R_2^{(8)}(0) = 919261925834668416; \quad R_3(0) = 2127921124617288, \quad R_3'(0) = 31808262868948566,$

$R_3''(0) = 422643933990433848, \quad R_3^{(3)}(0) = 5059127238393038844, \quad R_3^{(4)}(0) = 55311317976132200928,$

$R_3^{(5)}(0) = 559358124246419710080, \quad R_3^{(6)}(0) = 5291254778485384410240,$

$R_3^{(7)}(0) = 47275711643630010675648, \quad R_3^{(8)}(0) = 402327284442724791735744;$

$R_4(0) = 25145455277670299483484, \quad R_4'(0) = 180175476735597623990958,$

$R_4''(0) = 1233364679292092331540240, \quad R_4^{(3)}(0) = 8128711084294241029973952,$

$R_4^{(4)}(0) = 51948114141680687652017184, \quad R_4^{(5)}(0) = 323999742368987506576507680,$

$R_4^{(6)}(0) = 1983712454318091617900255520, \quad R_4^{(7)}(0) = 11984168286345820788017615520,$

$R_4^{(8)}(0) = 71752827530169076522475337120; \quad R_5(0) = 448455172063556728265470857,$

$R_5'(0) = 2222062109238254700527363412, \quad R_5''(0) = 10966377777101822969342094780,$

$R_5^{(3)}(0) = 54073563048190450871500371120, \quad R_5^{(4)}(0) = 266769748534576065247741326480,$

$R_5^{(5)}(0) = 1316887765697877890640258458880, \quad R_5^{(6)}(0) = 6499339666080758476651443438720,$

$R_5^{(7)}(0) = 32030330270749184706488159641920, \quad R_5^{(8)}(0) = 157419994070058290561168403698880;$
$R_6(0) = 2499301333480323924217097 3, \quad R_6'(0) = 97363392902732249529746070,$
$R_6''(0) = 3764394989756147170036178 84, \quad R_6^{(3)}(0) = 144272952872843989060050 8052,$
$R_6^{(4)}(0) = 547780834236163770736322 0348, \quad R_6^{(5)}(0) = 206024766817960801698651 57140,$
$R_6^{(6)}(0) = 767769237585163482443166 87404, \quad R_6^{(7)}(0) = 283621637079942700292207 261572,$
$R_6^{(8)}(0) = 103917831481286409094547 0799068; \quad R_7(0) = 2597945787032160227363676 99767,$
$R_7'(0) = 684880850316419203514174 536198, \quad R_7''(0) = 178152569994569916744517 7663968,$
$R_7^{(3)}(0) = 457863550517534895246453 3634608, \quad R_7^{(4)}(0) = 116403433781881974287224 27451648,$
$R_7^{(5)}(0) = 293040719381486065534535 33106688, \quad R_7^{(6)}(0) = 731136055161011905051982 64803328,$
$R_7^{(7)}(0) = 180921493369266920211167 500832768, \quad R_7^{(8)}(0) = 444290887449456146125411 745425408;$
$R_8(0) = 44273246915801991602101776 29, \quad R_8'(0) = 636782088564927911545658 7820,$
$R_8''(0) = 898395089393444957435681 2800, \quad R_8^{(3)}(0) = 124520067799218509923382 97600,$
$R_8^{(4)}(0) = 169780561115011529936754 81600, \quad R_8^{(5)}(0) = 228003448037996428684419 45600,$
$R_8^{(6)}(0) = 301918398664096524628192 89600, \quad R_8^{(7)}(0) = 394627973934041733794622 33600,$
$R_8^{(8)}(0) = 509633007582930917644699 77600.$

The proof of Theorem 1.1 is thus completed.

3. Remarks

Remark 3.1. The method in the proof of Theorem 1.1 has been effectively used in several articles such as [4, 7, 9, 10]. Generally speaking, this method may be applied to deal with the functions like

$$Q_{n;m_0,m_1,\dots,m_n}(t) = \sum_{k=0}^{n} e^{kt} P_{m_k}(t), \tag{3.1}$$

where $n, m_k \in \mathbb{N} \cup \{0\}$ and $P_{m_k}(t)$ are polynomials of degree m_k.

Remark 3.2. With the help of the famous software Mathematica, we may obtain that

$$R(t) = \frac{1}{216}t^{15} + \frac{1}{48}t^{16} + \frac{71}{1440}t^{17} + \frac{38599}{453600}t^{18} + \frac{1114147}{9072000}t^{19} + \frac{1414531}{9072000}t^{20} + \frac{200428987}{1143072000}t^{21} + \cdots. \tag{3.2}$$

This implies that Theorem 1.1 may also be proved by carefully expanding the function $R(t)$ into the power series at the point 0 and minutely verifying the positivity of the coefficients of t^k for $k \ge 15$.

Remark 3.3. This paper is a modified and extended version of [8, Appendix].

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